Editors' Suggestion

Topological Field Theory of Non-Hermitian Systems

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Non-Hermiticity gives rise to unique topological phases without Hermitian analogs. However, the effective field theory has yet to be established. Here, we develop a field-theoretical description of the intrinsic non-Hermitian topological phases. Because of the dissipative and nonequilibrium nature of non-Hermiticity, our theory is formulated solely in terms of spatial degrees of freedom, which contrasts with the conventional theory defined in spacetime. Our theory provides a universal understanding of non-Hermitian topological phenomena such as the unidirectional transport in one dimension and the chiral magnetic skin effect in three dimensions. Furthermore, it systematically predicts new physics; we illustrate this by revealing transport phenomena and skin effects in two dimensions induced by a perpendicular spatial texture. From the field-theoretical perspective, the non-Hermitian skin effect, i.e., the anomalous localization due to non-Hermiticity, is shown to be a signature of an anomaly.

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Topology plays a key role in modern physics. In particular, topological phases of matter have been arguably one of the most actively studied condensed-matter systems in recent years [1–3]. A universal understanding of topological phases is obtained by topological field theory in spacetime. For example, the Chern-Simons theory describes the quantum Hall effect [4–9], and the axion electrodynamics describes the magnetoelectric effect [10,11]. One of the consequences of the topological field theory description is the bulk-boundary correspondence: in the presence of a boundary, a topological field theory is gauge dependent at the boundary, and this gauge noninvariance must be compensated by an anomaly at the boundary [12].

While topological phases were mainly investigated for Hermitian systems at equilibrium, topological phases of non-Hermitian systems are attracting growing interest [13–57]. Non-Hermiticity arises from dissipation [58–60], and the interplay between non-Hermiticity and topology leads to new physics in open classical and quantum systems [61–74]. One of the remarkable consequences of non-Hermiticity is new types of topological phases without Hermitian analogs. For example, non-Hermitian topological phases arise generally in odd spatial dimensions [24,31], while no topological phases appear in these dimensions for Hermitian systems without symmetry. These unique topological phases arise from the complex-valued nature of spectra, which enables two types of energy gaps, i.e., point and line gaps [31]. In the presence of a boundary, such intrinsic non-Hermitian topology leads to the anomalous localization of an extensive number of eigenstates [48,49] that constitutes the non-Hermitian skin effect [18,25,26].

However, topological field theory has yet to be established for non-Hermitian systems. The Chern-Simons theory was shown to remain well defined even for non-Hermitian Chern insulators [34]. Still, this theory only describes non-Hermitian topological phases that are continuously deformed to Hermitian phases. Field-theoretical characterization of intrinsic non-Hermitian topology has remained elusive, although it is crucial for understanding and exploring non-Hermitian topological phenomena, including the skin effect.

In this Letter, we develop topological field theory of non-Hermitian systems. We show that field theory of intrinsic non-Hermitian topology is formulated solely by spatial degrees of freedom as a consequence of the dissipative and nonequilibrium nature of non-Hermiticity. This contrasts with the conventional theory defined by both spatial and temporal degrees of freedom. Our theory universally describes and systematically predicts unique non-Hermitian topological phenomena. Furthermore, we show that the non-Hermitian skin effect is a signature of an anomaly.

Non-Hermitian topology.—Non-Hermitian systems give rise to unique topological phases that have no counterparts in Hermitian systems. Such intrinsic non-Hermitian topology arises even in one dimension. Suppose that a non-Hermitian Bloch Hamiltonian $H(k)$ has a point gap, i.e., it is invertible in terms of reference energy $E \in \mathbb{C}$ (i.e., $\det[H(k) - E] \neq 0$ for all k) [24,31]. Then, the following winding number $W_1(E) \in \mathbb{Z}$ is well defined:

$$W_1(E) := -\oint_{\mathrm{BZ}} \frac{dk}{2\pi\mathrm{i}} \left\{ \frac{d}{dk} \log\det[H(k) - E] \right\}. \quad (1)$$

Notably, $W_1(E)$ vanishes when $H(k) - E$ is Hermitian. This is consistent with the absence of topological phases in one-dimensional Hermitian systems without symmetry [1–3].

A prototypical lattice model with nontrivial W_1 is given by [75]

$$\hat{H} = -\frac{1}{2}\sum_n [(1+\gamma)\hat{c}_{n+1}^\dagger \hat{c}_n + (1-\gamma)\hat{c}_n^\dagger \hat{c}_{n+1}], \quad (2)$$

where $\hat{c}_n$ ($\hat{c}_n^\dagger$) annihilates (creates) a particle on site n, and $\gamma \in \mathbb{R}$ denotes the asymmetry of the hopping amplitudes and describes the degree of non-Hermiticity. The corresponding Bloch Hamiltonian reads $H(k) = -\cos k + \mathrm{i}\gamma \sin k$ and winds around the origin in the complex energy plane. Consequently, we have $W_1(E) = \mathrm{sgn}(\gamma)$ as long as the reference energy E is inside the region surrounded by the loop of $H(k)$. Despite the presence of a point gap, an energy gap for the real part of the spectrum closes at $k = \pm\pi/2$, i.e., $\mathrm{Re}H(k = \pm\pi/2) = 0$; this type of energy gap is called a line gap [21,31]. To understand a universal feature of non-Hermitian topology, let us consider the continuum Dirac Hamiltonian around the line-gap-closing points:

$$H(k) = k + \mathrm{i}\gamma. \quad (3)$$

This non-Hermitian Dirac Hamiltonian, similarly to its lattice counterpart, is characterized by the nonzero winding number $W_1(E) = \mathrm{sgn}(\gamma - \mathrm{Im}E)/2$.

An important consequence of nontrivial W_1 is the non-Hermitian skin effect [48,49]. In the presence of a boundary, there appear $|W_1(E)|$ eigenstates with the eigenenergy E at the boundary. Notably, $W_1(E)$ can be nontrivial for many choices of the reference energy E, and consequently, an extensive number of eigenstates are localized at the boundary. In the lattice model in Eq. (2), all the eigenstates are localized at the right (left) edge for $\gamma > 0$ ($\gamma < 0$). Such anomalous localization is impossible in Hermitian systems and hence a unique non-Hermitian topological phenomenon.

Topological field theory.—Before developing effective field theory for intrinsic non-Hermitian topology, let us briefly recall the Hermitian case. The effective field theory for Hermitian systems is obtained by introducing a gauge potential $(\boldsymbol{A}, \phi)$ to a microscopic Hamiltonian and integrating out matter degrees of freedom. The quantum partition function of a Hamiltonian $H(\boldsymbol{k})$ is given by path integral as $Z[\boldsymbol{A}, \phi] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\mathrm{i}S}$ with the (real-time) action

$$\mathcal{S} = \int \bar{\psi}[\mathrm{i}\partial_t + \phi - H(-\mathrm{i}\partial_{\boldsymbol{x}} - \boldsymbol{A})]\psi d^d x dt. \quad (4)$$

Here, ψ and $\bar{\psi}$ describe matter degrees of freedom. Integrating over the matter field, we obtain the effective action $S[\boldsymbol{A}, \phi]$ for the external field, $e^{\mathrm{i}S[\boldsymbol{A},\phi]} \coloneqq Z[\boldsymbol{A}, \phi]/Z[0]$ with

$$Z[\boldsymbol{A}, \phi] = \det[\mathrm{i}\omega + \phi - H(\boldsymbol{k} - \boldsymbol{A})]. \quad (5)$$

In the presence of an energy gap, the effective action is given by a local functional of $(\boldsymbol{A}, \phi)$. The response of the system to the external field is read off from the current density $\boldsymbol{j} \coloneqq \delta S/\delta \boldsymbol{A}$. In this formulation, the topological invariant that appears in the topological term of the effective action is given by the Green's function [76]

$$G_0(\boldsymbol{k}, \omega) \coloneqq [\mathrm{i}\omega - H(\boldsymbol{k})]^{-1}, \quad (6)$$

which is a non-Hermitian matrix. For example, if we consider the Dirac Hamiltonian $H(\boldsymbol{k}) = k_x\sigma_x + k_y\sigma_y + m\sigma_z$ with Pauli matrices σ_i's, we obtain the $(2+1)$-dimensional Chern-Simons theory, which describes the quantum Hall effect.

The above path integral, in its Euclidean version, assumes the Gibbs state as an equilibrium density matrix. On the other hand, for the non-Hermitian case, the thermal equilibrium is no longer achieved, and it is generally unclear what kind of path integral one should set up. This constitutes a fundamental difficulty for developing effective field theory. This may be tackled, for example, by the Schwinger-Keldysh approach [77]. Nevertheless, as long as an energy gap for the real part of the spectrum (i.e., line gap) stays open, the above procedure gives rise to topological field theory even for non-Hermitian systems. In this case, non-Hermitian topological phases are continuously deformable to their Hermitian counterparts [31] and share the same topological field theory. For example, for non-Hermitian Chern insulators, the above procedure delivers the $(2+1)$-dimensional Chern-Simons theory [34]. However, this is not the case for intrinsic non-Hermitian topology. For the non-Hermitian Dirac Hamiltonian in Eq. (3), the line gap vanishes, and matter degrees of freedom cannot be integrated out safely; if we calculate $S[\boldsymbol{A}, \phi]$ from Eq. (5), it is ill defined. We also note that the quantization of $W_1(E)$ in Eq. (1) is guaranteed by the point gap $E(k) \neq E$ instead of the line gap $\mathrm{Re}E(k) \neq E$, which is a unique gap structure due to the complex-valued nature of the spectrum [24,31,37].

We thus seek a different formulation of field theory for intrinsic non-Hermitian topological phases. Since these phases arise out of equilibrium, the temporal degree of freedom should play a special role. Then, let us Fourier transform the field operator in time by $\psi(\boldsymbol{x}, t) = \int \psi_E(\boldsymbol{x}) e^{-\mathrm{i}Et} dE$ and focus on fixed energy E. We also switch off the scalar potential ϕ and focus on a time-independent vector potential $\boldsymbol{A}(\boldsymbol{x})$. The action in Eq. (4) reduces to

$$\mathcal{S}_E = \int \bar{\psi}_E [H(-\mathrm{i}\partial_{\boldsymbol{x}} - \boldsymbol{A}) - E] \psi_E d^d x. \tag{7}$$

In contrast to Eq. (4), this action is written solely in terms of the spatial degrees of freedom. The functional integral

$$Z_E[\boldsymbol{A}] = \int \mathcal{D}\bar{\psi}_E \mathcal{D}\psi_E e^{\mathrm{i}\mathcal{S}_E} = \det [H(\boldsymbol{k} - \boldsymbol{A}) - E] \tag{8}$$

serves as a generating functional of the single-particle Green's function $[E - H(\boldsymbol{k})]^{-1}$ with reference energy E. It is therefore expected to capture all physical information—including topological one such as the non-Hermitian skin effect. This type of spatial field theory is commonly used for Anderson localization [78,79] and also for Hermitian topological systems in odd dimensions [80]. It is discussed also for Floquet systems and their boundary unitary operators [81].

To further emphasize the special role played by the temporal direction, we note that one of the wave numbers, such as k in Eq. (3), plays a similar role to frequency ω for Hermitian systems; the inverse of the Green's function $G_0^{-1}(\boldsymbol{k}, \omega)$ in Eq. (6) for a Hermitian Hamiltonian is identified with a non-Hermitian Hamiltonian $H(\boldsymbol{k})$ in Eq. (8) by replacing ω with k. Thus, the effective action of non-Hermitian systems in $d + 0$ dimensions is mathematically equivalent to that of Hermitian systems in $(d-1) + 1$ dimensions. Consistently, d-dimensional non-Hermitian systems are topologically classified in the same manner as $(d-1)$-dimensional Hermitian systems in the same symmetry class [82]; the difference of one dimension corresponds to time [83]. The degree of a point gap, such as γ in Eq. (3), gives a relevant energy scale and ensures the local expansion of the effective action by the gauge potential.

One dimension.—Below, we explicitly provide field theories of intrinsic non-Hermitian topology and discuss unique phenomena, including the skin effect. For the non-Hermitian Dirac Hamiltonian in Eq. (3), the effective action is

$$S_E[A] \simeq \mathrm{i}\,\mathrm{tr}\{[H(-\mathrm{i}\partial_x) - E] A(x)\}, \tag{9}$$

where the vector potential A is assumed to be sufficiently small. After taking the sum explicitly, this further reduces to

$$S_E[A] = W_1(E) \int A(x) dx, \tag{10}$$

where the winding number $W_1(E)$ is defined for reference energy E as Eq. (1). This is the $(1+0)$-dimensional Chern-Simons theory. As discussed above, replacing x with t, we have the $(0+1)$-dimensional Chern-Simons theory, which describes Hermitian systems in zero dimension.

From this effective action, the current is obtained as

$$j(x, E) \coloneqq \frac{\delta S_E[A]}{\delta A(x)} = W_1(E). \tag{11}$$

Thus, particles unidirectionally flow from the left to the right (from the right to the left) for $W_1 > 0$ ($W_1 < 0$). Consistently, in the lattice model in Eq. (2), the hopping amplitude from the left to the right is greater (smaller) than that from the right to the left for $W_1 > 0$ ($W_1 < 0$). This type of directional amplification ubiquitously appears, for example, in open photonic systems [17,24,47,52] and parametrically driven bosonic systems [27]. The topological field theory in Eq. (10) underlies such unidirectional transport induced by asymmetric hopping.

Skin effect as an anomaly.—In the presence of a boundary, the topological action is no longer gauge invariant. Suppose that the system described by Eq. (10) lies in $x_L \leq x \leq x_R$, outside of which is the vacuum. Then, under the gauge transformation $A \to A + df/dx$ with an arbitrary function f, the effective action S_E changes into $S_E + W_1(E)[f(x_R) - f(x_L)]$ and explicitly depends on the choice of the gauge f. To retain gauge invariance, additional degree of freedom is required at the boundary $x = x_L, x_R$. This boundary system reads

$$S_E^{\text{boundary}} = -W_1(E)[\varphi(x_R) - \varphi(x_L)], \tag{12}$$

where $\varphi(x)$ denotes the phase of the wave function at x. Since φ changes to $\varphi + f$ through the gauge transformation, S_E^{boundary} changes into $S_E^{\text{boundary}} - W_1(E)[f(x_R) - f(x_L)]$. Thus, while S_E and S_E^{boundary} are individually gauge dependent, their combination $S_E + S_E^{\text{boundary}}$ is indeed gauge invariant.

The boundary action in Eq. (12) describes a pair of the charges $W_1(E)$ and $-W_1(E)$ at $x = x_R$ and $x = x_L$, respectively. These charges correspond to skin modes. Importantly, reference energy E can be chosen arbitrarily as long as $W_1(E)$ is nontrivial. An extensive number of the charges appear at the boundary, which correspond to an extensive number of skin modes. Thus, the skin effect originates from a non-Hermitian anomaly. This contrasts with Hermitian systems in one dimension, in which an anomaly results in only a finite number of symmetry-protected zero-energy modes at the boundary. We note that an anomaly discussed here is distinct from a dynamical anomaly in Refs. [42,51,53,84].

Three dimensions.—Topological field theories are also formulated in higher dimensions. In general, non-Hermitian systems in d dimensions are described by the $(d+0)$-dimensional Chern-Simons theory for odd d. This contrasts with Hermitian systems, which are described by the $(d+1)$-dimensional Chern-Simons theory for even d.

In three dimensions, for example, the non-Hermitian Dirac Hamiltonian $H(\boldsymbol{k}) = k_x\sigma_x + k_y\sigma_y + k_z\sigma_z + \mathrm{i}\gamma$ results in the $(3+0)$-dimensional Chern-Simons theory

$$S_E[\boldsymbol{A}] = \frac{W_3(E)}{4\pi} \sum_{ijk} \int \varepsilon_{ijk} A_i(\boldsymbol{x}) \partial_j A_k(\boldsymbol{x}) d^3x, \quad (13)$$

where W_3 is the three-dimensional winding number. The current density of this theory is

$$\boldsymbol{j}(\boldsymbol{x}, E) \coloneqq \frac{\delta S_E[\boldsymbol{A}]}{\delta \boldsymbol{A}(\boldsymbol{x})} = \frac{W_3(E)}{2\pi} \boldsymbol{B}(\boldsymbol{x}), \quad (14)$$

where $\boldsymbol{B} \coloneqq \nabla \times \boldsymbol{A}$ is a magnetic field. Thus, particles flow along the direction of the magnetic field $\boldsymbol{B}$. This is the chiral magnetic effect [85] in which non-Hermiticity induces chirality imbalance in a similar manner to an electric field. This further means that the direction of amplification can be controlled by a magnetic field, which is a unique property of three-dimensional systems. It is also remarkable that Ref. [53] recently constructed a lattice model that exhibits the non-Hermitian chiral magnetic effect. This Letter gives a field-theoretical understanding about it.

Under the open boundary conditions, S_E is gauge dependent. For the quantum Hall effect, which is described by the $(2+1)$-dimensional Chern-Simons theory, the gauge noninvariance is balanced with an anomaly of chiral fermions at the boundary [12]. In the non-Hermitian case, the boundary degrees of freedom are described by a Hamiltonian with a single exceptional point, $H(\boldsymbol{k}) = \pm k_x - \mathrm{i}k_y$, for $|W_3(E)| = 1$ [54,86], which reduces to the inverse of the Green's function of the conventional chiral fermions by replacing k_y with frequency ω. In $1+1$ dimensions, a chiral anomaly is described by $\partial_x j_x^A + \partial_t j_t^A = E/\pi$ with an axial current (j_x^A, j_t^A) and an electric field $E \coloneqq \partial_x A_t - \partial_t A_x$ [87–89]. Replacing time with another spatial component y, we have the following non-Hermitian analog of the anomaly equation [86]:

$$\nabla \cdot \boldsymbol{j}^A(\boldsymbol{x}, E) = \frac{W_3(E) B(\boldsymbol{x})}{\pi}, \quad (15)$$

where $B \coloneqq \partial_x A_y - \partial_y A_x$ is a magnetic field perpendicular to the surfaces. In terms of the global quantities such as the charge N_R (N_L) of the right-moving (left-moving) particle $H(\boldsymbol{k}) = k_x - \mathrm{i}k_y$ [$H(\boldsymbol{k}) = -k_x - \mathrm{i}k_y$], as well as the magnetic flux $\Phi \coloneqq \int B(\boldsymbol{x}) d^2x$, this anomaly equation reduces to

$$N_R(E) - N_L(E) = \frac{W_3(E)\Phi}{\pi}. \quad (16)$$

Combining it with the global conservation law $N_R + N_L = 0$ due to U(1) symmetry, we have $N_R = W_3\Phi/2\pi$ and $N_L = -W_3\Phi/2\pi$. Here, $\Phi/2\pi$ is the number of the fluxes since 2π denotes the flux quantum in the natural units (i.e., $e = \hbar = 1$). Thus, a signature of the topological action in three dimensions appears as the skin effect induced by a magnetic field. The number of the skin modes is given by the topological invariant W_3 and the number of fluxes. While Ref. [53] predicted this three-dimensional version of the skin effect—the chiral magnetic skin effect—on the basis of the bulk topological invariant, we here derive it from a chiral anomaly at a boundary. It occurs also in a lattice model [86,90].

Two dimensions.—For Hermitian systems, topological field theories of superconductors in $0+1$ and $1+1$ dimensions and insulators in $2+1$ and $3+1$ dimensions are derived from the Chern-Simons theories in $2+1$ and $4+1$ dimensions, respectively [10]. Topological field theories of non-Hermitian systems in even dimensions are also derived from higher-dimensional ones. For example, let us reduce the z direction of the $(3+0)$-dimensional theory in Eq. (13) by considering z to be a parameter and making the gauge potential $\boldsymbol{A}$ be independent of z. Then, the effective action in Eq. (13) reduces to

$$S_E[\boldsymbol{A}] = \frac{1}{2\pi} \sum_{ij} \int \theta(\boldsymbol{x}, E) \varepsilon_{ij} \partial_i A_j(\boldsymbol{x}) d^2x \quad (17)$$

with the Wess-Zumino term θ [86,91]. This is the effective action of non-Hermitian systems in two dimensions. Here, θ is a non-Hermitian analog of the electric polarization in $(1+1)$-dimensional Hermitian systems [92] and $\mathbb{Z}_2$ quantized in the presence of reciprocity or particle-hole symmetry.

The action in Eq. (17) generally describes non-Hermitian topological phenomena in two dimensions. The current density is

$$j_i(\boldsymbol{x}, E) = \frac{1}{2\pi} \sum_j \varepsilon_{ij} \partial_j \theta(\boldsymbol{x}, E), \quad (18)$$

which shows a particle flow in the direction perpendicular to the gradient of θ. Now, suppose that θ is spatially modulated along the y direction. Naively, such a y-dependent texture leads to a flow along the y direction and is irrelevant to transport along the x direction. However, Eq. (18) describes a perpendicular flow along the x direction as a result of non-Hermitian topology. To confirm this phenomenon, we investigate the lattice model $H = H_0 + V$ with $H_0(\boldsymbol{k}) = \sigma_x \sin k_x + \sigma_y \sin k_y + \mathrm{i}\gamma(\cos k_x + \cos k_y)$ and $V(\boldsymbol{x}) = \sigma_z \sin\phi(\boldsymbol{x}) + \mathrm{i}\gamma\cos\phi(\boldsymbol{x})$. Here, ϕ is given as $\phi(\boldsymbol{x}) = \pi/2 - 2\pi\Theta y/L_y$, leading to nontrivial θ [86]. While no topological features appear in the absence of the spatial texture, the y-dependent texture induces the complex-spectral winding [Fig. 1(a)]. Consequently, a particle flow along the x direction arises [Fig. 1(b)], which is consistent with the topological field theory description in Eq. (18). This perpendicular transport accompanies the perpendicular skin effect under the open boundary conditions [86]. The number of the skin modes is

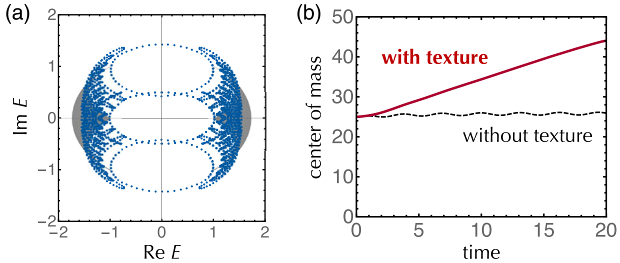

FIG. 1. Non-Hermitian topological phenomena in two dimensions. The periodic boundary conditions are imposed along both directions ($L_x = 50$, $L_y = 20$; $\gamma = 0.5$). (a) Complex spectra in the presence ($\Theta = 1$, blue dots) and absence ($\Theta = 0$, gray regions) of the spatial texture. (b) Time evolution of the wave packet center of mass along the x direction. The initial state is prepared to be $|\psi(0)\rangle \propto \sum_y |x = L_x/2, y\rangle$.

controlled by the spatial gradient Θ, which is also a unique feature of two-dimensional systems.

Discussions.—In this Letter, we develop topological field theory of non-Hermitian systems. Because of the dissipative and nonequilibrium nature of non-Hermiticity, the temporal degree of freedom is distinguished from the spatial degrees of freedom, and the field theory is formulated solely by the latter. This theory provides the universal understanding of non-Hermitian topological phenomena. We also demonstrate that the non-Hermitian skin effect originates from an anomaly. For Hermitian systems, topological field theory is relevant not only to noninteracting systems but also to disordered and interacting systems. Similarly, our theory should be applicable to non-Hermitian systems with disorder and interaction. Finally, it is noteworthy that other types of nonequilibrium topological field theory have recently been developed for Floquet operators [81] and Lindblad master equations [93].

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[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[3] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. **88**, 035005 (2016).
[4] A. N. Redlich, Phys. Rev. D **29**, 2366 (1984).
[5] A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983).
[6] S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).
[7] J. Fröhlich and T. Kerler, Nucl. Phys. **B354**, 369 (1991).
[8] A. Lopez and E. Fradkin, Phys. Rev. B **44**, 5246 (1991).
[9] X. G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992).
[10] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008).
[11] A. M. Essin, J. E. Moore, and D. Vanderbilt, Phys. Rev. Lett. **102**, 146805 (2009).
[12] C. G. Callan and J. A. Harvey, Nucl. Phys. **B250**, 427 (1985).
[13] M. S. Rudner and L. S. Levitov, Phys. Rev. Lett. **102**, 065703 (2009).
[14] M. Sato, K. Hasebe, K. Esaki, and M. Kohmoto, Prog. Theor. Phys. **127**, 937 (2012); K. Esaki, M. Sato, K. Hasebe, and M. Kohmoto, Phys. Rev. B **84**, 205128 (2011).
[15] Y. C. Hu and T. L. Hughes, Phys. Rev. B **84**, 153101 (2011).
[16] H. Schomerus, Opt. Lett. **38**, 1912 (2013).
[17] S. Longhi, D. Gatti, and G. D. Valle, Sci. Rep. **5**, 13376 (2015).
[18] T. E. Lee, Phys. Rev. Lett. **116**, 133903 (2016).
[19] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, Phys. Rev. Lett. **118**, 040401 (2017).
[20] Y. Xu, S.-T. Wang, and L.-M. Duan, Phys. Rev. Lett. **118**, 045701 (2017).
[21] H. Shen, B. Zhen, and L. Fu, Phys. Rev. Lett. **120**, 146402 (2018); V. Kozii and L. Fu, arXiv:1708.05841.
[22] K. Takata and M. Notomi, Phys. Rev. Lett. **121**, 213902 (2018).
[23] V. M. M. Alvarez, J. E. B. Vargas, and L. E. F. Foa Torres, Phys. Rev. B **97**, 121401(R) (2018).
[24] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X **8**, 031079 (2018); K. Kawabata, S. Higashikawa, Z. Gong, Y. Ashida, and M. Ueda, Nat. Commun. **10**, 297 (2019).
[25] S. Yao and Z. Wang, Phys. Rev. Lett. **121**, 086803 (2018); S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. **121**, 136802 (2018).
[26] F. K. Kunst, E. Edvardsson, J. C. Budich, and E. J. Bergholtz, Phys. Rev. Lett. **121**, 026808 (2018).
[27] A. McDonald, T. Pereg-Barnea, and A. A. Clerk, Phys. Rev. X **8**, 041031 (2018).
[28] C. H. Lee and R. Thomale, Phys. Rev. B **99**, 201103(R) (2019).
[29] T. Liu, Y.-R. Zhang, Q. Ai, Z. Gong, K. Kawabata, M. Ueda, and F. Nori, Phys. Rev. Lett. **122**, 076801 (2019).
[30] C. H. Lee, L. Li, and J. Gong, Phys. Rev. Lett. **123**, 016805 (2019).
[31] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, Phys. Rev. X **9**, 041015 (2019).
[32] H. Zhou and J. Y. Lee, Phys. Rev. B **99**, 235112 (2019).
[33] M. N. Chernodub and A. Cortijo, Symmetry **12**, 761 (2020).
[34] M. R. Hirsbrunner, T. M. Philip, and M. J. Gilbert, Phys. Rev. B **100**, 081104(R) (2019).
[35] H.-G. Zirnstein, G. Refael, and B. Rosenow, arXiv:1901.11241.
[36] D. S. Borgnia, A. J. Kruchkov, and R.-J. Slager, Phys. Rev. Lett. **124**, 056802 (2020).

[37] K. Kawabata, T. Bessho, and M. Sato, Phys. Rev. Lett. **123**, 066405 (2019).
[38] K. Yokomizo and S. Murakami, Phys. Rev. Lett. **123**, 066404 (2019).
[39] P. A. McClarty and J. G. Rau, Phys. Rev. B **100**, 100405(R) (2019).
[40] F. Song, S. Yao, and Z. Wang, Phys. Rev. Lett. **123**, 170401 (2019).
[41] E. J. Bergholtz and J. C. Budich, Phys. Rev. Research **1**, 012003(R) (2019).
[42] J. Y. Lee, J. Ahn, H. Zhou, and A. Vishwanath, Phys. Rev. Lett. **123**, 206404 (2019).
[43] T. Yoshida, K. Kudo, and Y. Hatsugai, Sci. Rep. **9**, 16895 (2019).
[44] H. Schomerus, Phys. Rev. Research **2**, 013058 (2020).
[45] L. Herviou, N. Regnault, and J. H. Bardarson, SciPost Phys. **7**, 069 (2019).
[46] P.-Y. Chang, J.-S. You, X. Wen, and S. Ryu, Phys. Rev. Research **2**, 033069 (2020).
[47] C. C. Wanjura, M. Brunelli, and A. Nunnenkamp, Nat. Commun. **11**, 3149 (2020).
[48] K. Zhang, Z. Yang, and C. Fang, Phys. Rev. Lett. **125**, 126402 (2020).
[49] N. Okuma, K. Kawabata, K. Shiozaki, and M. Sato, Phys. Rev. Lett. **124**, 086801 (2020).
[50] W. Xi, Z.-H. Zhang, Z.-C. Gu, and W.-Q. Chen, arXiv: 1911.01590.
[51] F. Terrier and F. K. Kunst, Phys. Rev. Research **2**, 023364 (2020).
[52] W.-T. Xue, M.-R. Li, Y.-M. Hu, F. Song, and Z. Wang, arXiv:2004.09529.
[53] T. Bessho and M. Sato, arXiv:2006.04204.
[54] M. M. Denner, A. Skurativska, F. Schindler, M. H. Fischer, R. Thomale, T. Bzdušek, and T. Neupert, arXiv:2008 .01090.
[55] Y. Ma and T. L. Hughes, arXiv:2008.02284.
[56] R. Okugawa, R. Takahashi, and K. Yokomizo, Phys. Rev. B **102**, 241202(R) (2020).
[57] K. Kawabata, M. Sato, and K. Shiozaki, Phys. Rev. B **102**, 205118 (2020).
[58] V. V. Konotop, J. Yang, and D. A. Zezyulin, Rev. Mod. Phys. **88**, 035002 (2016).
[59] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, Nat. Phys. **14**, 11 (2018).
[60] E. J. Bergholtz, J. C. Budich, and F. K. Kunst, Rev. Mod. Phys. **93**, 015005 (2021).
[61] C. Poli, M. Bellec, U. Kuhl, F. Mortessagne, and H. Schomerus, Nat. Commun. **6**, 6710 (2015).
[62] J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, M. S. Rudner, M. Segev, and A. Szameit, Phys. Rev. Lett. **115**, 040402 (2015).
[63] S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K. G. Makris, M. Segev, M. C. Rechtsman, and A. Szameit, Nat. Mater. **16**, 433 (2017).
[64] L. Xiao, X. Zhan, Z. H. Bian, K. K. Wang, X. Zhang, X. P. Wang, J. Li, K. Mochizuki, D. Kim, N. Kawakami, W. Yi, H. Obuse, B. C. Sanders, and P. Xue, Nat. Phys. **13**, 1117 (2017).
[65] P. St-Jean, V. Goblot, E. Galopin, A. Lemaître, T. Ozawa, L. L. Gratiet, I. Sagnes, J. Bloch, and A. Amo, Nat. Photon. **11**, 651 (2017).
[66] B. Bahari, A. Ndao, F. Vallini, A. E. Amili, Y. Fainman, and B. Kanté, Science **358**, 636 (2017).
[67] H. Zhou, C. Peng, Y. Yoon, C. W. Hsu, K. A. Nelson, L. Fu, J. D. Joannopoulos, M. Soljačić, and B. Zhen, Science **359**, 1009 (2018).
[68] G. Harari, M. A. Bandres, Y. Lumer, M. C. Rechtsman, Y. D. Chong, M. Khajavikhan, D. N. Christodoulides, and M. Segev, Science **359**, eaar4003 (2018); M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. Christodoulides, and M. Khajavikhan, Science **359**, eaar4005 (2018).
[69] H. Zhao, X. Qiao, T. Wu, B. Midya, S. Longhi, and L. Feng, Science **365**, 1163 (2019).
[70] M. Brandenbourger, X. Locsin, E. Lerner, and C. Coulais, Nat. Commun. **10**, 4608 (2019); A. Ghatak, M. Brandenbourger, J. van Wezel, and C. Coulais, Proc. Natl. Acad. Sci. U.S.A. **117**, 29561 (2020).
[71] T. Helbig, T. Hofmann, S. Imhof, M. Abdelghany, T. Kiessling, L. W. Molenkamp, C. H. Lee, A. Szameit, M. Greiter, and R. Thomale, Nat. Phys. **16**, 747 (2020); T. Hofmann, T. Helbig, F. Schindler, N. Salgo, M. Brzezińska, M. Greiter, T. Kiessling, D. Wolf, A. Vollhardt, A. Kabaši, C. H. Lee, A. Bilušić, R. Thomale, and T. Neupert, Phys. Rev. Research **2**, 023265 (2020).
[72] L. Xiao, T. Deng, K. Wang, G. Zhu, Z. Wang, W. Yi, and P. Xue, Nat. Phys. **16**, 761 (2020).
[73] S. Weidemann, M. Kremer, T. Helbig, T. Hofmann, A. Stegmaier, M. Greiter, R. Thomale, and A. Szameit, Science **368**, 311 (2020).
[74] L. Yamauchi, T. Hayata, M. Uwamichi, T. Ozawa, and K. Kawaguchi, arXiv:2008.10852.
[75] N. Hatano and D. R. Nelson, Phys. Rev. Lett. **77**, 570 (1996); Phys. Rev. B **56**, 8651 (1997).
[76] K. Ishikawa and T. Matsuyama, Nucl. Phys. **B280**, 523 (1987).
[77] A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, Cambridge, 2011).
[78] K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, England, 1997).
[79] A. Altland and B. Simons, *Condensed Matter Field Theory* (Cambridge University Press, Cambridge, England, 2006).
[80] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008).
[81] P. Glorioso, A. Gromov, and S. Ryu, Phys. Rev. Research **3**, 013117 (2021).
[82] Non-Hermitian Bloch Hamiltonians $H(\boldsymbol{k})$ in d dimensions with Altland-Zirnbauer symmetry have the same topological classification as Hermitian Bloch Hamiltonians in $d-1$ dimensions with the same symmetry class [31], in which time-reversal and particle-hole symmetry is respectively defined by $\mathcal{T}H^{*}(\boldsymbol{k})\mathcal{T}^{-1}=H(-\boldsymbol{k})$ and $\mathcal{C}H^{T}(\boldsymbol{k})\mathcal{C}^{-1}=-H(-\boldsymbol{k})$ with unitary matrices $\mathcal{T}$ and $\mathcal{C}$ (see also Appendix E in Ref. [31]).
[83] Reference [50] argued that topological classification of non-Hermitian systems in one dimension is the same as the Hermitian case from the field-theoretical perspective.

At face value, this result may contradict the intrinsic non-Hermitian topological phases in one dimension [24,31]. However, since Ref. [50] assumes a line gap and a spacetime formulation, it is compatible with our space formulation for a point gap.
[84] N. Okuma and M. Sato, Phys. Rev. B **103**, 085428 (2021).
[85] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D **78**, 074033 (2008).
[86] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.216405 for details about the skin effect of non-Hermitian Dirac Hamiltonians, the unidirectional transport in one dimension, the chiral anomaly, the chiral magnetic skin effect, and the transport and skin effect induced by a perpendicular spatial texture.
[87] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[88] J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).
[89] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Westview Press, Boulder, 1995).
[90] D. Nakamura, T. Bessho, and M. Sato (to be published).
[91] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).
[92] D. Vanderbilt, *Berry Phases in Electronic Structure Theory: Electric Polarization, Orbital Magnetization and Topological Insulators* (Cambridge University Press, Cambridge, 2018).
[93] F. Tonielli, J. C. Budich, A. Altland, and S. Diehl, Phys. Rev. Lett. **124**, 240404 (2020).