Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory

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Using the recently established formalism of a worldline quantum field theory description of the classical scattering of two spinless black holes, we compute the far-field time-domain waveform of the gravitational waves produced in the encounter at leading order in the post-Minkowskian (weak field but generic velocity) expansion. We reproduce the previous results of Kovacs and Thorne in a highly economic way. Then, using the waveform, we extract the leading-order total radiated angular momentum and energy (including differential results). Our work may enable crucial improvements of gravitational-wave predictions in the regime of large relative velocities.

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When two compact objects (black holes, neutron stars, or stars) fly past each other, their gravitational interactions not only deflect their trajectories but they also produce gravitational radiation, or gravitational bremsstrahlung, analogous to the electromagnetic case. The resulting waveform in the far field at leading order in Newton's constant G was constructed (in the spinless case) in a series of papers by Kovacs, Thorne, and Crowley in the 1970s [1–4] (see Refs. [5] for recent work on slow-motion sources). Today's gravitational wave (GW) observatories routinely detect quasicircular inspirals and mergers of binary black holes and neutron stars [6]. Yet bremsstrahlung events currently appear to be out of reach as the signal is not periodic and typically less intensive [7]. Still, these events represent interesting targets for GW searches, calling for accurate waveform models.

Indeed, the experimental success of GW astronomy brings up the need for high-precision theoretical predictions for the classical relativistic two-body problem [8]. A number of complementary classical perturbative approaches have been established over the years [9]. Yet quantum-field-theory-based techniques founded in a perturbative Feynman-diagrammatic expansion of the path integral in the classical limit have proven to be highly efficient. These come in two alternative approaches.

The first approach, the effective field theory (EFT) formalism [10], models the compact objects as pointlike massive particles coupled to the gravitational field. It has mostly been applied to a nonrelativistic post-Newtonian (PN) scenario for bound orbits in which an expansion in powers of Newton's constant G implies an expansion in velocities ($Gm/c^2 r \sim v^2/c^2$). Recently, it has also been extended to the post-Minkowskian (PM) expansion for unbound orbits [11,12] that are relevant for this work, an expansion in G for arbitrary velocities. In these EFT settings, the graviton field $h_{\mu\nu}(x)$ is integrated out successively (from small to large length scales) in the path integral, while the worldline trajectories of the black holes $x_i^\mu(\tau_i)$ are kept as classical background sources (see Ref. [13] for reviews).

The second and now blossoming approach starts out from scattering amplitudes of massive scalars—avatars of spinless black holes—minimally coupled to general relativity [14–17], thereby putting the younger innovations in on-shell techniques for scattering amplitudes (e.g., generalized unitarity [18] or the double copy [19]) to work. In order to obtain the conservative gravitational potential, one performs a subtle classical limit of the scattering amplitudes [20] in order to match to a nonrelativistic EFT for scalar particles with the desired potential [21] (see also Refs. [14,21]), which is known to the 3PM order [16] (complemented by certain radiation-reaction effects [20,22,23]). Very recently the 4PM conservative potential was also reported [24]. The so-obtained effective potential is then used to compute observables such as the scattering angle or the (PM-resummed) periastron advance in the bound system [11,24]. Further recent PM results exist for nonspinning particles [25], spin effects [26], tidal effects [27], and radiation effects [28].

In a recent work of three of the present authors, the synthesis of these two quantum-field-theory-based approaches to classical relativity was provided in the form of a worldline quantum field theory (WQFT) [29] in which quantizing both the graviton field $h_{\mu\nu}$ and the fluctuations about the bodies' worldline trajectories z_i^μ was shown to be an efficient approach yielding only the relevant classical contributions. In essence, the WQFT formalism provides an efficient diagrammatic framework for solving the equations of motion of gravity-matter systems perturbatively.

In this Letter, we employ this novel formalism to compute the time-domain gravitational waveform of a bremsstrahlung event at leading order in G, demonstrating its effectiveness. To our knowledge, the seminal result of Kovacs and Thorne [4] has not been verified in its entirety to date. As we shall see, our approach is far more efficient than the one employed back then, paving the way for calculations of higher orders. We stress that we are able to determine the far-field waveforms that are of direct relevance for GW observatories. As a check on these waveforms, we furthermore reproduce Damour's recent result for the total radiated angular momentum [22] at 2PM order. Our results also complement the recent result of the total radiated momentum at leading order in G (3PM) established with amplitude techniques [30]. We comment on how to achieve this result from our methods.

Worldline quantum field theory.—The classical gravitational scattering of two massive objects m_i moving on trajectories $x_i^\mu(\tau_i) = b_i^\mu + v_i^\mu \tau_i + z_i^\mu(\tau_i)$, is described by the WQFT with partition function [29]

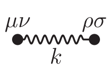
$$\mathcal{Z}_{\text{WQFT}} \coloneqq \text{const} \times \int D[h_{\mu\nu}] \int \prod_{i=1}^{2} D[z_i] e^{i(S_{\text{EH}}+S_{\text{gf}})} \exp\left\{-i \sum_{i=1}^{2} \int_{-\infty}^{\infty} \mathrm{d}\tau_i \frac{m_i}{2} [\eta_{\mu\nu} + \kappa h_{\mu\nu}(x)] \dot{x}_i^\mu \dot{x}_i^\nu \right\}, \tag{1}$$

where $S_{\text{EH}} + S_{\text{gf}}$ is the gauge-fixed Einstein-Hilbert action

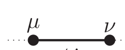
$$S_{\text{EH}} + S_{\text{gf}} = \int \mathrm{d}^4 x \left[-\frac{2}{\kappa^2} \sqrt{-g} R + \left(\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h^\nu{}_\nu\right)^2 \right], \tag{2}$$

with $\kappa^2 = 32\pi G$ the gravitational coupling. We have suppressed the ghost contributions in Eq. (1) as they are irrelevant in the classical setting. We work in mostly minus signature, $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, and set $c = \hbar = 1$.

Correlation functions in the WQFT $\langle \mathcal{O}(h, \{x_i\}) \rangle_{\text{WQFT}}$ result from an insertion of the operator $\mathcal{O}$ in the path integral and dividing by $\mathcal{Z}_{\text{WQFT}}$. Moving to momentum space for the graviton $h_{\mu\nu}(k)$ and energy space for the fluctuations $z^\mu(\omega)$, we have the retarded propagators



$$= i \frac{P_{\mu\nu;\rho\sigma}}{(k^0 + i\epsilon)^2 - \mathbf{k}^2}, \tag{3a}$$



$$= -i \frac{\eta^{\mu\nu}}{m\,(\omega + i\epsilon)^2}, \tag{3b}$$

with $P_{\mu\nu;\rho\sigma} \coloneqq \eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}$. The relevant vertices for the emission of a graviton off the worldline read

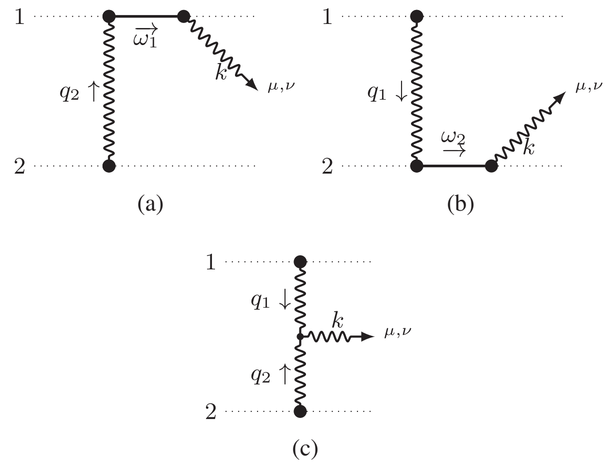
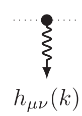
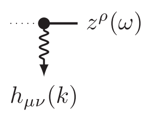


FIG. 1. The three diagrams contributing to the bremsstrahlung at 2PM order, where $\omega_i = k \cdot v_i$ by energy conservation at the worldline vertices. All three diagrams have the integral measure in Eq. (16); in the rest frame of black hole 1, diagram (a) does not contribute as soon as the outgoing graviton is contracted with a purely spatial polarization tensor, while (b) and (c) do.



$$= -i \frac{m\kappa}{2} e^{ik\cdot b} \delta\!\!\!{}^-(k \cdot v) v^\mu v^\nu, \tag{4}$$

with k outgoing, $\delta\!\!\!{}^-(\omega) \coloneqq (2\pi)\delta(\omega)$, and



$$= \frac{m\kappa}{2} e^{ik\cdot b} \delta\!\!\!{}^-(k \cdot v + \omega) \times \left(2\omega v^{(\mu}\delta_\rho^{\nu)} + v^\mu v^\nu k_\rho\right). \tag{5}$$

The energy ω is also taken as outgoing. One also has the standard bulk graviton vertices, of which we shall need only the three-graviton vertex (see, e.g., Ref. [31]).

To determine the bremsstrahlung of two traversing black holes, we compute the expectation value $k^2 \langle h^{\mu\nu}(k) \rangle_{\text{WQFT}}$. At leading (2PM) order, there are three diagrams contributing (cf. Fig. 1). We integrate over the momenta or energies of internal gravitons or fluctuations respectively; lack of three-momentum conservation at the worldline vertices leaves unresolved integrals for the tree-level diagrams.

Diagram (a) of Fig. 1 then takes the form [32]

$$k^2 \langle h_{\mu\nu}(k) \rangle_{\text{WQFT}}\big|_{(a)} = -\frac{m_1 m_2 \kappa^3}{8} \int_{q_1,q_2} \mu_{1,2}(k) \frac{[2\omega_1 v_1^{(\mu}\delta_\rho^{\nu)} - v_1^\mu v_1^\nu k_\rho][2\omega_1 v_1^{(\sigma}\eta^{\lambda)\rho} - v_1^\sigma v_1^\lambda q_2^\rho]}{(\omega_1 + i\epsilon)^2} \frac{P_{\sigma\lambda;\alpha\beta}}{[(q_2^0 + i\epsilon)^2 - \mathbf{q_2}^2]} v_2^\alpha v_2^\beta, \tag{6}$$

where $\omega_1 = k \cdot v_1$, $\int_{q_i} := \int d^4 q_i/(2\pi)^4$ and the integral measure is

$$\mu_{1,2}(k) = e^{i(q_1 \cdot b_1 + q_2 \cdot b_2)} \delta(q_1 \cdot v_1) \delta(q_2 \cdot v_2) \delta(k - q_1 - q_2), \tag{7}$$

with $\delta(k) := (2\pi)^4 \delta^{(4)}(k)$. The diagram (b) is naturally obtained by swapping $1 \leftrightarrow 2$. Diagram (c) includes the three-graviton vertex $V_3^{(\mu\nu)(\rho\sigma)(\lambda\tau)}(k, -q_1, -q_2)$:

$$k^2 \langle h_{\mu\nu}(k) \rangle_{\mathrm{WQFT}}|_{(c)} = -\frac{m_1 m_2 \kappa^3}{8} \int_{q_1,q_2} \mu_{1,2}(k) V_3^{(\mu\nu)(\rho\sigma)(\lambda\tau)} \frac{P_{\rho\sigma;\alpha\beta}}{[(q_1^0 + i\epsilon)^2 - \mathbf{q_1}^2]} \frac{P_{\lambda\tau;\gamma\delta}}{[(q_2^0 + i\epsilon)^2 - \mathbf{q_2}^2]} v_1^\alpha v_1^\beta v_2^\gamma v_2^\delta. \tag{8}$$

These integrands were already given in Ref. [29]. The sum of the three integrands also agrees with a previous amplitudes-based result [17] (see also Ref. [33] for the analog in dilaton gravity) and is gauge-invariant.

The waveform in spacetime in the wave zone is obtained from $\langle h^{\mu\nu}(k) \rangle_{\mathrm{WQFT}}$ as follows: we may identify

$$k^2 \langle h_{\mu\nu}(k) \rangle_{\mathrm{WQFT}} = \frac{\kappa}{2} S_{\mu\nu}(k), \tag{9}$$

where $S_{\mu\nu} = \tau_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\tau^\lambda{}_\lambda$ and $\tau_{\mu\nu}$ is the combined energy-momentum pseudotensor of matter and the gravitational field. Consider $S_{\mu\nu}(k)$ for a fixed GW frequency $k^0 = \Omega$. In the wave zone ($r \gg \{|b_i|, \Omega^{-1}, \Omega|b_i|^2\}$) the metric perturbation $h_{\mu\nu}(\mathbf{x}, t)$ takes the form of a plane wave (see, e.g., Chapter 10.4 of Weinberg [34]):

$$\kappa h_{\mu\nu}(\mathbf{x}, t) = \frac{4G}{r} S_{\mu\nu}(\Omega, \mathbf{k} = \Omega\hat{\mathbf{x}}) e^{-ik_\mu x^\mu} + c.c, \tag{10}$$

with the wave vector $k^\mu = \Omega(1, \hat{\mathbf{x}})$; $\hat{\mathbf{x}} = \mathbf{x}/r$ is the unit vector pointing in the direction of the observation point (hence $k^2 = 0$).

The total gauge-invariant frequency-domain waveform can be read off as $4GS_{ij}^{\mathrm{TT}}(\Omega, \mathbf{k} = \Omega\hat{\mathbf{x}})$, where TT denotes the transverse-traceless projection. The corresponding time-domain waveform $f_{ij}(u, \theta, \phi)$ is essentially its Fourier transform in Ω:

$$\kappa h_{ij}^{\mathrm{TT}} = \frac{f_{ij}}{r} = \frac{4G}{r} \int_\Omega e^{-ik \cdot x} S_{ij}^{\mathrm{TT}}(k)|_{k^\mu = \Omega(1, \hat{\mathbf{x}})}, \tag{11}$$

where $\int_\Omega := \int_{-\infty}^{\infty} \mathrm{d}\Omega/2\pi$. Note that $k \cdot x = \Omega(t - r)$ yields the retarded time $u = t - r$. Our task now is to perform the integrals; in a PM expansion $f_{ij} = \sum_n G^n f_{ij}^{(n)}$, and we seek the 2PM component $f_{ij}^{(2)}$. By focusing on the time-domain instead of the frequency-domain waveform, we considerably simplify the integration step. As we shall see, the integration over frequency Ω of the outgoing radiation coincides neatly with energy conservation along each worldline.

Kinematics.—We describe the waveform in a Cartesian coordinate system (t, x, y, z) where black hole 1 is initially at rest $v_1^\mu = (1, 0, 0, 0)$ and located at the spatial origin, i.e., we set $b_1^\mu = 0$. The orbit of black hole 2 we put in the $x - y$ plane with initial velocity $v_2^\mu = (\gamma, \gamma v, 0, 0)$ in the x direction; the impact parameter $b_2^\mu = (0, 0, b, 0) =: b^\mu$ points in the y direction. Introducing the polar angles θ and ϕ, we may write the unit (spatial) vector $\hat{x}^\mu$ pointing from black hole 1 to our observation point as

$$\hat{x}^\mu = \hat{e}_1^\mu \cos\theta + \sin\theta(\hat{e}_2^\mu \cos\phi + \hat{e}_3^\mu \sin\phi), \tag{12}$$

where $\hat{e}_i^\mu = (0, \hat{\mathbf{e}}_i)$ are spatial unit vectors. Also, we put $\rho^\mu = v_1^\mu + \hat{x}^\mu$.

The two additional unit spatial vectors orthogonal to $\hat{x}^\mu$ are

$$\hat{\theta}^\mu = \partial_\theta \hat{x}^\mu = (0, \hat{\boldsymbol{\theta}}), \qquad \hat{\phi}^\mu = \frac{1}{\sin\theta} \partial_\phi \hat{x}^\mu = (0, \hat{\boldsymbol{\phi}}). \tag{13}$$

Together with $\hat{x}^\mu$, they form a right-handed spatial coordinate system. GWs travel in the direction of $\hat{x}^\mu$, and we use $\hat{\theta}^\mu$ and $\hat{\phi}^\mu$ to define our polarization tensors in a linear basis:

$$e_+^{\mu\nu} = \hat{\theta}^\mu \hat{\theta}^\nu - \hat{\phi}^\mu \hat{\phi}^\nu, \qquad e_\times^{\mu\nu} = \hat{\theta}^\mu \hat{\phi}^\nu + \hat{\phi}^\mu \hat{\theta}^\nu. \tag{14}$$

The waveform $f_{ij}(u, \theta, \phi)$ is thus decomposed as

$$f_{ij} = f_+ (e_+)_{ij} + f_\times (e_\times)_{ij} \tag{15}$$

with $f_{+,\times} = \frac{1}{2}(e_{+,\times})_{ij} f_{ij}$.

The polarization tensors have zero time components, which conveniently implies the vanishing of diagram (a) in Fig. 1 once contracted with them. This observation follows directly from the expression for vertex, Eq. (5). In the case of diagram (a), the instance of this vertex that contracts with the outgoing graviton line carries an overall factor of $v_1^\mu = (1, 0, 0, 0)$, which is orthogonal to the spatial polarization tensors above.

Integration.—The two nonzero diagrams in Fig. 1 share the integration measure $\mu_{1,2}(k)$, Eq. (7). Including also the integration with respect to Ω in Eq. (11), the full measure becomes

$$\int_{\Omega,q_1,q_2} \mu_{1,2}(k) e^{-ik\cdot x} = \frac{1}{\rho\cdot v_2}\int_{\mathbf{q}} e^{i\mathbf{q}\cdot\tilde{\mathbf{b}}}, \tag{16}$$

where we recall that $k^\mu = \Omega\rho^\mu$; using the delta function constraints in $\mu_{1,2}(k)$, we can now identify

$$q_2 = k - q_1, \qquad q_1 = (0,\mathbf{q}), \qquad \Omega = -\frac{v\gamma}{\rho\cdot v_2}\mathbf{q}\cdot\hat{\mathbf{e}}_1. \tag{17}$$

We are left with a three-dimensional Euclidean integral involving the shifted τ-dependent impact parameter:

$$\tilde{\mathbf{b}}(\tau) = \mathbf{b} + \tau\hat{\mathbf{e}}_1, \qquad \tau = \frac{v\gamma}{\rho\cdot v_2}(u + \mathbf{b}\cdot\hat{\mathbf{x}}), \tag{18}$$

noting that $\rho\cdot v_2 = \gamma(1 - v\cos\theta)$. The polarizations of the waveform from Eq. (11) now take the schematic form [also using Eq. (15)]:

$$\frac{f^{(2)}_{+,\times}}{m_1 m_2} = 4\pi\int_{\mathbf{q}} e^{i\mathbf{q}\cdot\tilde{\mathbf{b}}}\left[\frac{\mathcal{N}^i_{+,\times}\mathbf{q}^i}{\mathbf{q}^2(\mathbf{q}\cdot\hat{\mathbf{e}}_1 - i\epsilon)} + \frac{\mathcal{M}^{ij}_{+,\times}\mathbf{q}^i\mathbf{q}^j}{\mathbf{q}^2(\mathbf{q}^2 + \mathbf{q}\cdot L\cdot\mathbf{q})}\right], \tag{19}$$

with the two terms corresponding to the nonzero diagrams (b) and (c) in Fig. 1 respectively. The rank-2 matrix L introduced here is

$$L^{ij} = 2\frac{v\gamma}{\rho\cdot v_2}\hat{\mathbf{e}}_1^{(i}\hat{\mathbf{x}}^{j)}. \tag{20}$$

Finally the vector and matrix insertions are explicitly given as the real and imaginary parts of [35]

$$\mathcal{N}^i = 2\frac{\gamma^2\sin^2\theta}{\rho\cdot v_2}\left[\frac{\gamma(1-3v^2)}{\rho\cdot v_2} + (1+v^2)\right]\hat{\mathbf{e}}_1^i + 2\frac{\gamma(1+v^2)\sin\theta}{\rho\cdot v_2}\left[\frac{(\rho\cdot v_2)^2 - 1}{v(\rho\cdot v_2)}\cos\phi + 2i\gamma\sin\phi\right]\hat{\mathbf{e}}_2^i, \tag{21a}$$

$$\mathcal{M}^{ij} = 8\frac{\gamma^4 v^4\sin^2\theta}{(\rho\cdot v_2)^3}\hat{\mathbf{e}}_1^i\hat{\mathbf{e}}_1^j + 16\frac{\gamma^3 v^2\sin\theta}{(\rho\cdot v_2)^2}\hat{\mathbf{e}}_1^{[i}(\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}})^{j]} + 4\frac{\gamma^2(1+v^2)}{\rho\cdot v_2}(\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}})^i(\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}})^j, \tag{21b}$$

where $\mathcal{N}^i = \mathcal{N}^i_+ + i\mathcal{N}^i_\times$ and $\mathcal{M}^{ij} = \mathcal{M}^{ij}_+ + i\mathcal{M}^{ij}_\times$. The insertions $\mathcal{N}^i$ and $\mathcal{M}^{ij}$ correspond to a helicity basis in which they have a particular simple expression. We integrate the two diagrams separately.

Integration of the first diagram is achieved using the simple result (true regardless of the vector $\tilde{\mathbf{b}}$):

$$\int_{\mathbf{q}} e^{i\mathbf{q}\cdot\tilde{\mathbf{b}}}\frac{\mathbf{q}^i}{\mathbf{q}^2(\mathbf{q}\cdot\hat{\mathbf{e}}_1 - i\epsilon)} = \frac{1}{4\pi}\left[\frac{\hat{\mathbf{e}}_1^i}{|\tilde{\mathbf{b}}|} - \frac{\tilde{\mathbf{b}}^i - (\tilde{\mathbf{b}}\cdot\hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1^i}{\tilde{\mathbf{b}}^2 - (\tilde{\mathbf{b}}\cdot\hat{\mathbf{e}}_1)^2}\left(1 + \frac{\tilde{\mathbf{b}}\cdot\hat{\mathbf{e}}_1}{|\tilde{\mathbf{b}}|}\right)\right], \tag{22}$$

which we prove in the Appendix. The other integral required corresponding to diagram (c) is somewhat more involved. The denominator of this integral is composed of an isotropic propagator together with an anisotropic one. The physical interpretation is a convolution between the potentials of the two black holes, where the potential of black hole 2 is boosted and leads to the anisotropic propagator. One compact representation is

$$\int_{\mathbf{q}} e^{i\mathbf{q}\cdot\tilde{\mathbf{b}}}\frac{\mathbf{q}^i\mathbf{q}^j}{\mathbf{q}^2(\mathbf{q}^2 + \mathbf{q}\cdot L\cdot\mathbf{q})} = \frac{1}{2\pi\Delta(G)}\left[\frac{(G_0 + \alpha G_1)A^{ij} - (G_1 + \alpha G_2)B^{ij}}{\sqrt{G(\alpha)}}\right]_{\alpha=0}^{\alpha=1}, \tag{23}$$

where we have introduced the quadratic polynomial

$$G(\alpha) = G_0 + 2\alpha G_1 + \alpha^2 G_2, \quad G_0 = \tilde{\mathbf{b}}^2,\ G_1 = \tilde{\mathbf{b}}^i\tilde{\mathbf{b}}^j\frac{\delta^{ij}L^{kk} - L^{ij}}{2},\ G_2 = (\tilde{\mathbf{b}}\cdot\hat{\boldsymbol{\phi}})^2\mathrm{Det}_2 L, \tag{24}$$

and $\Delta(G) = 4(G_1^2 - G_0 G_2)$ is the polynomial discriminant. We have also introduced the two matrices

$$A^{ij} = \mathrm{Det}_2(L)[-2(\tilde{\mathbf{b}}\cdot\hat{\boldsymbol{\phi}})(L^{-1}\cdot\tilde{\mathbf{b}})^{(i}\hat{\boldsymbol{\phi}}^{j)} + (\tilde{\mathbf{b}}\cdot\hat{\boldsymbol{\phi}})^2(L^{-1})^{ij} + (\tilde{\mathbf{b}}\cdot L^{-1}\cdot\tilde{\mathbf{b}})\hat{\boldsymbol{\phi}}^i\hat{\boldsymbol{\phi}}^j], \tag{25a}$$

$$B^{ij} = \tilde{\mathbf{b}}^2\delta^{ij} - \tilde{\mathbf{b}}^i\tilde{\mathbf{b}}^j. \tag{25b}$$

This integral is also discussed in the Appendix, where explicit forms of L^{-1} and $\mathrm{Det}_2(L)$ are given; again, both of the integrals, Eqs. (22) and (23), are solved for arbitrary $\tilde{\mathbf{b}}^i$ and L^{ij} (with the assumption that L^{ij} is rank 2).

Leading-order waveform.—By combining Eq. (19) with the insertions $\mathcal{N}^i_{+,\times}$ and $\mathcal{M}^{ij}_{+,\times}$ and the integrals above, we get the full 2PM waveform:

$$\frac{f^{(2)}_{+,\times}}{m_1 m_2} = \frac{\hat{\mathbf{e}}_1^i\mathcal{N}^i_{+,\times}}{\sqrt{\mathbf{b}^2 + \tau^2}} - \frac{\mathbf{b}^i\mathcal{N}^i_{+,\times}}{\mathbf{b}^2}\left(1 + \frac{\tau}{\sqrt{\mathbf{b}^2 + \tau^2}}\right) + \frac{2\mathcal{M}^{ij}_{+,\times}}{\Delta(G)}\left[\frac{(G_0 + \alpha G_1)A^{ij} - (G_1 + \alpha G_2)B^{ij}}{\sqrt{G(\alpha)}}\right]_{\alpha=0}^{\alpha=1}. \tag{26}$$

This is a rather compact representation of the gravitational bremsstrahlung waveform, which we have confirmed

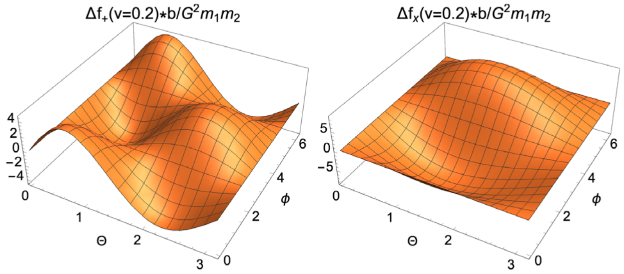


FIG. 2. Plots of the wave memories $\Delta f_{+,\times}$ for $v=0.2$. For a visualisation of the complete waveforms as they evolve with retarded time u see the Supplemental Material [36].

agrees with the (rather lengthy) result of Kovacs and Thorne [4]. The two values of α in the second line correspond to contributions from the two black holes. Note that there is also a leading (and nonradiating) 1PM contribution to the waveform which is independent of the retarded time $u=t-r$:

$$f_{+}^{(1)}=\frac{2m_2\gamma v^2\sin^2\theta}{1-v\cos\theta},\qquad f_{\times}^{(1)}=0. \tag{27}$$

Diagrammatically, this consists only of the vertex, Eq. (4), with emission from worldline 2; the contribution from worldline 1 again vanishes in our frame due to $(v_1\cdot e_{\pm}\cdot v_1)=0$.

To illustrate this result in Fig. 2, we present the gravitational wave memories $\Delta f_{+,\times}\coloneqq[f_{+,\times}]_{u=-\infty}^{u=+\infty}$. The beauty of our result, Eq. (26), lies in the fact that the memories only receive contributions from the second term and read

$$\Delta f_{+,\times}=-2G^2m_1m_2\frac{\mathbf{b}^i\mathcal{N}^i_{+,\times}}{\mathbf{b}^2}+\mathcal{O}(G^3). \tag{28}$$

Diagrammatically, they exclusively emerge from diagram (b) of Fig. 1. So they are manifestly insensitive to gravitational self-interactions—this was also pointed out in Ref. [22].

Radiated energy and angular momentum.—One may now use our result for the waveform, Eq. (26), to compute the total radiated momentum and angular momentum. Expressions for these quantities in terms of the asymptotic waveform are given in Refs. [22,37]:

$$P^{\mu}_{\rm rad}=\frac{1}{32\pi G}\int \mathrm{d}u\mathrm{d}\sigma[\dot f_{ij}]^2\rho^{\mu}, \tag{29}$$

$$J^{\rm rad}_{ij}=\frac{1}{8\pi G}\int \mathrm{d}u\mathrm{d}\sigma\left(f_{k[i}\dot f_{j]k}-\frac{1}{2}x_{[i}\partial_{j]}f_{kl}\dot f_{kl}\right), \tag{30}$$

where $\dot f_{ij}\coloneqq\partial_u f_{ij}$ and $\mathrm{d}\sigma=\sin\theta\mathrm{d}\theta\mathrm{d}\phi$ is the unit sphere measure.

We first concentrate on $J^{\rm rad}_{ij}$ as it contributes at leading order $\mathcal{O}(G^2)$ and was recently obtained in the center-of-mass frame [22]. The static nature of $f^{(1)}_{ij}$, Eq. (27), allows one to trivially perform the u integration and express the radiated angular momentum in terms of the wave memories $\Delta f_{+,\times}$. Inserting the basis of polarization tensors, Eq. (15), [and using $f_{\times}^{(1)}=0$] gives

$$J^{\rm rad}_{xy}=\frac{1}{8\pi}\int\mathrm{d}\sigma\left[\frac{\sin\phi}{\sin\theta}f_{+}^{(1)}\Delta f_{\times}-\frac{1}{2}\cos\phi\,\partial_\theta f_{+}^{(1)}\Delta f_{+}\right]+\mathcal{O}(G^3). \tag{31}$$

The spherical integral is elementary and yields

$$\frac{J^{\rm rad}_{xy}}{J^{\rm init}_{xy}}=\frac{4G^2m_1m_2}{b^2}\frac{(2\gamma^2-1)}{\sqrt{\gamma^2-1}}\mathcal{I}(v)+\mathcal{O}(G^3), \tag{32a}$$

$$\mathcal{I}(v)=-\frac{8}{3}+\frac{1}{v^2}+\frac{(3v^2-1)}{v^3}\mathrm{arctanh}(v), \tag{32b}$$

where we have normalized our result with respect to the initial angular momentum in the rest frame of black hole 1: $J^{\rm init}_{xy}=m_2|\mathbf{v}_2||\mathbf{b}|=m_2\gamma vb$. We find perfect agreement with Ref. [22,38].

Similarly, $P^{\mu}_{\rm rad}$ of Eq. (29) should reproduce the recent result of Ref. [30] contributing at $\mathcal{O}(G^3)$. So far we have only been able to perform the integral in the PN expansion recovering the result of Ref. [30] to order v^6. Yet it is straightforward to obtain differential quantities derived from the integrand of Eq. (29). The differential power spectrum (total energy radiated per unit frequency) as well as the total energy radiated per unit solid angle are collected in the Supplemental Material [36] to this Letter. These results go well beyond Kovacs and Thorne [4] and may be expanded to any desired order in v.

Conclusions.—Searching for GWs from scattering events over the full range of impact velocities requires precision predictions in the PM approximation. While the potential and radiation of bound systems was calculated to high PN order [39] (see Ref. [40] for spinning bodies), a resummation of PN results in the strong-field and fast-motion regimes is essential for building accurate waveform models [8]. The PM resummation is one promising recent attempt [21,41,42].

Our results provide a stepping stone for higher-order calculations, where a repertoire of advanced integration techniques can be put to use [16,24,30,43]. In fact, the 3PM integrand has essentially been presented in Ref. [29]. The present challenge lies in the multiscale integrals, which despite their tree-level structure are of higher loop three-momentum type as the worldline only preserves energy. Generalizations to spin and finite-size effects are possible and lead to the same families of integrations at 2PM. Also the extensions to bound systems using mappings between bound and unbound orbits [24,44] would be of great utility.

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