Bosonic Continuum Theory of One-Dimensional Lattice Anyons

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Anyons with arbitrary exchange phases exist on 1D lattices in ultracold gases. Yet, known continuum theories in 1D do not match. We derive the continuum limit of 1D lattice anyons via interacting bosons. The theory maintains the exchange phase periodicity fully analogous to 2D anyons. This provides a mapping between experiments, lattice anyons, and continuum theories, including Kundu anyons with a natural regularization as a special case. We numerically estimate the Luttinger parameter as a function of the exchange angle to characterize long-range signatures of the theory and predict different velocities for left- and right-moving collective excitations.

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Acquiring "any phase" when spatially exchanged [1,2], anyons break the dichotomic classification of quantum particles into bosons and fermions. Anyons have inspired theoretical and experimental physicists for decades [1–11]. Recently, scientific and technological interest increased again because of possible applications of non-Abelian anyons in topological quantum computing [12,13] and the apparent detection of Majorana zero modes [14–17]. Even though anyons are often considered to be exclusively two dimensional, they have also been discussed in a wide range of one-dimensional (1D) systems that differ strongly in phenomenology [18–27].

In 2D, Abelian anyons can exist because the possible continuous exchange of two indistinguishable particles provides topological nontriviality at a fundamental level. The wave function of two anyons acquires a phase factor $e^{i\theta}$, where the angle θ specifies the commutation up to a 2π periodicity. In 1D, the situation is more involved because particles cannot directly be exchanged without collision. This is only circumvented by networklike geometries [28], additional degrees of freedom [29], or measurement-based protocols [30]. Historically, several notions of one-dimensional anyons have emerged that are *a priori* disconnected to 2D anyons. As a fundamental approach, Leinaas and Myrheim have classified indistinguishable one-dimensional particles by generalized boundary conditions in terms of a statistical parameter $-\infty < \eta \leq \infty$ [2]. In the mathematically equivalent Lieb-Liniger model of interacting bosons [31,32], the statistical parameter is given by the bosonic on-site interaction, and hence differs strongly from a 2π-periodic exchange phase $e^{i\theta}$. Another approach to anyonic exchange in 1D is to consider a discontinuous jump in the wave function when two particles pass each other. This insight was used by Kundu, who proposed an integrable bosonic model on a continuum 1D system containing derivatives and squares of delta functions [33] as a model for 1D anyons. However, the bosonic form of the Kundu model is not 2π periodic in the statistical parameter either.

In contrast to the aforementioned continuum theories, a 2π-periodic model of anyons is constructed on a 1D lattice by anyonic creation and annihilation operators with the generalized commutation relation which ensures

$$a_i a_j^\dagger - e^{-i\theta \mathrm{sgn}(i-j)} a_j^\dagger a_i = \delta_{i,j}, \qquad (1)$$

that the particles acquire an anyonic phase factor $e^{i\theta}$ under exchange analogous to Abelian 2D anyons. Here, $\mathrm{sgn}(j) = j/|j|$ and $\mathrm{sgn}(0) = 0$. These lattice anyons of the so-called anyon Hubbard model attracted significant attention due to several proposals to realize it in optical lattices filled with ultracold bosons or fermions [34–42] and promising developments of artificial gauge fields and induced phase transitions by time-periodic forcing [43–55]. In particular, the anyonic exchange phase can be realized by an occupation-dependent Peierls-like phase using assisted Raman tunneling [34,35] or periodically modulated lattices [36–38]. While these lattice anyons faithfully recover the exchange phase from their two-dimensional continuous counterparts, this discrete construction sidesteps the most interesting aspects of anyon physics: The continuous exchange of two particles. To our knowledge, the relation between 1D Leinaas-Myrheim particles, the Kundu model, and the experimentally accessible lattice anyons has up to now been an open problem. In particular, no corresponding bosonic continuum Hamiltonian exists that is 2π periodic in the anyonic exchange angle.

In this Letter, we provide the continuum theory of the anyon Hubbard model. We thereby obtain an explicit mapping between experiments on one-dimensional bosonic lattices, lattice theories of anyons, and general theories in the continuum, including Kundu anyons as a special case. Deriving a naive long wavelength limit is insufficient. Instead, it is necessary to use the bosonic form of the Hamiltonian and to consider all orders of the phase angle. This leads to a statistically induced current density as well as two- and three-particle interactions, but as a reward results in the 2π periodicity in the anyonic phase angle even in 1D. We emphasize that the bosonic form of the Hamiltonian is experimentally directly accessible, in stark contrast to the purely anyonic description, which encodes the topological character of the exchange implicitly in the creation algebra or the boundary conditions of the wave functions. Additionally, the bosonic form facilitates theoretical calculations, since the anyonic exchange algebra is not preserved under unitary transformations. We furthermore provide numerical results using the density matrix renormalization group (DMRG) algorithm to illustrate how the continuum limit is approached and discuss the implications for the effective low-energy Tomonaga-Luttinger liquid theory. Non-Abelian extensions or unitary braided fusion categories and corresponding diagrammatic equations of anyon theories [12,56] are interesting prospects for future research and not discussed here.

Our starting point is the anyon Hubbard model on a 1D lattice with L sites [34]

$$H = -J\sum_{j=1}^{L-1}(a_j^\dagger a_{j+1} + \text{H.c.}) + \frac{U}{2}\sum_{j=1}^{L} n_j(n_j-1), \quad (2)$$

where the anyonic operators obey the algebra in Eq. (1) and $n_j = a_j^\dagger a_j$ is the particle number operator. Bosons are described by this model at $\theta = 0$. For $\theta = \pi$ the on-site quantum brackets Eq. (1) remain bosonic, so that so-called pseudofermions are originally included. Yet, as we show below, the low-density continuum limit captures the behavior of ordinary fermions as well. The anyon Hubbard model breaks spatial inversion symmetry and time reversal symmetry but obeys a generalized inversion symmetry [57], that is reduced to the combined action of time reversal and spatial inversion in the below continuum theories; for details, see the Supplemental Material [58]. By the Jordan-Wigner transformation [34]

$$a_j = e^{i\theta\sum_{k<j} n_k} b_j, \quad (3)$$

the relations in Eq. (1) can be exactly represented by bosonic operators b_j using the Hamiltonian

$$H = -J\sum_{j=1}^{L-1}(b_j^\dagger e^{i\theta n_j} b_{j+1} + \text{H.c.}) + \frac{U}{2}\sum_{j=1}^{L} n_j(n_j-1). \quad (4)$$

Here, the hopping depends on the occupation number in form of a Peierls-like factor $e^{i\theta n_j}$, which plays a central role for experimental realizations [34–38,55]. Interestingly, the Hubbard interaction $n_j(n_j-1)$ is independent of the anyonic phase because $n_j = a_j^\dagger a_j = b_j^\dagger b_j$. In the continuum limit, it leads to a simple two-body interaction term [59,60], which can be added to the effective interaction terms arising from the anyonic exchange derived in the following.

The continuum limit is defined as a process of a systematic renormalization of all terms when reducing the lattice spacing d with increasing number of sites L such that the physical length $l = Ld$ remains finite. Following the procedure of Ref. [59], the bosonic field operator in the continuum is

$$\Psi_B^\dagger(x) = \lim_{d\to 0} b_j^\dagger/\sqrt{d}, \quad (5)$$

which results in the bosonic commutator

$$[\Psi_B(\tilde{x}), \Psi_B^\dagger(x)] = \lim_{d\to 0}\frac{\delta_{i,j}}{d} \equiv \delta(x-\tilde{x}). \quad (6)$$

Because of the delta function, it is important that all expressions are normal ordered before taking the continuum limit in order to avoid divergences. Furthermore, we expand the bosonic operator as

$$\Psi_B(x+d) \approx \Psi_B(x) + d\partial_x\Psi_B(x) + \frac{d^2}{2}\partial_x^2\Psi_B(x), \quad (7)$$

since higher orders renormalize to zero in the continuum limit. In order to keep the full dependence on the anyonic phase angle θ, it is crucial to express the Peierls-like factor in normal ordered form, which is possible to all orders according to [61]

$$e^{i\theta n_j} = \sum_{q=0}^{\infty}\frac{(i\theta)^q}{q!}n_j^q = \sum_{q=0}^{\infty}\frac{(i\theta)^q}{q!}\sum_{m=0}^{q} S(q,m)(b_j^\dagger)^m(b_j)^m, \quad (8)$$

where $S(q,m)$ are the Stirling numbers of second kind [62]. Taking the continuum limit according to Eqs. (5) and (7), we observe that each operator b_i carries a factor of $\sqrt{d}$ and each derivative a factor of d, which yields an overall scale $Jd^2 = \hbar^2/2m_{\text{eff}}$ that is set to unity in the following. By neglecting higher powers in d and resumming the Peierls-like factor in Eq. (8), we finally obtain the Hamiltonian in the continuum limit

$$\mathcal{H}_{\text{cont}} = \int dx\,[\mathcal{H}_{\text{kin}}(x) + \mathcal{H}_{\text{int}}(x) + \mathcal{H}_J(x)], \quad (9)$$

with

$$\mathcal{H}_{\rm kin}(x) = \partial_x \Psi_B^\dagger(x) \partial_x \Psi_B(x),$$
$$\mathcal{H}_{\rm int}(x) = [V_2(\theta) + c] \mathpunct{:}\rho_B^2(x)\mathpunct{:} + V_3(\theta)\mathpunct{:}\rho_B^3(x)\mathpunct{:},$$
$$\mathcal{H}_J(x) = V_J(\theta)\mathpunct{:}\rho_B(x) J_B(x)\mathpunct{:}, \tag{10}$$

where $J_B(x) = -i[\Psi_B^\dagger(x)\partial_x\Psi_B(x) - \text{H.c.}]$ is the current operator, $\rho_B(x) = \Psi_B^\dagger(x)\Psi_B(x)$ the density operator, and $: \cdots :$ denotes normal order. The parameter c is related to the Hubbard interaction $U = 2Jdc$, and we have furthermore dropped the energy dependence on the total density $\rho_0 = N/Ld$, since the particle number N is a conserved quantity. The coupling constants of the theory are

$$V_J(\theta) = \text{Im}[\tilde{V}_2(\theta)] = -\sin(\theta), \tag{11}$$

$$V_2(\theta) = -2d^{-1}\text{Re}[\tilde{V}_2(\theta)] = 2d^{-1}[1 - \cos(\theta)], \tag{12}$$

$$V_3(\theta) = -\text{Re}[\tilde{V}_2(\theta)^2] = 1 + 2\cos(\theta) - \cos(2\theta), \tag{13}$$

with $\tilde{V}_2(\theta) = 1 - e^{i\theta}$. The two-body interaction $V_2(\theta)$ and the rescaled Hubbard interaction $c = U/(2Jd)$ strongly renormalize when $d \to 0$ if $\theta \neq 0$, which is known to occur also for the ordinary Fermi-Hubbard and Bose-Hubbard interaction U in the continuum limit [59,63]. In the Hubbard model this can be resolved by rescaling U with the overall density, leading to a renormalized theory which recovers the so-called Tonks-Girardeau limit with infinitely strong interactions for low densities [64,65]. However, in the anyonic theory, the angle θ cannot simply be rescaled or renormalized without changing the anyonic phase angle, which would violate the topological character. As we see below, and justified by our numerical results, the dependence on the lattice spacing d allows us to define regularized coupling constants, which are fixed by experimental parameters. Moreover, we recognize a current-density interaction potential V_J and a three-body interaction V_3, so that the full theory cannot be derived by only considering two-particle scattering processes.

We furthermore observe that the resulting bosonic model in Eq. (9) has the same structure as the integrable Kundu model [33] albeit with different coupling constants. The Kundu model is described by

$$V_J^{\rm Kundu}(\theta) = -\theta, \tag{14}$$

$$V_2^{\rm Kundu}(\theta) = \theta^2 \int dx\, \delta^2(x) + c, \tag{15}$$

$$V_3^{\rm Kundu}(\theta) = \theta^2. \tag{16}$$

Up to second order in θ, the parameters in Eqs. (14)–(16) recover the anyonic Kundu model [33]

$$\mathcal{H}_K = \int_{-\infty}^{\infty} dx\, \partial_x \Psi_A^\dagger(x) \partial_x \Psi_A(x) + \int_{-\infty}^{\infty} dx\, c\, \Psi_A^\dagger(x)\Psi_A^\dagger(x)\Psi_A(x)\Psi_A(x). \tag{17}$$

Here, the anyonic operators Ψ_A are related to the bosonic ones Ψ_B by the continuum version of the Jordan-Wigner transformation [33]

$$\Psi_A(x) = e^{i\theta \int_{-\infty}^{x} n(y) dy} \Psi_B(x), \tag{18}$$

where $n = \Psi_B^\dagger \Psi_B$. This relation naively looks like a straightforward continuum limit of Eq. (3), but such an approximation does not capture the full topological character since the crucial symmetry $\theta \to \theta + 2\pi$ is lost.

By comparing the coupling constants, we see that the Kundu model $\mathcal{H}_K$ corresponds to the special case of small anyon angles θ in the continuum model $\mathcal{H}_{\rm cont}$ in Eq. (9). Moreover, the singular behavior of a double delta function is replaced by the $1/d$ divergence of V_2 in Eq. (12). Therefore, our derived continuum model is more general and introduces a well-defined limiting procedure how the controversially discussed [66,67] double delta function in the original Kundu model must be interpreted. Namely, we find that one has to extrapolate the experimental results when lowering the lattice spacing d toward zero while keeping Ld finite. Obviously, changing d directly is difficult in an optical lattice, but it is possible to reach this limit by noting that the density $\rho_0 = N/Ld$ remains finite in the continuum limit. For a given density ρ_0, the continuum limit can then be effectively achieved by extrapolating $d = N/L\rho_0 \to 0$ by systematically lowering the number of particles per site N/L.

We illustrate this procedure by means of a numerical experiment using the DMRG algorithm [68], which is a powerful tool to analyze the properties of the proposed continuum theory in Eq. (9). We simulate noninteracting anyons by bosons with an occupation-dependent hopping in the continuum limit of Eq. (4) with $U = 0$ using up to 500 DMRG states in finite systems with fixed boundary conditions at the edges. For the values of θ and the relatively low densities used for the simulations below, we find that the numerical restriction to at most two bosons per site gives good results. It is well established that the local density can be considered as a convenient observable to analyze the interaction strength in 1D [36,69–73], since characteristic density oscillations develop near the edges due to collective modes, which ultimately are related to Friedel oscillations in the fermion limit [74]. An interacting bosonic gas gradually develops density oscillations near edges [60] that grow with increasing interactions and with decreasing densities analogous to the Fermi-Hubbard model [73].

In Fig. 1(a), we see that the corresponding density oscillations in a noninteracting anyon gas grow with θ,

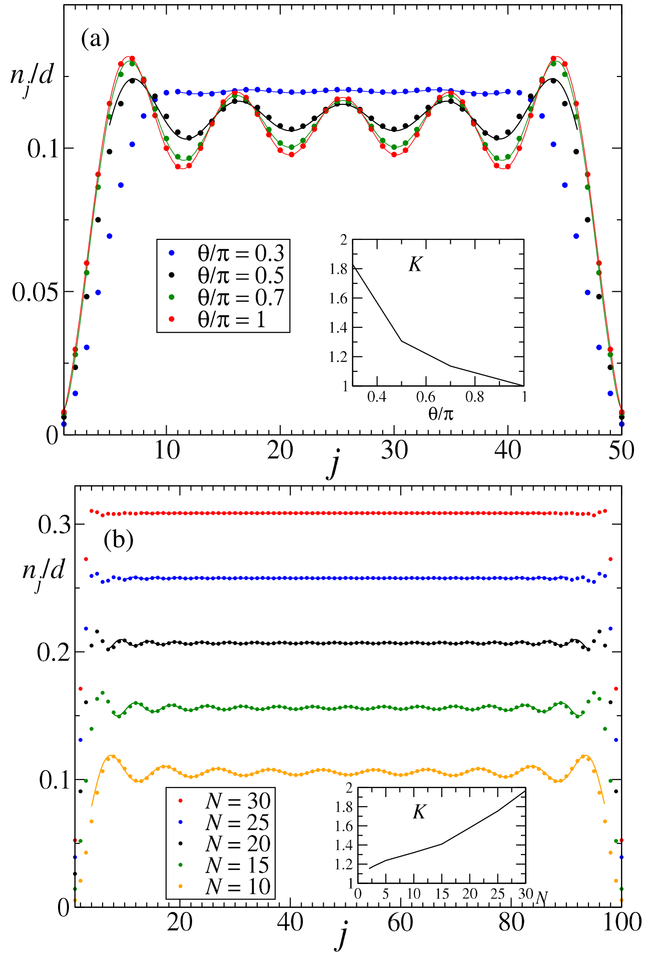

FIG. 1. DMRG results for the local particle density n_j/d. (a) For a chain of $L = 50$ sites with $N = 5$ anyons with different θ. (b) For $\theta = 0.5\pi$ and a chain length $L = 100$ the Friedel oscillations increase with decreasing particle number N. Solid lines are fits to power-law-decaying oscillations in Eq. (20), from which the Luttinger parameters K are extracted (insets).

which plays the role of an effective interaction as expected. Using $N = 5$ particles and $L = 50$ sites, the density waves gradually build up with increasing values of θ. Keeping $\theta = 0.5\pi$ fixed in Fig. 1(b), we observe how the characteristic oscillations steadily increase with lower densities. We see that choosing a small but finite density N/L creates a natural cutoff similar to choosing a finite d in the bosonization procedure to renormalize diverging terms in impurity problems [75].

In order to understand the oscillations on a more quantitative level, we derive the corresponding Tomonaga-Luttinger liquid (TLL) theory [76] describing the low-energy excitation of interacting bosons. Using the phase-density representation of the bosonic fields from the harmonic fluid approach [77,78] for the interactions of Eq. (10), we immediately arrive at a TLL Hamiltonian together with a special parity-breaking interaction Δ [58]

$$\mathcal{H}_{\mathrm{cont}} \approx \frac{u}{2\pi} \int dx \left[\frac{1}{K} (\partial_x \phi)^2 + K (\partial_x \Theta)^2 + \Delta \partial_x \phi \partial_x \Theta \right], \tag{19}$$

where energy shifts due to conserved quantities (total current and density) have again been omitted [58]. The dependence of the Luttinger parameter K, the Fermi-velocity u, and the interaction Δ on the parameters c, V_2, V_3, and V_J can be read off to lowest order $\pi^2/K^2 = (V_2 + c)/\rho_0 + 3V_3$, $u = 2\pi\rho_0/K$, and $\Delta = 8\rho_0 V_J/u$. But note that the corresponding expressions from the harmonic fluid approach are known to be only reliable for very small coupling constants [78,79], i.e., for small θ and c. Therefore, other methods are required to obtain a quantitative estimate of K for all θ as discussed below. The nontrivial interaction Δ has interesting physical consequences. Under spatial parity inversion, the fields transform $\partial_x \phi \to \partial_x \phi$ and $\partial_x \Theta \to -\partial_x \Theta$ [80], so this interaction can be traced back to the parity-breaking nature of the anyonic description. However, the Δ term is not affected by the Luttinger rescaling of the fields $\phi \to \sqrt{K}\phi$ and $\Theta \to \Theta/\sqrt{K}$ and becomes diagonal using the known mode expansions in terms of collective density waves [78] as shown in the Supplemental Material [58]. So no additional transformation is required. Therefore, the vacuum and the excitations of a TLL are preserved, but the interaction Δ leads to different velocities and energies of left- and right-moving density excitations $\epsilon_q = [u + \Delta \mathrm{sgn}(q)/2]|q|$, which becomes relevant for time-dependent correlations [58].

In order to provide a quantitative estimate of the Luttinger parameter K for all θ, the numerical data in Fig. 1 prove useful. It is known that the long-range decay of the oscillations away from the edges is governed by a power law [69,70], where the exponent, for spinless models, is the Luttinger parameter K [71]. In particular, the oscillations in the local density of a finite-size TLL follow the analytic expression [73,78,81]

$$n_j/d \approx \rho + A \cos\left[2\rho\pi(jd - \ell/2)\right] \left(\ell \sin \frac{\pi j d}{\ell}\right)^{-K}, \tag{20}$$

where ρ is the average density near the middle of the chain $\ell/2 \equiv (L+1)d/2$. For the ground state expectation value of n_j the parity-breaking interaction Δ does not contribute [58]. As shown in Fig. 1, Eq. (20) describes the local density very well for all $\theta > 0$ and results in a nontrivial exponent K which approaches unity for small densities and $\theta \to \pi$. The fits are spatially limited by a cutoff distance from the edges, which indicates the range of validity of the TLL theory. Notably, the cutoff distance increases with the Luttinger parameter K. It hence becomes increasingly difficult to describe weaker oscillations, but the data are still consistent with the expected behavior of free bosons $K \to \infty$ for $\theta \to 0$, see inset of Fig. 1(a). In the opposite limit of $\theta = \pi$, the solid line in Fig. 1(a) is given by the analytic expression for free fermions [73] over the entire system, which means that the pseudofermions are well described by ordinary fermions with $K = 1$, in this case [35,36]. The TLL parameter in the insets of Fig. 1 determines relevant long-range correlations [76] including energy-dependent quantities like the local density of states [82]. The numerical data therefore not only confirm the stability of a TLL ground state but also predict the dominant correlations of the continuum anyon model quantitatively as a function of θ and ρ_0. In experimental setups, the confinement is commonly given by a harmonic trap, which also gives rise to density oscillations [60] described by corresponding fit functions [83]. Thus, experimental measurements of the local densities [84] in optical traps can be used to extract these characteristic exponents.

In conclusion, capturing the essential feature of a 2D anyonic exchange phase by 1D anyons on a lattice inspired our development of a continuum theory of 1D anyons in terms of interacting bosons, which keeps the topological character of Abelian anyonic exchange in contrast to previously discussed Hamiltonians in 1D. The representation of anyons as normal ordered bosons with modified hopping is crucial when taking the continuum limit. We take all orders of the anyonic phase into account, resulting in a Hamiltonian that includes current density as well as two- and three-particle bosonic interactions with 2π-periodic coefficients in the anyonic phase angle. The description of 1D continuum anyons, that captures this topological hallmark, solves an open problem and the theory extends the two-dimensional concept of a 2π-periodic anyonic exchange phase to 1D. The known Kundu model is contained in the limit of a vanishing statistical angle. Our work therefore uncovers a unifying, physically motivated continuum theory of one-dimensional Abelian anyons that derives from the original idea of an anyonic exchange phase in 2D. We thereby connect recent experiments on ultracold atomic gases to the seminal considerations of Leinaas and Myrheim as well as Wilczek, which inspired the name "anyon" [1]. In physical systems, we quantitatively predict the appearance of characteristic density oscillations, which reflect the TLL correlations. Our studies also show that a hallmark of anyonic physics are different velocities of left- and right-moving density excitations, which could be observable in dynamic experiments on the structure factor or time-dependent correlators [58].

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