Quantifying the Difference between Many-Body Quantum States

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The quantum state overlap is the textbook measure of the difference between two quantum states. Yet, it is inadequate to compare the complex configurations of many-body systems. The problem is inherited by the widely employed quantum state fidelity and related distances. We introduce the weighted distances, a new class of information-theoretic measures that overcome these limitations. They quantify how hard it is to discriminate between two quantum states of many particles, factoring in the structure of the required measurement apparatus. Therefore, they can be used to evaluate both the theoretical and the experimental performances of complex quantum devices. We also show that the newly defined "weighted Bures length" between the input and output states of a quantum process is a lower bound to the experimental cost of the transformation. The result uncovers an exact quantum limit to our ability to convert physical resources into computational ones.

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Introduction.—Quantum particles are the building blocks of light and matter, but they can display very complex configurations. An important goal of quantum theory is to describe their differences with simple metrics. The state overlap $|\langle i|j\rangle|$ is the standard proxy to compare two wave functions $|i\rangle$, $|j\rangle$, and it has a compelling statistical meaning: it quantifies how hard it is to discriminate two pure states via a single quantum measurement [1]. The overlap is instrumental to build the Fubini-Study distance $\cos^{-1}|\langle i|j\rangle|$ [2,3], which evaluates the distinguishability of two quantum states in terms of how far they are in the system Hilbert space.

Unfortunately, the state overlap is not fully adequate to compare many-body wave functions. Very similar states can be flagged as maximally different. For example, there is zero overlap between the N-qubit states $|0\rangle^{\otimes N}$, $|0\rangle^{\otimes N-1}|1\rangle$, for arbitrarily large N. Moreover, geometrically close states can have very different properties. Transforming $|0\rangle^{\otimes N}$ into the entangled Greenberger-Horne-Zeilinger (GHZ) state $a|0\rangle^{\otimes N}+b|1\rangle^{\otimes N}$, $|a|, |b|\neq 0, 1$ takes experimental resources that grow with the system size [4], e.g., $O(N)$ operations in gate-based quantum computers [5], however big their overlap $|a|$ may be.

The same issues plague the generalizations of the state overlap that quantify the difference between two mixed states ρ and σ, e.g., the quantum fidelity $F(\rho,\sigma)=\mathrm{Tr}|\rho^{1/2}\sigma^{1/2}|_1$ [6,7] and related distances [8]. This fact is troublesome. As we expect to steadily up-size quantum technologies, we need trustworthy tools to evaluate the performances of large noisy quantum machines [9]. Reconstructing the fidelity between, say, the target and the output states of a computation is often the only way to certify that a device is truly quantum without accessing its inner workings [10–13].

In this Letter, we introduce the weighted distances, a class of measures for comparing many particle states. A standard, overlap-based distance quantifies the ability to discriminate two states of a system via a single optimal measurement. Here, we consider a more general scenario. Cooperating observers independently monitor different subsystems, evaluating the difference between two preparations of the assigned subsystem by a standard distance. We construct a weighted sum of these distances, such that the importance of each observer contribution is *inversely* proportional to the size of the assigned subsystem. Since the difficulty of performing measurements is arguably related to the size of the required apparatuses, these quantities weight each contribution in terms of how easy is it to experimentally implement the related measurement. We define a weighted distance as the maximum over all these kinds of weighted sums. The weighted distances satisfy a set of desirable mathematical properties, certifying that they are robust information measures. We perform explicit calculations of interesting case studies, showing that the newly defined weighted Bures length is more informative than the related standard Bures length [14,15]. For example, if a large measurement apparatus is needed to discriminate between two states, their weighted distance is short, because it is experimentally difficult to distinguish one state from the other.

Then, we show that the weighted Bures length between the input and output states of a quantum process is a lower bound to the physical resources that are needed to implement the transformation. That is, the ability to discriminate two quantum states is never greater than the experimental cost of transforming one state into the other. The result is surprising: state distinguishability and state transformation are considered "quite different" tasks [16]. We demonstrate that they are related. Previous works established the minimum time and energy time ("action") to perform state transformations [17–22]. The input-output weighted Bures length is a lower bound to a newly defined index, which factors the required energy, time, and size of gates for quantum state preparation. While proving the optimality of quantum algorithms is notoriously hard [23], the result highlights a fundamental quantitative limit to quantum information processing. The bound is also valid for mixed states and nonunitary state transformations. Hence, it applies to realistic, noisy quantum dynamics.

Definition and justification of weighted distances.—Let us call ρ_N and σ_N two arbitrary density matrices that represent different preparations of an N-particle quantum system. It is well known that full reconstruction of quantum states is a daunting task [24]. It is therefore interesting to build an information measure that captures the difficulty to discriminate between the two states with a single measurement. Suppose one can perform all possible positive operator-valued measures on the system: $\mathcal{M} = \{\mathcal{M}_i \geq 0, \sum_i \mathcal{M}_i = I_N\}$ [5]. The ability to distinguish between ρ_N and σ_N is customarily quantified via maximization of a certain classical statistical distance $d_{\rm cl}$ for probability distributions [8],

$$d(\rho_N, \sigma_N) \coloneqq \max_{\mathcal{M}} \sum_i d_{\rm cl}(\mathrm{Tr}\{\mathcal{M}_i\rho_N\}, \mathrm{Tr}\{\mathcal{M}_i\sigma_N\}) \\ \coloneqq \sum_i d_{\rm cl}(\mathrm{Tr}\{\tilde{\mathcal{M}}_i\rho_N\}, \mathrm{Tr}\{\tilde{\mathcal{M}}_i\sigma_N\}), \quad (1)$$

in which $\tilde{\mathcal{M}} = \{\tilde{\mathcal{M}}_i\}$ is the most informative measurement. Given three arbitrary density matrices ρ_N, σ_N, and τ_N, we assume that the quantity meets the following criteria:

$$d(\rho_N, \sigma_N) \geq 0 (\text{non-negativity}), \\ d(\rho_N, \sigma_N) = 0 \Leftrightarrow \rho_N = \sigma_N \quad (\text{faithfulness}), \\ d(\rho_N, \sigma_N) \geq d(\Lambda(\rho_N), \Lambda(\sigma_N)), \ \forall \ \Lambda \quad (\text{contractivity}), \\ d(\rho_N, \sigma_N) \leq d(\rho_N, \tau_N) + d(\tau_N, \sigma_N) \quad (\text{triangle inequality}), \quad (2)$$

in which Λ is a completely positive trace-preserving (CPTP) map, the most general kind of quantum operation [5]. The distance is normalized such that it takes the maximal value M_d for orthogonal states $d(\rho_N, \sigma_N) = M_d \Leftrightarrow \mathrm{Tr}\{\rho_N\sigma_N\} = 0$. Indeed, these states can be discriminated with certainty. Contractivity under CPTP maps implies that the distance is nonincreasing under partial trace, $d(\rho_N, \sigma_N) \geq d(\rho_k, \sigma_k)$, in which ρ_k and σ_k are the states of a $k < N$-particle subset. The ability to extract information from quantum systems depends on the size of the measurement setup. However, the distance function is not explicitly dependent on the number of particles N, nor the size of the optimal measurement apparatus $\tilde{\mathcal{M}}$. Indeed, there are, in general, several solutions of the maximization in Eq. (1). This degeneracy is maximal for pairs like the N-qubit states $|0\rangle^{\otimes N}, |1\rangle^{\otimes N}$: they are perfectly discriminated by projecting on the computational bases $\{0, 1\}^{\otimes k}, \ \forall\, k \in [1, N]$.

Consider therefore a more general scenario, in which there is a set of cooperating observers that want to discriminate between ρ_N and σ_N. Each of them performs the optimal measurements $\tilde{\mathcal{M}}^{k_\alpha}$ to discriminate the states ρ_{k_α} and σ_{k_α} of subsystems composed of $k_\alpha \leq N$ particles (Fig. 1), then computing $d(\rho_{k_\alpha}, \sigma_{k_\alpha})$. The setup defines a measurement partition

$$P_{k_\alpha} \coloneqq \left\{\tilde{\mathcal{M}}^{k_\alpha}, \sum_\alpha k_\alpha = N\right\}.$$

For example, given $N = 3$, there are the following options: three observers perform single-site detections, determining the partition $\{\tilde{\mathcal{M}}^1, \tilde{\mathcal{M}}^1, \tilde{\mathcal{M}}^1\}$; an observer makes a bipartite measurement, and another one performs a single-particle measurement, inducing three possible partitions $\{\tilde{\mathcal{M}}^2, \tilde{\mathcal{M}}^1\}$ [25]; a single observer implements a three-site measurement $\tilde{\mathcal{M}}^3$. The measurements on different subsystems are independent and compatible, $[\tilde{\mathcal{M}}^{k_{\alpha_i}}, \tilde{\mathcal{M}}^{k_{\alpha_j}}] = 0, \ \forall\, \tilde{\mathcal{M}}^{k_{\alpha_i}}, \ \tilde{\mathcal{M}}^{k_{\alpha_j}} \in P_{k_\alpha}$. Then, we might pick the sum of all the contributions $\sum_\alpha d(\rho_{k_\alpha}, \sigma_{k_\alpha})$ to quantify the information that is extractable from P_{k_α}. Consequently, the maximal value of the arithmetic sum over all the system partitions could be a new measure of state distinguishability. Unfortunately, this quantity would not take into account that each measurement is performed on a different number of particles k_α. It is experimentally harder to implement $\tilde{\mathcal{M}}^k$ than any $\tilde{\mathcal{M}}^{l<k}$. An extreme case

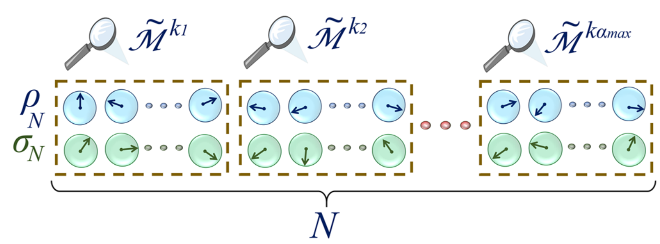

FIG. 1. Consider two N-particle states ρ_N and σ_N. A set of observers compute the distance between the marginal s tates of subsystems with size k_α, $\sum_\alpha k_\alpha = N$, given by $d(\rho_{k_\alpha}, \sigma_{k_\alpha}) = \sum_i d_{\rm cl}(\mathrm{Tr}\{\tilde{\mathcal{M}}_i^{k_\alpha}\rho_{k_\alpha}\}, \mathrm{Tr}\{\tilde{\mathcal{M}}_i^{k_\alpha}\sigma_{k_\alpha}\})$. We quantify the difficulty to discriminate the two states by a weighted sum of each observer contribution.

is the discrimination of the GHZ state from the classically correlated state $|a|^2|0\rangle\langle 0|^{\otimes N} + |b|^2|1\rangle\langle 1|^{\otimes N}$: they are found to be identical by all measurement setups but a full scale N-particle detection. By increasing N, it becomes harder to distinguish the two preparations. Yet, the maximal distance sum is $d(\rho_N, \sigma_N)$, which does not depend on N. A better choice is, for each partition P_{k_α}, to sum all the observer contributions, while weighting their relative importance by the *inverse* of the size of the measured subsystem

$$\delta_{d,P_{k_\alpha}}(\rho_N, \sigma_N) \coloneqq \sum_\alpha \frac{1}{k_\alpha} d(\rho_{k_\alpha}, \sigma_{k_\alpha}). \tag{3}$$

This more refined quantity filters out system degeneracy, which manifests when two or more particles are in the same state. Comparing the two states $\rho_N = |0\rangle\langle 0|^{\otimes N}$ and $\sigma_N = |1\rangle\langle 1|^{\otimes k}|0\rangle\langle 0|^{\otimes N-k}$, one has $\delta_{d,P_{k_\alpha}}(\rho_N, \sigma_N) \le kM_d$. Note that, conversely, the weighted sum $\sum_\alpha k_\alpha d(\rho_{k_\alpha}, \sigma_{k_\alpha})$ overvalues the difference between states. For example, by choosing the N-particle detection $\tilde{\mathcal{M}}^N$, one would have $Nd(\rho_N, \sigma_N) = Nd(\rho_k, \sigma_k) = NM_d, \ \forall\, k$.

We are now ready to quantify the ability to discriminate two arbitrary N-partite quantum states by a single index:

We define the d weighted distance between two states ρ_N and σ_N as

$$D_d(\rho_N, \sigma_N) \coloneqq \max_{P_{k_\alpha}} \delta_{d,P_{k_\alpha}}(\rho_N, \sigma_N). \tag{4}$$

We further justify the definition. Since it is a (weighted) sum of distances with positive weights, the weighted distance inherits the first and fourth properties of the distance function in Eq. (1), which we listed in Eq. (2). The second property, the faithfulness, is satisfied because it is the maximal one among all the weighted sums in Eq. (3). The third property, contractivity, holds for local CPTP maps performed on a single subsystem. See the full proof in the Supplemental Material [26]. The weighted distance is invariant only under single-particle unitary maps, while the standard distance d is invariant under all unitaries. This property is crucial for comparing many-body configurations, capturing the fact that the states $|00\rangle$, $a|00\rangle + b|11\rangle$ are more different than $|00\rangle$, $a|00\rangle + b|10\rangle$. The weighted distance is bounded via the chain of inequalities

$$\frac{1}{N} d(\rho_N, \sigma_N) \le D_d(\rho_N, \sigma_N) \le Nd(\rho_N, \sigma_N) \le NM_d, \tag{5}$$

being maximal for "maximally different" preparations, such that both the global states and all their marginal states are orthogonal. Note that the importance of the largest measurement setup does not increase under trivial extensions of the system. For example, consider the N-partite states $|0\rangle^{\otimes N}$, $|x_1x_2, \ldots, x_N\rangle$. By adding a Q-particle register in $|0\rangle^{\otimes Q}$, the new states are $|0\rangle^{\otimes N+Q}$, $|x_1x_2, \ldots, x_N\rangle|0\rangle^{\otimes Q}$. One has $(N+Q)d(\rho_{N+Q}, \sigma_{N+Q}) \ge Nd(\rho_N, \sigma_N)$, while $D_d(\rho_{N+Q}, \sigma_{N+Q}) = D_d(\rho_N, \sigma_N)$, since an N-particle detection $\tilde{\mathcal{M}}^N$ is still maximally informative.

We test the usefulness of the notion of weighted distance. Adopting as standard distance the Bures length $B(\rho_N, \sigma_N) \coloneqq \cos^{-1} F(\rho_N, \sigma_N)$ [14,15,27], motivated by the considerations detailed via Eqs. (1)–(4), we define the weighted Bures length as

$$D_B(\rho_N, \sigma_N) \coloneqq \max_{P_{k_\alpha}} \delta_{B,P_{k_\alpha}}(\rho_N, \sigma_N). \tag{6}$$

We compare the two quantities via explicit calculations in some interesting case studies, see Table I. The results confirm that the weighted Bures length is more informative than the standard Bures length. For pure states, the latter is equal to the Fubini-Study distance [28]. Consequently, Eq. (6) defines a weighted Fubini-Study distance for pure states. In general, the full knowledge of the quantum states under study is required for exact calculations of both standard and weighted distances, but statistical methods for estimating standard distances from incomplete data are readily applicable, by construction, to weighted distance estimation [29–31].

The weighted Bures length lower bounds the experimental cost of quantum processes.—The weighted distances have a clear metrological meaning, being more sophisticated proxies than standard distances for state discrimination [33]. An important related question asks what the cost is of creating very different configurations in terms of physical resources, such as energy and time. Specifically, generating highly correlated states from $|0\rangle^{\otimes N}$, transforming an initial state in a very different output, is a requisite of all quantum algorithms. Establishing the physical limits to quantum programming, i.e., how small state preparation circuits can be, is therefore of great interest, as environmental noise quickly corrupts them [34]. The results in Table I highlight that, when calculated between an initial state $|0\rangle^{\otimes N}$ and highly correlated outputs, the weighted Bures length is monotonically increasing with the size of the system. We show that, indeed, the weighted Bures length between the initial and final states of a quantum process is the minimum experimental cost of the state transformation. We employ a geometric argument to rigorously prove the claim (Fig. 2).

A quantum dynamics from an N-qubit input state ρ_N to a final state σ_N is a path in the stratified Riemannian manifold of density matrices [8,35]. The state of the system at time t has spectral decomposition $\rho_{N,t} = \sum_{r=1}^{2^N} \lambda_r(t)|r(t)\rangle\langle r(t)|,\ t \in [0, T]$, with $\rho_{N,0} \equiv \rho_N$, $\rho_{N,T} \equiv \sigma_N$. Its rate of change is the time derivative $\dot{\rho}_{N,t}$. One builds a distance measure between two quantum states ρ_N and σ_N by calculating the minimum of the length functional $\int_0^T ||\dot{\rho}_{N,t}|| dt$ for some given norm. In particular, the input-output Bures length is the distance induced by the Fisher norm [36]

TABLE I. We calculate the standard Bures length and the weighted Bures length, as defined in Eq. (6), for N-qubit states (full details in the Supplemental Material [26]). Here $|\mathrm{GHZ}_k\rangle = (a|0\rangle^{\otimes k} + b|1\rangle^{\otimes k})$, $\mathrm{class}_k = (|a|^2|0\rangle\langle 0|^{\otimes k} + |b|^2|1\rangle\langle 1|^{\otimes k})$, and $|\mathrm{Dicke}_{N,k}\rangle = (1/\sqrt{\binom{N}{k}}\sum_i \mathcal{P}_i|0\rangle^{\otimes N-k}|1\rangle^{\otimes k}$ is the N-qubit Dicke state with k excitations [32], in which $\mathcal{P}_i$ are the possible permutations. The weighted Bures length is a better descriptor of the difference between multipartite quantum states. If two states become more different by increasing N, i.e., there are more measurement setups that discriminate between them, the quantity increases. If discriminating two states becomes harder, the weighted Bures length decreases.

ρ_N, σ_N	$B(\rho_N, \sigma_N)$	$D_B(\rho_N, \sigma_N)$
$\lvert 0\rangle^{\otimes N}$, $\lvert 1\rangle^{\otimes k}\lvert 0\rangle^{\otimes N-k}$	$\pi/2,\ \forall\, k$	$k(\pi/2)$
$\lvert 0\rangle^{\otimes N}$, $\lvert \mathrm{GHZ}_k\rangle \otimes \lvert 0\rangle^{\otimes N-k}$	$\cos^{-1}\lvert a\rvert$	$k\cos^{-1}\lvert a\rvert$
$\lvert 0\rangle^{\otimes N}$, $\lvert \mathrm{GHZ}_l\rangle^{\otimes k}\lvert 0\rangle^{\otimes N-kl}$	$\cos^{-1}\lvert a\rvert^k$	$kl\cos^{-1}\lvert a\rvert$
$\lvert 0\rangle\langle 0\rvert^{\otimes N}$, $\mathrm{class}_k \otimes \lvert 0\rangle\langle 0\rvert^{\otimes N-k}$	$\cos^{-1}\lvert a\rvert,\ \forall\, l$	$k\cos^{-1}\lvert a\rvert$
$\lvert 0\rangle\langle 0\rvert^{\otimes N}$, $\mathrm{class}_l^{\otimes k} \otimes \lvert 0\rangle\langle 0\rvert^{\otimes N-kl}$	$\cos^{-1}\lvert a\rvert^k,\ \forall\, l$	$kl\cos^{-1}\lvert a\rvert$
$\lvert 0\rangle^{\otimes N}$, $\lvert \mathrm{Dicke}_{N,k}\rangle$	$(\pi/2)\ \forall\, k$	$N\cos^{-1}(1-k/N)$
$\lvert 0\rangle\langle 0\rvert^{\otimes N}$, $I_k/2^k \otimes \lvert 0\rangle\langle 0\rvert^{\otimes N-k}$	$\cos^{-1}(1/\sqrt{2^k})$	$k\cos^{-1}(1/\sqrt{2})$
$\lvert \mathrm{GHZ}_N\rangle\langle \mathrm{GHZ}_N\rvert, I_N/2^N, \lvert a\rvert, \lvert b\rvert \neq 1/\sqrt{2}$	$\cos^{-1}[(\lvert a\rvert+\lvert b\rvert)/\sqrt{2^N}]$	$N\cos^{-1}[(\lvert a\rvert+\lvert b\rvert)/\sqrt{2}]$
$\mathrm{class}_N, I_N/2^N, \lvert a\rvert, \lvert b\rvert \neq 1/\sqrt{2}$	$\cos^{-1}[(\lvert a\rvert+\lvert b\rvert)/\sqrt{2^N}]$	$N\cos^{-1}[(\lvert a\rvert+\lvert b\rvert)/\sqrt{2}]$
$\lvert \mathrm{GHZ}_N\rangle\langle \mathrm{GHZ}_N\rvert, I_N/2^N, N\text{even}, \lvert a\rvert = \lvert b\rvert = 1/\sqrt{2}$	$\cos^{-1}(1/\sqrt{2^{N-1}})$	$N\pi/16$
$\mathrm{class}_N, I_N/2^N, N\text{even}, \lvert a\rvert = \lvert b\rvert = 1/\sqrt{2}$	$\cos^{-1}(1/\sqrt{2^{N-1}})$	$N\pi/16$
$\mathrm{class}_N, \lvert \mathrm{GHZ}_N\rangle\langle \mathrm{GHZ}_N\rvert$	$\cos^{-1}\sqrt{a^4+b^4}$	$\cos^{-1}\sqrt{a^4+b^4}/N$

$$B(\rho_N, \sigma_N) = \min_{\rho_{N,t}} \int_0^T ||\dot{\rho}_{N,t}||_{\mathcal{F}} dt,$$

$$||\dot{\rho}_{N,t}||^2_{\mathcal{F}} \coloneqq \sum_r \frac{\dot{\lambda}_r^2(t)}{4\lambda_r(t)} + \sum_{r<s} \frac{|\langle r(t)|\dot{\rho}_{N,t}|s(t)\rangle|^2}{\lambda_r(t)+\lambda_s(t)}. \quad (7)$$

The first term in Eq. (7) is the classical Fisher norm. The second one is a purely quantum contribution (related to the state eigenbasis evolution), being the only term surviving for unitary maps (the two terms coexist for generic CPTP operations). We evaluate the cost of eigenbasis changes, adopting the viewpoint that classical computations are free. The transformation can be split into two steps: the eigenvalue change and the eigenbasis change: $\rho_N \to \tau_N \to \sigma_N$, in which $\tau_N = \sum_{r=1}^{2N} \lambda_r(T)|r(0)\rangle\langle r(0)|$ [37]. The first step can be always completed via a classical process [38], while the second one can be implemented by a unitary path $\tau_{N,t}, \tau_{N,0} \equiv \tau_N, \tau_{N,T} \equiv \sigma_N$. For unitary processes, the first step is redundant, $\rho_N = \tau_N$. Hence, we quantify the "quantum cost" for implementing an arbitrary (even nonunitary) transformation $\rho_N \to \sigma_N$ as

$$B^q(\rho_N, \sigma_N) \coloneqq \min_{\text{unitary paths}\tau_{N,t}} \int_0^T ||\dot{\tau}_{N,t}||_{\mathcal{F}} dt. \quad (8)$$

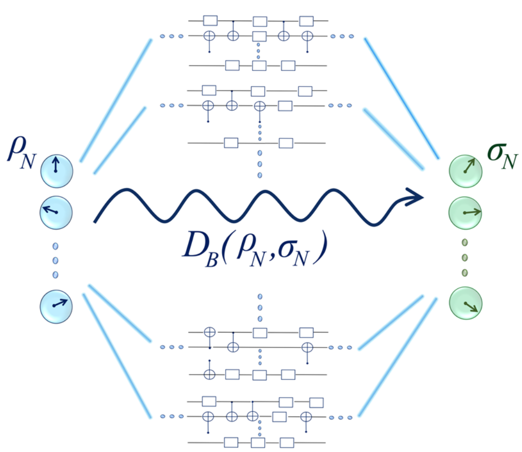

FIG. 2. We prove that the weighted Bures length $D_B(\rho_N, \sigma_N)$ is a lower bound to the experimental cost of the state transformation $\rho_N \to \sigma_N$. The bound is also valid for nonunitary quantum processes.

Suppose we carry out the second step via a sequence of quantum gates $U = \Pi_l U_l, U_l = e^{-iH_l T_l}$ (we run U_1, then U_2, and so on). The spectral decomposition of each time-independent Hamiltonian is $H_l = \sum_{x_l=1}^{2^{k_l}} h_{x_l}|h_{x_l}\rangle\langle h_{x_l}|, h_{x_{l>m}} \geq h_{x_m}, \forall\, l, m,$ and T_l is the runtime of each gate. Note that any Hamiltonian H_l affects $k_l \leq N$ particles. Call τ^l_{N,t_l} the intermediate state at time $t_l \in [0, T_l]$ while implementing U_l, with $\tau^l_{N,0} \equiv \tau^l_N, \tau^1_{N,0} \equiv \tau_N$. Since time-independent Hamiltonian dynamics are constant speed processes, one has

$$B^q(\rho_N, \sigma_N) \leq \int_0^{\sum_l T_l} ||\dot{\tau}_{N,t}||_{\mathcal{F}} dt$$
$$= \sum_l \int_0^{T_l} ||\dot{\tau}^l_{N,t_l}||_{\mathcal{F}} dt_l = \sum_l ||\dot{\tau}^l_N||_{\mathcal{F}} T_l. \quad (9)$$

The inequality can be saturated when σ_N (and therefore τ_N) is a pure state. The squared speed of the process lower bounds the variance of the generating Hamiltonian, which is also constant in time [39],

$$V_{\tau_N^l}(H_l) \coloneqq \mathrm{Tr}\{H_l^2 \tau_N^l\} - \mathrm{Tr}\{H_l \tau_N^l\}^2 \geq ||\dot{\tau}_N^l||_{\mathcal{F}}^2, \quad \forall\ l. \tag{10}$$

By employing the (halved) seminorm $E_l \coloneqq (h_{x_l=2^{k_l}} - h_{x_l=1})/2$ [40], we quantify the cost of the state transformation in terms of physical resources by

$$\mathcal{R}_{U_l} \coloneqq k_l E_l T_l \Rightarrow \mathcal{R}_U \coloneqq \sum_l \mathcal{R}_{U_l}. \tag{11}$$

The first term k_l represents the size of each quantum gate U_l. The second term quantifies the energy requirement for each gate. Note that $E_l^2 \geq V_{\rho_l}(H_l)$, $\forall\, l$. The third contribution is the allowed time interval for each gate. Factoring in the gate size is essential. A single-qubit Hamiltonian of spectrum $(x, -x)$ is easier to implement, in some given time T_l, than a $k > 1$-partite interaction generated by $(\underbrace{x, 0, \ldots, -x}_{2^k})$, even though the eigenvalue gap E_l is equal. By remembering Eq. (5) and exploiting the triangle inequality of the weighted distances, it follows that the experimental cost $\mathcal{R}_U$ of a state transformation $\rho_N \to \sigma_N$ is lower bounded by the weighted Bures length between initial and final states,

$$\begin{aligned} \mathcal{R}_{U_l} &\geq k_l B^q(\tau_N^l, \tau_N^{l+1}) \geq D_B(\tau_N^l, \tau_N^{l+1}), \quad \forall\ l \\ &\Rightarrow \text{for unitary processes}: \mathcal{R}_U \geq D_B(\rho_N, \sigma_N), \\ &\quad \text{for general quantum processes}: \mathcal{R}_U \geq D_B(\tau_N, \sigma_N). \end{aligned} \tag{12}$$

The bounds are formally similar to energy-time uncertainty relations and quantum speed limits [17–22], yet they can be more informative, as they provide a more nuanced resource count for quantum processes. For example, they determine the minimum time to complete state transformations at fixed energy and gate size. Note that the right-hand side is zero if and only if $[\rho_N, \sigma_N] = 0$. That is, if and only if there exists a classical dynamics that transforms the input into the output state [38]. The left-hand inequality in Eq. (12) is saturated when the intermediate states τ_N^l are the most sensitive ones to the unitary perturbations U_l; i.e., they are coherent superpositions $(|h_{2^{k_l}}\rangle + e^{i\phi}|h_{x_l=1}\rangle)/\sqrt{2}, \phi \in [0, 2\pi]$. The result in Eq. (12) advances our understanding of many-body quantum processes in three ways. First, it provides a lower limit to the difficulty to run quantum computations in terms of an exact, analytical bound, rather than an order of magnitude [41–43]. Second, it applies to mixed states and nonunitary processes, beyond the idealized scenario of perfectly controllable quantum dynamics. Third, the right-hand side of the bound, the weighted Bures distance, is not just a numerical value, but it has a physical meaning. Specifically, the bound highlights that our ability to manipulate quantum states, e.g., generating entangled configurations from the input state $|0\rangle^{\otimes N}$, is never greater than the instrumental experimental cost.

Conclusion.—We have introduced the weighted distances [Eq. (4)], a new class of information measures. They capture the difficulty in distinguishing many-body quantum states. Moreover, we uncovered a fundamental bound to quantum information processing [Eq. (12)]. The size of state preparation algorithms is never smaller than the weighted Bures length between the input and the output states, i.e., our ability to discriminate between the two states. We anticipate that the weighted distances will help evaluate the theoretical and experimental performance of quantum technologies [44] and explore critical properties of open quantum systems [45].

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[26] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.170502 for full details.

[27] Note that the Bures length is different from the Bures distance, which is defined as $2 - 2F(\rho_N, \sigma_N)$.

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Correction: The description of properties given in the paragraph after Eq. (4) contained minor errors and required a change to the second sentence and an insertion of what now is the fourth sentence.