Energy-Constrained Discrimination of Unitaries, Quantum Speed Limits, and a Gaussian Solovay-Kitaev Theorem

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We investigate the energy-constrained (EC) diamond norm distance between unitary channels acting on possibly infinite-dimensional quantum systems, and establish a number of results. First, we prove that optimal EC discrimination between two unitary channels does not require the use of any entanglement. Extending a result by Acín, we also show that a finite number of parallel queries suffices to achieve zero error discrimination even in this EC setting. Second, we employ EC diamond norms to study a novel type of quantum speed limits, which apply to pairs of quantum dynamical semigroups. We expect these results to be relevant for benchmarking internal dynamics of quantum devices. Third, we establish a version of the Solovay-Kitaev theorem that applies to the group of Gaussian unitaries over a finite number of modes, with the approximation error being measured with respect to the EC diamond norm relative to the photon number Hamiltonian.

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Introduction.—The task of distinguishing unknown objects is arguably a fundamental one in experimental science. Quantum state discrimination, one of the simplest examples of a problem of this sort, has gained a central role in the flourishing field of quantum information science. The optimal measurement for discriminating between two quantum states via quantum hypothesis testing was found by Holevo and Helstrom [1–4]. Subsequent fundamental contributions related to state discrimination include the operational interpretation of quantum relative entropy [5] and of a related entanglement measure via quantum generalizations of Stein's lemma [6–8], the identification of a quantum Chernoff bound for symmetric hypothesis testing [9–11], and the discovery of quantum data hiding [12–16].

While quantum states are simpler objects, quantum processes, or channels, are more fundamental [17]. The basic primitive in distinguishing them is that of binary channel discrimination: two distant parties, Alice and Bob, are granted access to one query of one of two channels $\mathcal{N}$ and $\mathcal{M}$, with *a priori* probabilities p and $1-p$, and they have to guess which channel was chosen. The best strategy consists of Alice preparing a (possibly entangled) bipartite state $|\Psi\rangle_{AA'}$, sending the system A through the noisy channel, and the auxiliary system (or ancilla) A' through an ideal (noiseless) channel to Bob, who then performs state discrimination on the bipartite system AA' that he receives. When both $\mathcal{N}$ and $\mathcal{M}$ are unitary channels, however, the auxiliary system is not needed [18] (cf. Theorem 3.55 in Ref. [19]). Experimentally, this simplification is helpful, as it exempts us from using: (a) an ancilla and entanglement; and (b) an ideal side channel, which might be technologically challenging.

More insight into the channel distinguishability problem can be gained by looking at multiquery discrimination [20–22]. When the channels are unitary, a seminal result by Acín states that perfect discrimination is possible with only a finite number of queries [23,24], a phenomenon that has no analogue for states [25]. The same result can be achieved by using an adaptive strategy that requires no entanglement [26].

It is common to assume that any arbitrary quantum operation can be employed for the discrimination task at hand. This is, however, often unrealistic, due to technological as well as physical limitations. This is the case, e.g., when the quantum states (respectively, the channels) to be discriminated are distributed among (respectively, connect) two parties who can only employ local operations assisted by classical communication. Such a restriction could severely hinder the discrimination power, both for states [12–16] and for channels [27,28].

Another example of physical restriction comes about, for instance, when one studies continuous-variable (CV) quantum systems, e.g., collections of electromagnetic modes traveling along an optical fiber. This setting, which constitutes the basis of practically all proposed protocols for quantum communication, is of outstanding technological and experimental relevance [29–32]. Accordingly, the theoretical study of CV quantum channels is a core area of quantum information [33–35]. CV channel discrimination can be thought of as a fundamental primitive for benchmarking such channels.

When accessing a CV quantum system governed by a Hamiltonian H, one only has access to states ρ with bounded mean energy $\mathrm{Tr}[\rho H] \leq E$. This fundamentally unavoidable restriction motivates us to look into energy-constrained (EC) channel discrimination [21,36–38]. In our setting, we separate the energy cost of manufacturing probes from that of measuring the output states [39], and only account for the former. This is justified operationally by thinking of the unknown channel (either $\mathcal{N}$ or $\mathcal{M}$) as connecting an EC client to a quantum computing server that has access to practically unlimited energy. In the above context, the figure of merit is the so-called EC diamond norm distance $\|\mathcal{N} - \mathcal{M}\|_\diamond^{H,E}$ [36,37,40].

In this Letter, we (1) study the EC diamond norm distance between unitary channels, and employ it to establish (2) operationally meaningful quantum speed limits [41] for experimentally relevant Hamiltonians, as well as (3) a Solovay-Kitaev theorem [42,43] for Gaussian (i.e., symplectic) unitaries. Our first result states that optimal EC discrimination of two unitary channels does not require any entanglement (Theorem 1). This extends the analogous result for unconstrained discrimination (c.f. Theorem 3.55 in Ref. [19]). In the same setting, we then generalize Acín's result [23], proving that a finite number of parallel queries suffices to achieve zero error (Theorem 2).

We then employ the EC diamond norm distance to quantify in an operationally meaningful way the speed at which time evolutions under two different Hamiltonians drift apart from each other (Theorem 3). Our result amounts to a quantum speed limit [41] that applies to a more general setting than previously investigated [44–57], namely, that involving two different unitary groups. As a special case, we study evolutions induced by quadratic Hamiltonians on a collection of harmonic oscillators (Corollary 4). Analogous estimates are then given for the case in which one of the two channels models an open quantum system (Theorem 5) [58].

Our last result is a Solovay-Kitaev theorem [42,43] for Gaussian unitaries (Theorem 6). It states that any finite set of gates generating a dense subgroup of the symplectic group can be used to construct short gate sequences that approximate well, in the EC diamond norm corresponding to the photon number Hamiltonian, any desired Gaussian unitary. The significance of our result rests on the compelling operational interpretation of the EC diamond norm in terms of channel discrimination: the action of the constructed gate will be almost indistinguishable from that of the target on all states with a certain maximum average photon number.

The setting.—Quantum states on a Hilbert space $\mathcal{H}$ are represented by density operators, i.e., positive trace-class operators with trace one, on $\mathcal{H}$. Quantum channels are modeled by completely positive and trace preserving (CPTP) maps acting on the space of trace-class operators on $\mathcal{H}$. A Hamiltonian on $\mathcal{H}$ is a densely defined self-adjoint operator H whose spectrum $sp(H)$ is bounded from below. Up to redefining the ground state energy, we can assume that $\min \mathrm{sp}(H) = 0$, in which case we call H grounded. In what follows, for a pure state $|\psi\rangle \in \mathcal{H}$, we will denote with $\psi \coloneqq |\psi\rangle\langle\psi|$ the corresponding density matrix.

CV quantum systems, i.e., finite collections of harmonic oscillators, or modes, are central for applications [33,34]. The Hilbert space of an m-mode system is formed by all square-integrable functions on $\mathbb{R}^m$, and is denoted by $\mathcal{H}_m \coloneqq L^2(\mathbb{R}^m)$. The creation and annihilation operators corresponding to the jth mode ($j = 1, \ldots, m$) will be denoted by $a_j^\dagger$ and a_j, respectively. They satisfy the canonical commutation relations (CCRs) $[a_j, a_k^\dagger] = \delta_{jk}$. In the (equivalent) real picture, one defines the position and momentum operators $x_j \coloneqq (a_j + a_j^\dagger)/\sqrt{2}$ and $p_j \coloneqq (a_j - a_j^\dagger)/(\sqrt{2}i)$, organized in the vector $R \coloneqq (x_1, p_1, \ldots, x_m, p_m)^\intercal$. The CCRs now read $[R, R^\intercal] = i\Omega_m$, with $\Omega_m \coloneqq \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^{\oplus m}$. Gaussian unitaries are products of exponentials $e^{-(i/2)R^\intercal Q R}$, where Q is an arbitrary $2m \times 2m$ symmetric matrix, and $\frac{1}{2} R^\intercal Q R$ is called a quadratic Hamiltonian. Gaussian unitaries are in one-to-one correspondence with symplectic matrices via the relation $U_S \leftrightarrow S$ defined by $U_S^\dagger R_j U_S = \sum_k S_{jk} R_k$. The corresponding unitary channel will be denoted with $\mathcal{U}_S(\cdot) \coloneqq U_S(\cdot)U_S^\dagger$. Recall that a $2m \times 2m$ real matrix S is called symplectic if $S\Omega_m S^\intercal = \Omega_m$, and that symplectic matrices form a group, hereafter denoted by $\mathrm{Sp}_{2m}(\mathbb{R})$ [59].

The energy cost of a channel discrimination protocol comes from two main sources: first, the preparation of the probe state to be fed into the unknown channel, and, second, the subsequent quantum measurement, which inescapably requires energy to be carried out [39]. In this Letter we consider only the first contribution, i.e., the energy cost of the probe. Operationally, we can separate the above two contributions by considering the following setting. An unknown channel, either $\mathcal{N}_{A\to B}$ (with *a priori* probability p) or $\mathcal{M}_{A\to B}$ (with *a priori* probability $1 - p$) connects two distant parties, Alice (the sender) and Bob (the receiver). We assume that Alice's equipment only allows for the preparation of probe states with an average energy at most E, as measured by some positive Hamiltonian $H_A \geq 0$ on the input system. No such restriction is placed on Bob, who can carry out any measurement he desires, and whose task is that of guessing the channel. We can further distinguish two possibilities: (i) Alice is limited to preparing states ρ_A on the input system A, to be sent to Bob via the unknown channel; or (ii) she can prepare a (possibly entangled) state $\rho_{AA'}$, where A' is an arbitrary ancilla, and send also A' to Bob via an ideal (noiseless) channel. The energy constraint reads $\mathrm{Tr}[\rho_A H_A] \le E$, where in case (ii) we set $\rho_A := \mathrm{Tr}_{A'}\rho_{AA'}$. The error probability corresponding to (ii) takes the form $P_e^{H,E}(\mathcal{N},\mathcal{M};p) = \frac{1}{2}(1 - \|p\mathcal{N} - (1-p)\mathcal{M}\|_\diamond^{H,E})$, where for a superoperator $\mathcal{L}_A$ that preserves self-adjointness the EC diamond norm is defined by

$$\|\mathcal{L}_A\|_\diamond^{H,E} = \sup_{\substack{|\Psi\rangle_{AA'}: \\ \mathrm{Tr}\,\Psi_A H_A \le E}} \|(\mathcal{L}_A \otimes id_{A'})(\Psi_{AA'})\|_1, \qquad (1)$$

where $\|\cdot\|_1$ is the trace norm, while the supremum is over all states $|\Psi\rangle_{AA'}$ on AA', with A' being an ancilla, whose reduced state on A has energy bounded by E. A similar expression but without A' holds in setting (i).

Results.—Throughout this section we discuss our main findings. Complete proofs as well as additional technical details can be found in the Supplemental Material [60].

(1) *EC discrimination of unitaries:* Our first result states that the above settings (i) and (ii) are equivalent in the case of two unitary channels. This generalizes the seminal result of Aharonov *et al.* [18] (cf. Theorem 3.55 in Ref. [19]), and implies that optimal EC discrimination of unitaries can be carried out without the use of any entanglement.

Theorem 1. Let U, V be two unitaries acting on a Hilbert space of dimension $\dim\mathcal{H} \ge 3$, and call $\mathcal{U}(\cdot) := U(\cdot)U^\dagger$, $\mathcal{V}(\cdot) := V(\cdot)V^\dagger$ the associated channels. Let H be a grounded Hamiltonian, and fix $E > 0$. Then

$$\begin{aligned} \|\mathcal{U}-\mathcal{V}\|_\diamond^{H,E} &= \sup_{\langle\psi|H|\psi\rangle \le E} \|(\mathcal{U}-\mathcal{V})(\psi)\|_1 \\ &= 2\sqrt{1 - \inf_{\langle\psi|H|\psi\rangle \le E} |\langle\psi|U^\dagger V|\psi\rangle|^2}. \end{aligned} \qquad (2)$$

In other words, in this case the supremum in Eq. (1) can be restricted to unentangled pure states.

The above result can be used to estimate the EC diamond norm distance between displacement channels. These are defined for $z \in \mathbb{R}^{2m}$ by $\mathcal{D}_z(\cdot) := \mathscr{D}(z)(\cdot)\mathscr{D}(z)^\dagger$, where $\mathscr{D}(z) := e^{-i\sum_j (\Omega_m z)_j R_j}$. Letting $N := \sum_j a_j^\dagger a_j$ be the total photon number Hamiltonian, one has that

$$\begin{aligned} \sqrt{1-e^{-\|z-w\|^2 f(E)^2}} &\le \frac{1}{2}\|\mathcal{D}_z - \mathcal{D}_w\|_\diamond^{N,E} \\ &\le \sin\left(\min\left\{\|z-w\| f(E), \frac{\pi}{2}\right\}\right), \\ f(E) &:= \frac{1}{\sqrt{2}}(\sqrt{E} + \sqrt{E+1}). \end{aligned} \qquad (3)$$

Using the structure of the symplectic group, we also obtain the following upper bound for the difference of two symplectic unitaries: given S, $S' \in \mathrm{Sp}_{2m}(\mathbb{R})$,

$$\begin{aligned} \frac{1}{2}\|\mathcal{U}_S - \mathcal{U}_{S'}\|_\diamond^{N,E} &\le \sqrt{(\sqrt{6}+\sqrt{10}+5\sqrt{2}m)(E+1)} \\ &\quad g(\|(S')^{-1}S\|_\infty)\sqrt{\|(S')^{-1}S - I\|_2}, \\ g(x) &:= \sqrt{\frac{\pi}{x+1}} + \sqrt{2x}, \end{aligned} \qquad (4)$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_2$ denote the operator norm and the Hilbert-Schmidt norm, respectively. We can also exploit Theorem 1 to immediately extend a celebrated result by Acín [23] (see also Refs. [25,26]), and establish that even in the presence of an energy constraint (which is particularly relevant in the case of unitaries acting on CV quantum systems), a finite number of parallel queries achieves zero-error discrimination.

Theorem 2. In the setting of Theorem 1, there exists a positive integer n such that n parallel uses of $\mathcal{U}$ and $\mathcal{V}$ can be discriminated perfectly using inputs of finite total energy E, i.e.,

$$\|\mathcal{U}^{\otimes n} - \mathcal{V}^{\otimes n}\|_\diamond^{H_{(n)},E} = 2, \qquad (5)$$

where $H_{(n)} := \sum_{j=1}^n H_j$ is the n-copy Hamiltonian, and $H_j := I \otimes \cdots \otimes I \otimes H \otimes I \otimes \cdots \otimes I$, with the H in the jth location.

(2) *Quantum speed limits:* Our first application deals with the problem of quantifying the relative drift caused by two different unitary dynamics on a quantum system. This may be important, for instance, in benchmarking internal Hamiltonians of quantum devices.

In what follows, our findings are generally presented in the form of an upper bound on the EC diamond norm distance between time evolution channels. This is an alternative yet completely equivalent reformulation of a quantum speed limit. To recover the standard one [41], one has to turn the inequality around and recast it as a lower bound on the time taken to reach a certain prescribed distance [60]. Our first result extends previous findings by Winter [[37], Theorem 6] and some of us [[57], Proposition 3.2] by tackling the case of two different unitary groups.

Theorem 3. Let H, H' be self-adjoint operators. Without loss of generality, assume that 0 is in the spectrum of H. Let the "relative boundedness" inequality

$$\|(H-H')|\psi\rangle\| \le \alpha\|H|\psi\rangle\| + \beta \qquad (6)$$

hold for some constants $\alpha, \beta > 0$ and for all (normalized) states $|\psi\rangle$. Then the unitary channels

$$\mathcal{U}_t(\cdot) := e^{-iHt}(\cdot)e^{iHt}, \qquad \mathcal{V}_t(\cdot) := e^{-iH't}(\cdot)e^{iH't} \qquad (7)$$

satisfy the following: for all $t \ge 0$ and $E > 0$,

$$\|\mathcal{U}_t - \mathcal{V}_t\|_\diamond^{|H|,E} \le 2\sqrt{2}\sqrt{\alpha E t} + \sqrt{2}\beta t. \qquad (8)$$

Let us note that Eq. (8) admits a simple reformulation in terms of the Loschmidt echo operator $M_t \coloneqq e^{iH't}e^{-iHt}$ [60,91]. The relative boundedness condition (6) is not merely an artefact of the proof, and is there to ensure that low energy eigenvectors of H do not have very high energies relative to H', which would trivialize the bound Eq. (8). The estimate in Eq. (8) can be shown to be optimal up to multiplicative constants: in general, the diffusive term proportional to $\sqrt{t}$ cannot be removed even for very small times (see Sec. III.B in the SM [60]).

A special case of Theorem 3 that is particularly relevant for applications is that of two quadratic Hamiltonians on a collection of m harmonic oscillators, or modes.

Corollary 4. On a system of m modes, consider the two Hamiltonians $H = \sum_{j=1}^m d_j a_j^\dagger a_j$ and $H' = \sum_{j,k=1}^m (X_{jk} a_j^\dagger a_k + Y_{jk} a_j a_k + Y_{jk}^* a_j^\dagger a_k^\dagger)$, where $d_j > 0$ for all j, and X, Y are two $m\times m$ matrices, with X Hermitian. Then the corresponding unitary channels in Eq. (7) satisfy Eq. (8) for all $t \geq 0$ and $E > 0$, with

$$\begin{aligned} \alpha &= \|D^{-1}\| \left(\sqrt{\frac{3}{2}}\|X-D\|_2 + \left(1+\sqrt{\frac{3}{2}}\right)\|Y\|_2 \right), \\ \beta &= \frac{m-1}{\sqrt{2}}\|X-D\|_2 + \sqrt{\frac{(2m+1)^2}{2} + 2m^2}\|Y\|_2, \end{aligned} \tag{9}$$

where $D_{jk} \coloneqq d_j \delta_{jk}$.

We now look at the more general scenario where the discrimination is between a closed-system unitary evolution and an open-system quantum dynamics. We expect this task to be critical, e.g,. in benchmarking quantum memories, where the effects of external interactions are detrimental and must be carefully controlled. Open quantum systems are described by quantum dynamical semigroups (QDSs) [92,93], i.e., families of channels $(\Lambda_t)_{t\geq 0}$ that (i) obey the semigroup law, $\Lambda_{t+s} = \Lambda_t \circ \Lambda_s$ for t, $s \geq 0$, and (ii) are strongly continuous, in the sense that $\lim_{t\to 0^+} \|\Lambda_t(\rho) - \rho\|_1 = 0$ for all ρ. QDSs take the form $\Lambda_t = e^{t\mathcal{L}}$, where the generator $\mathcal{L}$ is assumed to be of Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) type [94–96] and acts on an appropriate dense subspace of the space of trace class operators as

$$\mathcal{L}(X) = -i[H,X] + \frac{1}{2}\sum_\ell (2L_\ell X L_\ell^\dagger - L_\ell^\dagger L_\ell X - X L_\ell^\dagger L_\ell). \tag{10}$$

Here, H is the internal Hamiltonian, while the Lindblad operators L_ℓ ($\ell = 1,2,\ldots$) model dissipative processes. In our approach these can be unbounded, and hence our results significantly generalize previous works on quantum speed limits in open systems [58].

Theorem 5. Let H be a self-adjoint operator with 0 in its spectrum, and set $\mathcal{U}_t(\cdot) \coloneqq e^{-iHt}(\cdot)e^{iHt}$. Let $(\Lambda_t)_{t\geq 0}$ be a QDS whose generator $\mathcal{L}$ is of GKLS type and satisfies the relative boundedness condition

$$\frac{1}{2}\left\| \sum_\ell L_\ell^\dagger L_\ell |\psi\rangle \right\| \leq \alpha \|H|\psi\rangle\| + \beta \tag{11}$$

for all (normalized) states $|\psi\rangle$, where $\beta \geq 0$ and $0 \leq \alpha < 1$ are two constants. Then it holds that

$$\|\mathcal{U}_t - \Lambda_t\|_\diamond^{|H|,E} \leq 4\left[\sqrt{\sqrt{2}\alpha E t} + \beta t\right] \tag{12}$$

for all $t \geq 0$ and $E > 0$.

Once again, the role of condition (11) is that of ensuring that the Lindblad operators do not make low energy levels decay too rapidly, an effect that we could exploit to design a simple discrimination protocol with a small energy budget. We now demonstrate the applicability of our result by looking at the example of quantum Brownian motion [61,62]. Consider a single quantum particle in one dimension, subjected to a harmonic potential and to a diffusion process. The Hilbert space is $\mathcal{H}_1 = L^2(\mathbb{R})$; we set $H = \frac{1}{2}(x^2 + p^2)$ and $L_\ell = \gamma_\ell x + i\delta_\ell p$ ($\ell = 1,2$), where $p \coloneqq -i(d/dx)$ is the momentum operator, and γ_ℓ, $\delta_\ell \in \mathbb{C}$. In this case Eq. (11) is satisfied, e.g., with $\alpha = (|\gamma_1| + |\delta_1|)^2 + (|\gamma_2| + |\delta_2|)^2$, provided that the right-hand side is smaller than 1, and $\beta = |\gamma_1||\delta_1| + |\gamma_2||\delta_2| + \kappa$, where $\kappa = 0.2047$ is a constant [60]. Therefore, Eq. (12) yields an upper estimate on the operational distinguishability between closed and open dynamics for given waiting time and input energy.

(3) A Gaussian Solovay-Kitaev theorem: The celebrated Solovay-Kitaev theorem [42,43] is a fundamental result in the theory of quantum computing. In layman's terms, it states that any finite set of quantum gates that generates a dense subgroup of the special unitary group is capable of approximating any such desired unitary by means of short sequences of gates. In practice, many of the elementary gates that form the toolbox of CV platforms for quantum computing [29,97] are modeled by Gaussian unitaries. Therefore, a Gaussian version of the Solovay-Kitaev theorem is highly desirable. In establishing our result, we measure the approximation error for gates on an m-mode quantum system by means of the operationally meaningful EC diamond norm distance relative to the total photon number Hamiltonian $N = \sum_{j=1}^m a_j^\dagger a_j$.

Theorem 6. Let $m \in \mathbb{N}$, $r > 0$, $E > 0$ and define $\widetilde{Sp}_{2m}^r(\mathbb{R})$ to be the set of all symplectic transformations S such that $\|S\|_\infty \leq r$. Then, given a set $\mathcal{G}$ of gates that is closed under inverses and generates a dense subset of $\widetilde{Sp}_{2m}^r(\mathbb{R})$, for any symplectic transformation $S \in \widetilde{Sp}_{2m}^r(\mathbb{R})$ and every $0 < \delta$, there exists a finite concatenation S' of poly$(\log \delta^{-1})$ elements from $\mathcal{G}$, which can be found in time poly$(\log \delta^{-1})$ and such that

$$\|\mathcal{U}_S - \mathcal{U}_{S'}\|_\diamond^{N,E} \le F(m)G(r)\sqrt{E+1}\sqrt{\delta}, \qquad (13)$$

where $\mathcal{U}_S(\cdot) \coloneqq U_S(\cdot)U_S^\dagger$, and

$$F(m) \coloneqq 2\sqrt{\sqrt{2m}(\sqrt{6}+\sqrt{10}+5\sqrt{2}m)},$$
$$G(r) \coloneqq (\sqrt{\pi}+\sqrt{2}(r+2))\sqrt{r+2}.$$

The above result guarantees that any Gaussian unitary can be approximated with a relatively short sequence of gates taken from our base set. Note that the sequence length increases with both the squeezing induced by S (quantified by the parameter $\|S\|_\infty$) and the energy threshold E. Theorem 6 also guarantees that finding the relevant gate sequence is a computationally feasible task, thus bolstering the operational significance of the result. Finally, in the Supplemental Material [60] we show that sets of the form $\mathcal{G} = \mathcal{K} \cup \{S\}$, where $\mathcal{K}$ generates a dense subgroup of the passive Gaussian unitary group and S is an arbitrary nonpassive Gaussian unitary, satisfy the denseness assumption of Theorem 6.

Conclusions.—We investigated the EC diamond norm distance between channels, which has a compelling operational interpretation in the context of EC channel discrimination. For the case of two unitary channels, we showed that optimal discrimination can be carried out without using any entanglement, and with zero error upon invoking finitely many parallel queries. An open question here concerns the possibility of obtaining the same result by means of adaptive rather than parallel strategies. This is known to be possible in the finite-dimensional, energy-unconstrained scenario [26].

We then studied some problems where the EC diamond norm can be employed to quantify in an operationally meaningful way the distance between quantum operations. We provided quantum speed limits that apply to the conceptually innovative setting where one compares two different time evolution (semi-)groups, instead of looking at a single one, as previously done.

Finally, we established a Gaussian version of the Solovay-Kitaev theorem, proving that any set of Gaussian unitary gates that is sufficiently powerful to be capable of approximating any desired Gaussian unitary can do so also efficiently, i.e., by means of a relatively small number of gates. Our result bears a potential impact on the study of all those quantum computing architectures that rely on optical platforms.

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