

Elliptic, Yangian-Invariant “Leading Singularity”

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We derive closed formulas for the first examples of nonalgebraic, elliptic “leading singularities” in planar, maximally supersymmetric Yang-Mills theory and show that they are Yangian invariant.

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Introduction.—The *residues* of (the loop integrands of) scattering amplitudes on iterated, simple poles have played a critical and starring role through most of the incredible advances in our understanding of perturbative quantum field theory in recent years. Such residues, when they are of maximal codimension, have been called “leading singularities” [1,2] and have been used in the context of generalized unitarity [3–6] to construct integrands for amplitudes to impressive orders of perturbation [6–21]. Leading singularities have also appeared as the individual terms generated by Britto-Cachazo-Feng-Witten recursion relations for amplitudes at tree level [22,23] and beyond [24–26].

In the case of maximally supersymmetric Yang-Mills theory in the planar limit (SYM), leading singularities were discovered to enjoy an infinite-dimensional Yangian symmetry algebra [27–30] and were later found to be classified according to the “positroid” stratification of Grassmannian manifolds [31–33]. In Ref. [34], the correspondence between Yangian invariants and compact contour integrals over the positroid-canonical top form [35–37] in the Grassmannian was established.

As maximal codimension residues, leading singularities can be viewed as the coefficients of loop amplitude integrands that are locally (in loop-momentum space) “ $d \log$ ” differential forms—wedge products of simple poles in all loop-momentum variables. (A multidimensional residue can in fact be defined by this fact alone—given by the inverse Jacobian of the requisite change of variables, evaluated at this point.) It is no surprise, therefore, that leading singularities also appear as the coefficients of polylogarithmic functions arising from “loop integration”

(over the Feynman contour of real momenta). Indeed, the connection between the *sufficiency* of leading singularities as information required to construct amplitudes and the existence of $d \log$ representations of loop integrands has been the subject of much research in recent years [14,18,38,39].

Nevertheless, an increasing body of research has shown that many perturbative scattering amplitudes are not in fact polylogarithmic [40–43] but involve a much richer variety of functions—such as elliptic multiple polylogarithms [44–46] or even integrals over Calabi-Yau manifolds [47–50]. When the integrands of loop amplitudes are not $d \log$ somewhere locally, leading singularities as they have been so-far defined represent insufficient information (for reconstruction via unitarity) about perturbative amplitudes. The attempts to use generalized unitarity in such cases have relied on less intrinsically well-defined (or well-motivated) strategies—such as matching amplitude integrands functionally through some number of off-shell evaluations (see, e.g., [16,18,51]).

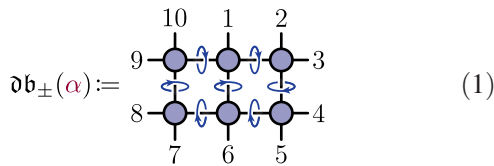
In this Letter, we argue that the notion of leading singularity must be broadened to include *any* full-dimensional compact, contour integral of an amplitude. Up to factors of $(2\pi i)$, this definition automatically includes those functions covered previously but expands their scope to include contour integrals that do not just compute the residues around simple poles. We conjecture that this new, broadened definition of leading singularities represents *complete* unitarity-level information for amplitudes in perturbative SYM (and perhaps considerably beyond).

The first appearance of nonpolylogarithmic structure for planar SYM theory occurs in the ten-particle $N^3\text{MHV}$ amplitude at two loops [42]. In this Letter, we derive closed formulas for the elliptic leading singularities of this amplitude. We have checked that they are Yangian invariant. This check is very nontrivial since they involve the periods of complete elliptic integrals. Thus, they are not any

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elliptic analogs of “residues”—which would be necessarily algebraic—but rather contour integrals directly, carrying (some notion of) transcendental weight. Interestingly, the loop integrand whose coefficient would be an elliptic leading singularity takes the form of what has been defined as a “pure” elliptic integral [52,53]. We strongly suspect that this magic is not accidental: namely, that for any basis of loop integrands diagonalized on maximal-dimension, compact contours, the coefficients of amplitudes (in SYM) will be Yangian invariant and all integrands will be pure (in a sense suitably generalized to integrals involving higher-dimensional Calabi-Yau periods). But we leave such speculations to future work.

Elliptic subleading singularities of SYM.—We begin our analysis with the “double-box” cut of the two-loop, ten-particle N^3 MHV amplitude in planar SYM:



$$\mathbf{db}_{\pm}(\alpha) := \text{Diagram} \quad (1)$$

It is a “next-to-leading singularity” in the ordinary sense (a codimension-7 residue) and corresponds to a contour which encircles the seven poles from the propagators shown. This residue depends on one internal (on-shell) degree of freedom denoted α and a discrete label “ \pm ” signifying which of the two branches of the solution to the cut equations is taken. [As a positroid configuration, it corresponds to a 13-dimensional cell in $G_+(3, 10)$ labeled by the decorated permutation $\{6, 5, 4, 8, 7, 11, 10, 9, 13, 12\}$, integrated against the 12 constraints $\delta^{3 \times 4}(C \cdot Z)$ —leaving a one-form over the remaining variable (see, e.g., [33,54].)]

We can give an explicit formula for the double-box cut $\mathbf{db}_{\pm}(\alpha)$ as a single residue of (the relevant term of) the codimension-6 “kissing-triangle” function given in Ref. [16]:

$$d\alpha d\beta \frac{1}{\alpha\beta} R[1, \hat{2}, \hat{6}, \hat{7}, \hat{1}] R[\hat{2}, 3, 4, 5, 6] R[\hat{7}, 8, 9, 10, 1], \quad (2)$$

where $R[a, b, c, d, e]$ is the R invariant [55,56] with momentum supertwistor arguments $\{\mathcal{Z}_a, \dots, \mathcal{Z}_e\}$ [57] and the “hatted” arguments are defined geometrically (see, e.g., [38]) by

$$\begin{aligned} \hat{2}(\alpha) &:= \mathcal{Z}_2 + \alpha \mathcal{Z}_1, & \hat{7}(\beta) &:= \mathcal{Z}_7 + \beta \mathcal{Z}_6, \\ \hat{6}(\alpha) &:= (56) \cap (\hat{2}34), & \hat{1}(\beta) &:= (101) \cap (\hat{7}89). \end{aligned} \quad (3)$$

[Notice that the variables α and β both carry nontrivial little-group weights. These could be absorbed by rescaling them (and the form) appropriately; but any such rescaling would cancel out of any complete compact contour integral (or residue).]

To obtain (1) from (2), we must compute the residue of (2) associated with the contour encircling the pole $\langle \hat{2} \hat{6} \hat{7} \hat{1} \rangle = 0$. We may take this residue with respect to either variable, and we choose to eliminate β . Because $\langle \hat{2} \hat{6} \hat{7} \hat{1} \rangle$ is quadratic in β , there are two branches available.

For the sake of concreteness, we note that

$$\text{Res}_{\langle \hat{2} \hat{6} \hat{7} \hat{1} \rangle=0} \left[\frac{d\beta}{\langle \hat{2} \hat{6} \hat{7} \hat{1} \rangle} \right] = \pm \frac{c_y}{y(\alpha)}, \quad (4)$$

where $y(\alpha)^2$ is the (by-construction *monic*) quartic polynomial

$$\begin{aligned} \frac{1}{c_y^2} y^2(\alpha) &:= (\langle \hat{2} \hat{6} 6(789) \cap (101) \rangle + \langle \hat{2} \hat{6} 7(689) \cap (101) \rangle)^2 \\ &\quad - 4 \langle \hat{2} \hat{6} 6(689) \cap (101) \rangle \langle \hat{2} \hat{6} 7(789) \cap (101) \rangle, \end{aligned} \quad (5)$$

where c_y^2 is defined to be the inverse of the coefficient of α^4 on the right-hand side of (5) [so as to render $y^2(\alpha)$ monic], and the particular solutions β_{\pm}^* to $\langle \hat{2} \hat{6} \hat{7} \hat{1} \rangle = 0$ are given by

$$\beta_{\pm}^* := \frac{\langle \hat{2} \hat{6} 6(789) \cap (101) \rangle + \langle \hat{2} \hat{6} 7(689) \cap (101) \rangle \pm y(\alpha)/c_y}{2 \langle \hat{2} \hat{6} (689) \cap (101) 6 \rangle}.$$

To define the double-box residue (1), we may therefore replace the pole $\langle \hat{2} \hat{6} \hat{7} \hat{1} \rangle$ (appearing in the denominator of $R[1, \hat{2}, \hat{6}, \hat{7}, \hat{1}]$) with $y(\alpha)$ and replace β with $\beta_{\pm}^*(\alpha)$ everywhere else in (2).

As $y^2(\alpha)$ is an irreducible quartic, the differential forms $\mathbf{db}_{\pm}(\alpha)$ should be understood as involving the geometry of an elliptic curve. In general, any such differential form may be represented in the form

$$\mathbf{db}_{\pm}(\alpha) := d\alpha \left[Q(\alpha) \pm \frac{1}{y(\alpha)} R(\alpha) \right], \quad (6)$$

where Q and R are algebraic (super)functions of α . As we are interested in taking a contour integral over the cycles of the elliptic curve, only the term involving $1/y(\alpha)$ matters; we may extract this piece by writing

$$[\mathbf{db}_+(\alpha) - \mathbf{db}_-(\alpha)] := \mathbf{db}(\alpha) := \frac{d\alpha}{y(\alpha)} \widehat{\mathbf{db}}(\alpha). \quad (7)$$

Analytic structure of the elliptic form $\mathbf{db}(\alpha)$.—It is not hard to see that the differential form $\mathbf{db}(\alpha)$ has many simple poles—corresponding to the various factorization channels of the six four-particle amplitudes appearing in (1). Every such factorization channel has the topology of a “pentabox” leading singularity; counting every distinct solution to the cut equations for each topology, there are altogether 24 such “boundary” on-shell functions; let us denote them \mathbf{pb}_i .

Each of these “factorizations” of the double-box cut (1) corresponds to a simple pole located at a_i in the α plane with residue *equal* to the corresponding pentabox on-shell function \mathfrak{pb}_i . For example, near

$$\alpha \rightarrow a_1 := \frac{\langle 2(34) \cap (1012)56 \rangle}{\langle (34) \cap (1012)561 \rangle}, \quad (8)$$

the differential form $\mathfrak{db}(\alpha)$ has a simple pole with residue

$$\text{Res}_{\alpha=a_1} [\mathfrak{db}(\alpha)] = \begin{array}{c} \text{Diagram of a double box with 10 vertices labeled 1-10} \\ \text{Residue value} \end{array} =: \mathfrak{pb}_1. \quad (9)$$

This function corresponds to one of the ordinary codimension-one boundaries of the positroid configuration of the double box; as such, it can easily be computed as the canonical (12-dimensional) form in the Grassmannian integrated against $\delta^{3 \times 4}(C \cdot Z)$. Regardless of how it is represented or computed, the location of each pole and its residue is easy to determine explicitly. Interestingly, it is worth noting that all of the poles a_i lie on the real axis, and are negative semidefinite, for positive kinematics (see, e.g., [58] for a discussion of positive kinematics). These have been given explicitly in Supplemental Material [59].

Expanding $\mathfrak{db}(\alpha)$ into a basis of forms with manifest simple poles results in a representation of $\mathfrak{db}(\alpha)$ which may be written

$$\mathfrak{db}(\alpha) =: \frac{d\alpha}{y(\alpha)} \mathfrak{db}_0 + d\alpha \sum_{i=1}^{24} \frac{y(a_i)}{(\alpha - a_i)y(\alpha)} \mathfrak{pb}_i, \quad (10)$$

where \mathfrak{db}_0 —the coefficient of the differential form $d\alpha/y(\alpha)$ without any simple poles—is therefore defined indirectly (but explicitly and without ambiguity) by

$$\mathfrak{db}_0 := \widehat{\mathfrak{db}}(\alpha) - \sum_{i=1}^{24} \frac{y(a_i)}{(\alpha - a_i)} \mathfrak{pb}_i. \quad (11)$$

[Actually, for (10) and (11), there is in fact (exactly) one pole at $\alpha = -\infty$; for this term, the differential form in the sum should be understood as being $d\alpha[\alpha/y(\alpha)]$.]

Importantly, since \mathfrak{db}_0 has no poles in α (including at infinity), it must be *independent* of α —a fact that we have checked analytically. As such, it is worthwhile to express it in the form

$$\mathfrak{db}_0 := \widehat{\mathfrak{db}}(\alpha^*) - \sum_{i=1}^{24} \frac{y(a_i)}{(\alpha^* - a_i)} \mathfrak{pb}_i, \quad (12)$$

for any choice of α^* . As every expression appearing in the right-hand side of (12) is fully known analytically

[as superfunctions (or expressed in terms of R invariants)], this provides a concrete definition for \mathfrak{db}_0 .

Elliptic leading singularities of SYM.—We are now in a position to determine the elliptic leading singularity—the integral of the form $\mathfrak{db}(\alpha)$ over some choice of elliptic cycle, say, Ω_a :

$$e_a := \int_{\Omega_a} \mathfrak{db}(\alpha) = \int_{\Omega_a} \frac{d\alpha}{y(\alpha)} \widehat{\mathfrak{db}}(\alpha) = \pm 2 \int_{\Omega_a} \mathfrak{db}_{\pm}(\alpha). \quad (13)$$

To specify the particular elliptic cycle Ω_a , it is worthwhile to note that, for positive (nondegenerate) momentum-twistor kinematics, it turns out that the roots r_i of the quartic $y^2(\alpha)$,

$$y^2(\alpha) =: (\alpha - r_1)(\alpha - r_2)(\alpha - r_3)(\alpha - r_4), \quad (14)$$

always come in complex-conjugate pairs—between which we may introduce branch cuts. To be clear, we have chosen to order the roots such that $r_1^* = r_2$ and $r_3^* = r_4$, with $\text{Re}(r_1) > \text{Re}(r_3)$ and $\text{Im}(r_{1,3}) > 0$; with this ordering of the roots (the reverse of the default ordering from `Solve[]` in *Mathematica*), the cross-ratio

$$\phi := \frac{(r_2 - r_1)(r_3 - r_4)}{(r_2 - r_3)(r_1 - r_4)} \quad (15)$$

is always real; moreover, $\phi \in (0, 1)$ for positive \mathcal{Z} 's. With these conventions, we define Ω_a to be the contour “enclosing” the branch cut between the complex-conjugate pair of roots $r_{1,2}$ which does *not* encircle any of the simple poles of $\mathfrak{db}(\alpha)$.

In order to compute the elliptic leading singularity (13), therefore, we merely need to note the basic period integrals appearing in (10):

$$\int_{\Omega_a} d\alpha \frac{1}{y(\alpha)} = \frac{4i}{\sqrt{(r_3 - r_2)(r_4 - r_1)}} K[\phi] \quad (16)$$

and

$$\int_{\Omega_a} d\alpha \frac{y(a_i)}{(\alpha - a_i)y(\alpha)} = \frac{4i}{\sqrt{(r_3 - r_2)(r_4 - r_1)}} \left(\frac{y(a_i)}{(r_4 - a_i)} K[\phi] + \frac{y(a_i)(r_4 - r_2)}{(r_2 - a_i)(r_4 - a_i)} \Pi \left[\frac{(r_4 - a_i)(r_2 - r_1)}{(r_2 - a_i)(r_4 - r_1)}, \phi \right] \right), \quad (17)$$

where $K[\phi]$ and $\Pi[q, \phi]$ are the complete elliptic integrals of the first and third kinds, respectively, defined in accordance with the conventions of *Mathematica*. Of particular importance is that both (16) and (17) are pure imaginary for positive kinematics; for the latter integral (17), this statement is nontrivial [even for $\phi \in (0, 1)$], as the coefficients of both complete elliptic integrals appearing in (17)

have nonzero real and imaginary parts and only the combination is pure imaginary. The full integral in (13) is obtained by integrating each term in (10) and using the explicit formula for $\widehat{\mathfrak{db}}_0$ in (12) (for any choice of α^*). As a consequence of the above discussion, this representation of e_a is term-by-term purely imaginary.

One reason for our preference for the a cycle (and also for our conventions regarding the roots) is that, in the space of positive kinematics, the only possible kinematic degenerations at codimension one result in the collision of one of the two pairs of complex-conjugate roots. When this happens, it is easy to see that *both* integrals (16) and (17) become *equal to* $(2\pi i)$ times the *residue* around the corresponding simple pole generated by the colliding pair of roots. (Recall that $K[0] = \Pi[0, 0] = \pi/2$.) The b -cycle integrals, in contrast, diverge upon such degenerations.

More concise formulas for the leading singularities.—In the discussion above, the reader should notice that every pentabox on-shell function \mathfrak{pb}_i appears twice: once in the definition of $\widehat{\mathfrak{db}}_0$ in (12) (where α^* may be taken as arbitrary) and once as the coefficient of the particular (α -dependent) differential form in $\widehat{\mathfrak{db}}(\alpha)$ in (10). From the first, (16) results in a contribution to e_a of

$$e_a \supset -\frac{4i}{\sqrt{(r_3 - r_2)(r_4 - r_1)}} K[\phi] \times \frac{y(a_i)}{(\alpha^* - a_i)} \mathfrak{pb}_i; \quad (18)$$

from the second, (17) results in a contribution of

$$e_a \supset \frac{4i}{\sqrt{(r_3 - r_2)(r_4 - r_1)}} \left(\frac{y(a_i)}{(r_4 - a_i)} K[\phi] + \frac{y(a_i)(r_4 - r_2)}{(r_2 - a_i)(r_4 - a_i)} \Pi \left[\frac{(r_4 - a_i)(r_2 - r_1)}{(r_2 - a_i)(r_4 - r_1)}, \phi \right] \right) \mathfrak{pb}_i. \quad (19)$$

From these two contributions, mere pattern recognition suggests a “preferential” choice for the arbitrary point α^* . In particular, if we were to take α^* to be r_4 , all the terms involving $K[\phi] \times \mathfrak{pb}_i$ would cancel. As α^* is indeed arbitrary, this would result in a final, more compact expression:

$$e_a = \frac{4i}{\sqrt{(r_3 - r_2)(r_4 - r_1)}} \left(K[\phi] \widehat{\mathfrak{db}}(\alpha^* \rightarrow r_4) + \sum_{i=1}^{24} \frac{y(a_i)(r_4 - r_2)}{(r_2 - a_i)(r_4 - a_i)} \Pi \left[\frac{(r_4 - a_i)(r_2 - r_1)}{(r_2 - a_i)(r_4 - r_1)}, \phi \right] \mathfrak{pb}_i \right). \quad (20)$$

The reader may be worried about the fact that taking α^* to be r_4 sets $y(\alpha^*) \rightarrow 0$. As such, the first term in (20) may appear ill defined. However, $y(\alpha^*)$ also appears (manifestly) in the denominator in the definition of the

differential form $\widehat{\mathfrak{db}}(\alpha^*)$; as such, the evaluation—for $\widehat{\mathfrak{db}}(\alpha)$ —may be performed without taking limits (and turns out to be extremely stable, numerically). This simplified form is included in Supplemental Material [59].

It is worth noting that, upon any physical degeneration (at codimension one), the elliptic function e_a in fact vanishes identically. This can be understood by noting that any such physical degeneration would correspond to an ordinary “residue” contour about the simple pole generated by the collision of the roots— $[(2\pi i)$ times the *residue*] about the point $\alpha = r_1$ or r_4 ; as there is no corresponding on-shell function to draw, the amplitude must vanish on such a contour. We have checked that it does.

The b -cycle elliptic leading singularity—the one encircling a branch cut between the roots r_1 and r_3 , say (and *not* encircling any of the simple poles)—is easily obtained from our work above [replacing $r_2 \leftrightarrow r_3$ in (16) and (17)]. Using these expressions for the b -cycle integrals of the relevant differential forms and choosing α^* to be r_4 as before, the resulting expression becomes

$$e_b = \frac{4}{\sqrt{(r_3 - r_2)(r_4 - r_1)}} \left(K[1 - \phi] \widehat{\mathfrak{db}}(\alpha^* \rightarrow r_4) + \sum_{i=1}^{24} \frac{y(a_i)(r_4 - r_3)}{(r_3 - a_i)(r_4 - a_i)} \Pi \left[\frac{(r_4 - a_i)(r_3 - r_1)}{(r_3 - a_i)(r_4 - r_1)}, 1 - \phi \right] \mathfrak{pb}_i \right). \quad (21)$$

One interesting aspect of these formulas is that the b -cycle leading singularity e_b is invariant under a fivefold cyclic rotation (in a highly nontrivial way), while the a -cycle integral e_a is not—reflecting the asymmetry of the contour (analogous to the noncyclic invariance of the four-mass box leading singularities).

Explicit, computer-usable (*Mathematica*) expressions for both elliptic leading singularities e_a and e_b are included as Supplemental Material [59]. This code makes use of tools made available in Refs. [14, 16, 60].

Yangian invariance of the elliptic leading singularities.—Among the most interesting aspects of our results so far is that Yangian invariance *requires* the complete elliptic integrals $K[\phi]$ and $\Pi[q, \phi]$ as coefficients appearing in their definition. The easiest way to see this is to consider one of the level-one generators (see, e.g., [30]):

$$J_B^A := \sum_{a=1}^n \mathcal{Z}_a^A \frac{\partial}{\partial \mathcal{Z}_a^B}, \quad (22)$$

where \mathcal{Z}_a is a supermomentum twistor and the component A is taken to be fermionic and B bosonic. This operator turns out to be surprisingly powerful. For example, it tells us that *any* nontrivial function of cross-ratios times a Yangian invariant will not be Yangian invariant. In particular, direct application of this operator demonstrates that

the four-mass box coefficient as defined in Ref. [14], which is not simply a product of R invariants but includes as part of its definition a particular function of the relevant cross-ratios, is *only* Yangian invariant with these peculiar prefactors included.

It turns out that no combination or subset of the terms (with constant coefficients) that appear in the two formulas for $e_{a,b}$ in (20) and (21), respectively, is Yangian invariant except for the $e_{a,b}$ themselves. We have checked this explicitly using numerical approximations for the derivatives appearing in the Yangian generator (22).

That any integral over a compact, full-dimensional cycle in the Grassmannian should be Yangian invariant may not be surprising: indeed, it seems to be a consequence of the arguments described in Ref. [34]. However, the fact that these integrals, in the case of elliptic contours, *require* the nonalgebraic content of complete elliptic integrals is very surprising, as this is in stark contrast with the notion of taking residues (a purely algebraic operation).

The lesson here has very obvious consequences for generalization beyond elliptic contours—which we must leave to future work.

Conclusions and discussion.—In this Letter, we have motivated a broader definition of leading singularity to be any full-dimensional compact, contour integral of a perturbative amplitude’s loop integrand. This definition differs from the previous one merely by factors of $(2\pi i)$ in the case of simple (logarithmic) poles but includes also the contour integrals of elliptic curves. We have given closed formulas for elliptic-containing contour integrals for the first non-polylogarithmic structure that arises in planar SYM theory, and we have checked that they are Yangian invariant. We conjecture that, with this broader definition, all leading singularities of SYM will be Yangian invariant and that the set of all leading singularities will represent complete information for perturbative amplitudes in this theory. We will have more to say about this in a forthcoming work.

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