Genuine High-Dimensional Quantum Steering

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High-dimensional quantum entanglement can give rise to stronger forms of nonlocal correlations compared to qubit systems, offering significant advantages for quantum information processing. Certifying these stronger correlations, however, remains an important challenge, in particular in an experimental setting. Here we theoretically formalize and experimentally demonstrate a notion of genuine high-dimensional quantum steering. We show that high-dimensional entanglement, as quantified by the Schmidt number, can lead to a stronger form of steering, provably impossible to obtain via entanglement in lower dimensions. Exploiting the connection between steering and incompatibility of quantum measurements, we derive simple two-setting steering inequalities, the violation of which guarantees the presence of genuine high-dimensional steering, and hence certifies a lower bound on the Schmidt number in a one-sided device-independent setting. We report the experimental violation of these inequalities using macropixel photon-pair entanglement certifying genuine high-dimensional steering. In particular, using an entangled state in dimension $d = 31$, our data certifies a minimum Schmidt number of $n = 15$.

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Introduction.—The possibility of having entanglement between quantum systems with a large number of degrees of freedom opens interesting perspectives in quantum information science [1]. In particular, high-dimensional quantum systems can lead to stronger forms of correlations [2,3], featuring increased resilience to noise and losses [4–6]. This makes them a promising alternative to qubits for applications in quantum technology, in particular for quantum communications [7–12]. Experimentally, impressive progress has been achieved in recent years towards the generation and manipulation of high-dimensional entanglement [13–17]. A key problem is then to certify and characterize this entanglement. This is challenging not only due to the large number of parameters in the Hilbert space, but also because experimentally available data is typically limited. Nevertheless, significant progress has been reported in scenarios assuming fully characterized measurement devices [18–24].

It turns out that quantum theory allows one to certify high-dimensional entanglement, as quantified by the notion of Schmidt number [25,26] (see below), based only on the nonlocal correlations it produces, hence relaxing the requirement of a perfectly calibrated or trusted measurement device. That is, given some observed data, one can, in principle, certify the presence of high-dimensional entanglement (i.e., infer a lower bound on the Schmidt number) without making any assumptions about the workings of the measurement devices used. Beyond their fundamental interest, such black-box tests are also relevant for device-independent quantum information processing [27–29]. Previous works have discussed these questions for Bell nonlocality mostly on the theoretical level [30,31], with proof-of-principle experiments certifying entangled states of Schmidt number $n = 3$ [16] and $n = 4$ [32]. The experimental certification of higher-dimensional entanglement via nonlocality is extremely demanding technologically, requiring very high state fidelities and offering extremely low tolerance to noise.

In this work, we address these questions from the point of view of quantum steering, a form of quantum correlations intermediate between entanglement and Bell nonlocality [33,34]. Quantum steering relaxes the strict technological requirements of Bell nonlocality by assuming an uncharacterized or untrusted measurement device only on one side. However, steering tests developed so far can only witness the presence of entanglement (i.e., certify a Schmidt number $n > 1$) but do not characterize the entanglement dimensionality [35–37]. Here we develop a notion of genuine high-dimensional quantum steering. This leads to effective methods for certifying a minimal entanglement dimensionality, i.e., a lower bound on the Schmidt number n, in a one-sided device-independent setting. We demonstrate this experimentally with photon pairs entangled in their discretized transverse position-momentum and certify Schmidt numbers up to $n = 15$.

Consider a scenario featuring two distant parties, Alice and Bob, sharing a bipartite quantum state ρ_{AB}. In the task of steering [see Fig. 1(a)], Alice performs several possible quantum measurements on her subsystem, thus remotely "steering" the state of Bob's subsystem to

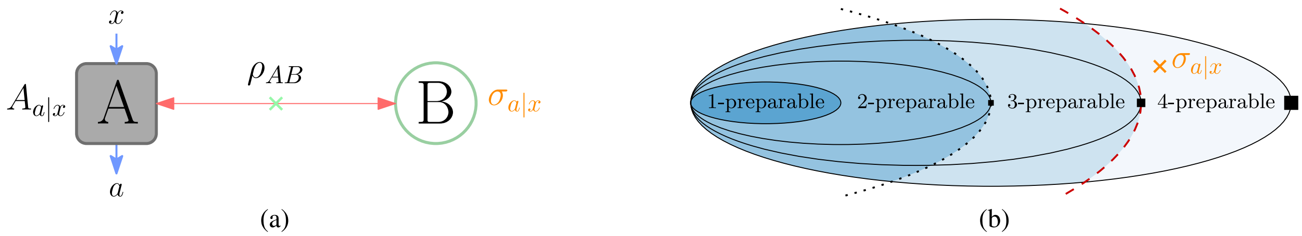

FIG. 1. High-dimensional quantum steering. (a) Alice and Bob share an entangled state ρ_{AB}. By performing local measurements $A_{a|x}$, Alice remotely steers Bob's subsystem, described by the assemblage $\sigma_{a|x}$. (b) As the entanglement dimensionality (the Schmidt number n) of the state ρ_{AB} increases, stronger correlations can be created. More precisely, by performing well-chosen measurements, Alice can generate for Bob an assemblage $\sigma_{a|x}$ that can provably not have been obtained via any lower-dimensional entangled state. To prove this, we define the notion of n-preparable assemblages, i.e., that can be produced via ρ_{AB} with Schmidt number n. This leads to a hierarchy of sets, shown here for $n \leq 4$. First, one-preparable assemblages (with $\mathrm{SR}(\sigma_{a|x}) = 0$ thus $\delta(\sigma_{a|x}) = 1$) feature no quantum steering. Next, the two- and three-preparable sets contain assemblages achievable with entangled states of Schmidt number $n = 2$ and $n = 3$, respectively. Beyond this, there exist assemblages that are not three-preparable, hence featuring genuine four-dimensional steering, as witnessed by violation of a steering inequality (red dashed line corresponding to $\delta(\sigma_{a|x}) > 3$). This guarantees the presence of an entangled state of Schmidt number $n = 4$ in a one-sided device-independent setting.

$$\sigma_{a|x} = \mathrm{Tr}_A[(A_{a|x} \otimes \mathbb{1}_B)\rho_{AB}], \qquad (1)$$

where x denotes Alice's choice of measurement and a its outcome. Alice's measurements are represented by a set of positive operators $A_{a|x}$ satisfying $\sum_a A_{a|x} = \mathbb{1}_A$ for all x. The collection $\{\sigma_{a|x}\}_{a,x}$ of the possible (unnormalized) steered states is termed an assemblage, referred to as $\sigma_{a|x}$ in the following. When this assemblage can be produced without the use of entanglement, i.e., via a so-called local hidden state (LHS) model [33], the assemblage is called unsteerable. If this is not possible, then the assemblage demonstrates steering. This effect has been investigated experimentally, mostly with qubit entanglement [38–41].

Here we are specifically interested in the situation where Alice and Bob share a quantum state featuring high-dimensional entanglement. Consider, for instance, a $d \times d$ maximally entangled state $|\phi_d\rangle = \sum_{j=0}^{d-1} |j, j\rangle/\sqrt{d}$. By using any set of incompatible measurements, Alice can generate on Bob's side an assemblage featuring steering [42,43]. Moreover, for large dimensions, the robustness to noise and losses of these assemblages is known to increase [44–47]. This suggests that high-dimensional entanglement can in fact lead to assemblages featuring a stronger form of quantum correlations. In particular, by using well-chosen measurements, Alice may generate an assemblage for Bob that could not have been created using lower-dimensional entanglement. Below, we formalize this intuition and define the notion of genuine high-dimensional steering.

Specifically, we characterize the entanglement dimensionality through the concept of Schmidt number [25,26]. The Schmidt number of a state ρ_{AB} is the minimum n such that there exists a decomposition $\rho_{AB} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, where all $|\psi_j\rangle$ are pure entangled states of Schmidt rank at most n. For pure states, the Schmidt number is simply equal to the Schmidt rank. This motivates us to define the notion of n-preparable assemblages.

Definition 1: An assemblage $\sigma_{a|x}$ acting on $\mathbb{C}^d$ is n-preparable, with $1 \leq n \leq d$, when it can be decomposed as $\sigma_{a|x} = \mathrm{Tr}_A[(A_{a|x} \otimes \mathbb{1}_B)\rho_{AB}]$, where $\rho_{AB} \in \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B)$ has Schmidt number n and d is the dimension of the Hilbert space $\mathcal{H}^B \equiv \mathbb{C}^d$.

In other words, an n-preparable assemblage $\sigma_{a|x}$ can be prepared via suitable operations on an entangled state of Schmidt number n. However, one could also prepare this assemblage via operations on a state with a larger Schmidt number. This implies that any n-preparable assemblage is also straightforwardly $(n+1)$-preparable, which leads to a nested structure of assemblages, as shown in Fig. 1(b). For $n = 1$ we recover the usual definition of steering: any one-preparable assemblage can be reproduced via a LHS model [33] or, equivalently, via a separable state (i.e., with Schmidt number $n = 1$) of arbitrary dimension on Alice's side [48,49]. On the other hand, for $n = d$ we obtain the full set of quantum assemblages on $\mathbb{C}^d$, as any decomposition of a density matrix can be remotely generated via shared entanglement and well-chosen local measurements [50,51].

Interestingly, there exist assemblages that are n-preparable without being $(n-1)$-preparable, thus featuring genuine n-dimensional steering. For instance, in the case $n = 4$, there are assemblages that cannot be created by only using entangled states with Schmidt number 3. Such assemblages thus feature genuine four-dimensional steering and guarantee that the underlying states have Schmidt number $n = 4$ [see Fig. 1(b)]. It turns out, however, that the general characterization of the set of n-preparable assemblages is challenging. Notably, standard methods that allow for a full characterization of unsteerable assemblages do not work here: determining whether an assemblage is one-preparable can be cast as a semidefinite programming [35], which does not appear to be the case for n-preparability when $n > 1$.

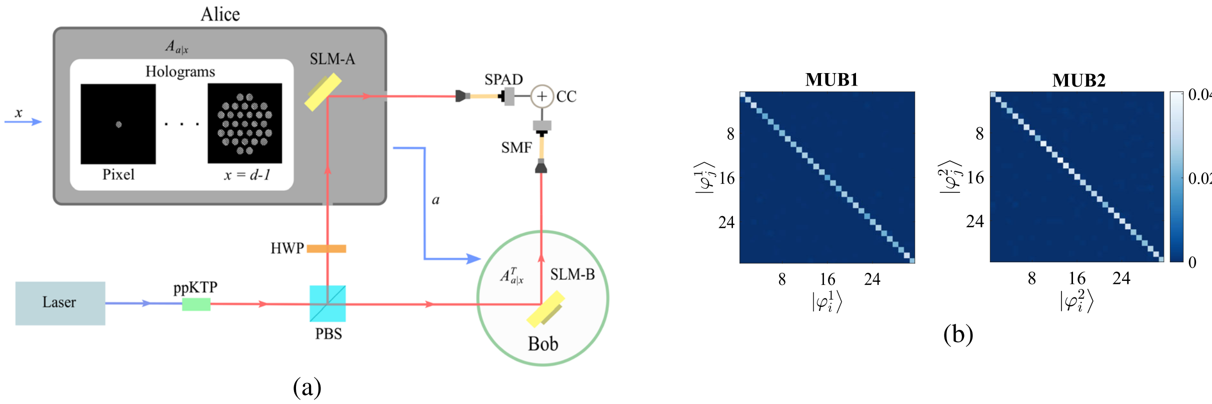

FIG. 2. Experimental realization. (a) One photon from an entangled photon pair and a classical bit x are distributed to the untrusted party, Alice. She generates holograms on a spatial light modulator (SLM-A) to perform the projection $A_{a|x}$ onto outcome a for the given basis x and passes a to the trusted party, Bob. Bob receives the other photon along with this classical information, forming the conditional state $\sigma_{a|x}$, and he performs the projection $A^T_{a|x}$ according to his chosen steering inequality. Coincident photon detection events are then used to evaluate the steering inequality, under the fair sampling assumption, allowing us to certify genuine high-dimensional steering. (b) Normalized two-photon coincidence counts in a pair of $d = 31$ dimensional mutually unbiased pixel bases ($x = 1$ and 2). Using these measurements we obtain the maximum value of $\delta(\sigma_{a|x}) \geq 14.1 \pm 0.6$ that demonstrates genuine 15-dimensional steering (i.e., Schmidt number $n = 15$).

Nevertheless, we can derive a necessary criterion for n-preparability in the case where Alice has two possible measurements. We use the notion of steering robustness (SR), a convex quantifier of quantum steering [35,52]:

$$\mathrm{SR}(\sigma_{a|x}) = \min_{t,\tau_{a|x}}\left\{ t \geq 0 \,\middle|\, \frac{\sigma_{a|x} + t\tau_{a|x}}{1+t} \text{ unsteerable} \right\}, \quad (2)$$

where the minimization is over all assemblages $\tau_{a|x}$ with the same dimension and numbers of inputs and outputs as $\sigma_{a|x}$. The SR quantifies the robustness of $\sigma_{a|x}$ to an arbitrary noise $\tau_{a|x}$ before becoming unsteerable. Specifically, since any assemblage $\sigma_{a|x}$ that is n-preparable can by definition be written as in Eq. (1) where ρ_{AB} has Schmidt number n, we can use the convexity of the steering robustness to upper bound $\mathrm{SR}(\sigma_{a|x})$ by

$$\sum_j p_j \mathrm{SR}\{\mathrm{Tr}_A[(A_{a|x} \otimes \mathbb{1}_B)|\psi_j\rangle\langle\psi_j|]\} \quad (3)$$

$$\leq \max_{|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n} \mathrm{SR}\{\mathrm{Tr}_A[(A_{a|x} \otimes \mathbb{1}_B)|\psi\rangle\langle\psi|]\}. \quad (4)$$

Importantly, the assemblages relating to this last upper bound act on $\mathbb{C}^n$. Using the connection between quantum steering and measurement incompatibility [42,43,53,54] and a recent result identifying the most incompatible pairs of quantum measurements in a given dimension [55], we find that the steering robustness of any n-preparable assemblage is upper bounded by $(\sqrt{n}-1)/(\sqrt{n}+1)$; all details are in the Supplemental Material [56]. Hence, any n-preparable assemblage $\sigma_{a|x}$ (with two inputs for Alice) satisfies

$$n \geq \left(\frac{1 + \mathrm{SR}(\sigma_{a|x})}{1 - \mathrm{SR}(\sigma_{a|x})}\right)^2 \equiv \delta(\sigma_{a|x}), \quad (5)$$

so that violating this inequality amounts to certifying that the assemblage is not n-preparable, i.e., the shared state has at least a Schmidt number of $n+1$.

The inequality (5) turns out to be tight for all $n \leq d$: if ρ_{AB} is a maximally entangled state of dimension $n \times n$, and Alice performs projective measurements onto two mutually unbiased bases (MUBs), then the resulting assemblage (embedded in $\mathbb{C}^d$) saturates the bound. Recall that two orthonormal bases are called mutually unbiased if the scalar product of any vector from the first basis with any vector from the second basis is equal to $1/\sqrt{n}$. Below, we use a standard construction of sets of MUBs when d is prime (see Sec. II of the Supplemental Material [56]).

From the above, in particular inequality (5), we see that given the steering robustness of an assemblage, we obtain a lower bound on the dimension n such that $\sigma_{a|x}$ is n-preparable. Full tomography on Bob's side would make the exact computation of the steering robustness possible via semidefinite programming [35]. A more effective (and experimentally friendly) method consists in using the results of Refs [57,58] showing that the steering robustness can be lower bounded via a steering functional involving only a pair of MUBs measurements (denoted $M_{a|x}$) for both Alice and Bob, namely,

$$\frac{1}{\lambda}\sum_{a,x}\mathrm{Tr}[(M_{a|x} \otimes M^T_{a|x})\rho_{AB}] - 1 \leq \mathrm{SR}(\sigma_{a|x}) \leq \frac{\sqrt{n}-1}{\sqrt{n}+1}, \quad (6)$$

where $\lambda = 1 + 1/\sqrt{d}$ (see Sec. III of the Supplemental Material [56]). The second inequality in Eq. (6) corresponds to the result of Eq. (5) and gives a steering inequality valid for all n-preparable assemblages. For $n = 1$, we recover the inequality $(\sum_{a,x} \mathrm{Tr}[(M_{a|x} \otimes M_{a|x}^T)\rho_{AB}] \leq 1 + 1/\sqrt{d})$, the violation of which certifies the presence of entanglement (i.e., Schmidt number $n > 1$) [16,35–37]. Clearly, the inequality (6) is more general, and provides a lower bound on the Schmidt number n depending on the amount of violation.

Note that this steering inequality as well as the relation (5) apply to general assemblages (involving, e.g., mixed states). Note also that the inequality can be saturated by using a $d \times d$ pure maximally entangled state and projective measurements onto MUBs. In Sec. III of Ref. [56] we derive the critical noise threshold for violating the inequality for isotropic states (mixture of a maximally entangled state with white noise). The above method is well adapted to experiments and allows us to certify genuine high-dimensional steering in practice. We use photon pairs entangled in their discrete transverse position-momentum, also known as "pixel" entanglement [24]. This platform allows us to access generalized d-dimensional measurements with a very high quality in dimensions up to $d = 31$. As shown in Fig. 2(a), a nonlinear ppKTP crystal is pumped with a continuous-wave ultraviolet laser (405 nm) to produce a pair of pixel-entangled infrared photons (810 nm) via type-II spontaneous parametric down-conversion (SPDC). The photon pairs are separated by a polarizing beam splitter (PBS) and directed to Alice and Bob, who each have access to a holographic spatial light modulator (SLM) for performing generalized projective measurements in the pixel basis or any of its MUBs. The holograms used for performing projective measurements are optimized by tailoring the size and spacing of the pixels based on the knowledge of the joint-transverse-momentum-amplitude (JTMA) of the generated biphoton state [59]. This choice of basis warrants that the state well approximates a maximally entangled state and that, in addition to the strong correlations in the pixel basis, pixel-MUB are also strongly correlated owing to momentum conservation of the narrow-band, weakly focused pump. The SLM holograms ensure that only photons carrying pixel or pixel-MUB modes of interest couple efficiently to single-mode fibers (SMF) and are subsequently detected by single-photon avalanche detectors (SPADs). This allows us to reconstruct the results of the measurement operators $M_{a|x}$ and $M_{a|x}^T$, and implement them in the steering inequality (6).

It is important to note that the measurements actually performed in the experiment only have two outcomes, depending on whether the photon detector clicks or not. While it is common practice to reconstruct full projective measurements out of these dichotomic ones [13,16,32,46], the underlying assumption is strong since the corresponding

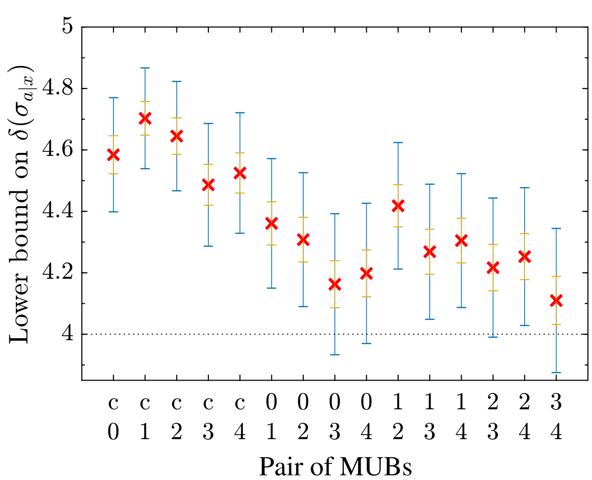

FIG. 3. Experimental certification of genuine five-dimensional steering. A photon pair with entanglement in dimension $d = 5$ is generated. The complete set of MUBs features $d + 1 = 6$ bases, labeled by c for computational and 0…4, which leads to 15 possible pairs of MUBs to be measured by both Alice and Bob. For each pair, the certified dimension is given by the ceiling of the quantity $\delta(\sigma_{a|x})$; the steering robustness being estimated via the steering inequality (6). Here all pairs of MUBs certify the presence of genuine five-dimensional steering, and hence maximal Schmidt number $n = d = 5$, within 1 standard deviation. The second error bar represents 3 standard deviations.

TABLE I. Experimental results for higher dimensions. Entanglement is prepared in prime dimensions d from 5 to 31. For each d, we provide the minimum and maximum values of the quantity $\delta(\sigma_{a|x})$, the ceiling of which gives a lower bound on the certified Schmidt number n. For $d = 5$, these values correspond to those of Fig. 3. For $d = 19$, a Schmidt number of $n = 14$ can be certified (for the best pair of MUBs), while all 190 possible pairs certify (at least) $n = 11$. Moreover, for $d = 31$, the data certifies a Schmidt number $n = 15$, i.e., genuine 15-dimensional quantum steering. Note that for $d \geq 23$, the time required for measuring all $d + 1$ MUBs scales unfavorably (in particular for the computational basis), thus only one pair of MUBs was measured. In higher dimensions, the errors are larger due to higher count rates.

	Lower bound on $\delta(\sigma_{a|x})$		
Dimension d	Minimum	Maximum	Certified Schmidt number n
5	4.1 ± 0.1	4.7 ± 0.1	5
7	5.1 ± 0.2	6.4 ± 0.1	7
11	6.3 ± 0.3	9.1 ± 0.2	10
13	7.0 ± 0.3	10.1 ± 0.3	11
17	9.3 ± 0.3	12.4 ± 0.3	13
19	10.1 ± 0.5	13.6 ± 0.5	14
23	11.4 ± 0.5		12
29	12.1 ± 0.6		13
31	14.1 ± 0.6		15

steering scenarios are inherently different, having a different number of inputs and outputs. Note also that due to detector and system inefficiencies (see Sec. IV of the Supplemental Material [56]) we are working under the fair-sampling hypothesis; however, no subtraction of background or accidental counts is performed.

The results are given in Fig. 3 and Table I. Note that since there are $d+1$ MUBs in (prime) dimensions d, there are $d(d+1)/2$ possible pairs of them, giving rise to potentially different certified dimensions. For $d=5$, we consider all 15 possible pairs of MUBs, for all of which we find $\delta(\sigma_{a|x}) > 4$ (see Fig. 3), thus certifying genuine five-dimensional steering (i.e., Schmidt number $n=5$). That is, none of this data could be reproduced with entangled states of Schmidt number $n \leq 4$. Of all possible pairs, those utilizing the pixel basis (also referred to as computational or simply "c") exhibit slightly better bounds owing to the higher visibility in this basis, since it is the natural Schmidt basis resulting from momentum conservation.

Next we investigate higher dimensions, up to $d=31$. Note that for $d \geq 23$, we measured only one pair of MUBs to optimize the total data acquisition time, as the number of single-outcome measurements required increases with $O(d^2)$. In Table I we only show, for simplicity, the minimum and maximum values obtained for the parameter $\delta(\sigma_{a|x})$; a Schmidt number of $n=15$ can be certified when using an entangled state in dimension $d=31$. Moreover, for $d=19$, all 190 possible pairs of MUBs certify (at least) Schmidt number $n=11$ and up to $n=14$. The total measurement time for measuring two MUBs (excluding the computational basis) was 40 sec for $d=5$ and 16 min for $d=31$.

We have developed the concept of genuine high-dimensional steering, leading to effective methods for certifying a lower bound on the entanglement dimensionality (the Schmidt number) in a one-sided device-independent setting, as demonstrated in a photonic experiment. Moreover, our approach can be readily applied to other quantum platforms using different degrees of freedom (see Sec. V of the Supplemental Material [56]). Our work could be of significant interest for information-theoretic tasks such as randomness generation and cryptography. More generally, this represents an important step towards the realization of noise-robust, high-capacity quantum networks in the near future.

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