Breakdown of Tan's Relation in Lossy One-Dimensional Bose Gases

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In quantum gases with contact repulsion, the distribution of momenta of the atoms typically decays as $\sim 1/|p|^4$ at large momentum p. Tan's relation connects the amplitude of that $1/|p|^4$ tail to the adiabatic derivative of the energy with respect to the coupling constant or scattering length of the gas. Here it is shown that the relation breaks down in the one-dimensional Bose gas with contact repulsion, for a peculiar class of stationary states. These states exist thanks to the infinite number of conserved quantities in the system, and they are characterized by a rapidity distribution that itself decreases as $1/|p|^4$. In the momentum distribution, that rapidity tail adds to the usual Tan contact term. Remarkably, atom losses, which are ubiquitous in experiments, do produce such peculiar states. The development of the tail of the rapidity distribution originates from the ghost singularity of the wave function immediately after each loss event. This phenomenon is discussed for arbitrary interaction strengths, and it is supported by exact calculations in the two asymptotic regimes of infinite and weak repulsion.

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Introduction.—In a quantum gas, contact interactions can impart large momenta to the particles: the singularity of the many-body wave function when two particles are at the same position is reflected in the tails of their momentum distribution $w(p)$, which decay as $w(p) \sim 1/|p|^4$. It contrasts with the Gaussian decay that would be expected from the Boltzmann distribution in an ideal gas. The $1/|p|^4$ tails were noticed in hard-core one-dimensional (1D) bosons by Minguzzi *et al.* [1] (see also Ref. [2]), then studied in 1D gases of arbitrary interaction strength by Olshanii and Dunjko [3], and by Tan in three-dimensional (3D) fermionic gases [4–6]. (For a general analysis in two and three dimensions for bosons, fermions, and mixtures see Refs. [7,8].) Remarkably, the amplitude of the tail, $C \coloneqq \lim_{p\to\infty} |p|^4 w(p)$, is a thermodynamic quantity [3,5]. Tan's "adiabatic sweep theorem" [5], or simply "Tan's relation," connects the amplitude C to the adiabatic derivative [9] of the energy with respect to the two-body interaction parameter. For Bose gases, Tan's relation reads [8]

$$C = C_c, \quad \text{with} \quad C_c \coloneqq \frac{m^2}{(2\pi\hbar)^d} 2g^2 \frac{\partial(E/V)}{\partial g}. \tag{1}$$

Here m is the particles' mass, E is the energy of the gas, V is its volume, and g is the interaction coupling constant [10]. The momentum distribution is normalized as $\int d^d p w(p) = N/V$, where N is the total number of atoms and d is the dimension of the system. The contact density C_c is defined by the second equality of Eq. (1), for any density matrix diagonal in the eigenbasis. Tan's relation $C = C_c$ has been proved with wide generality and applies to many states of the gas [11,12].

Tails in the momentum distribution have been observed experimentally in 3D fermionic gases and Tan's relation has been verified [13,14]. It has also been verified, using spectroscopy, in 3D Bose gases [15]. On the theory side, Tan's relation and its extensions have been thoroughly investigated [7,8,11,12,16–18]. Recent works have focused on the 1D Bose gas [19–22], exploiting the relation between the contact density and the zero-distance two-body correlation function [Eq. (3)].

Tan's relation (1) is based on the assumption that the tails of the momentum distribution are due entirely to the contact two-body interaction. In this Letter, we point out that this assumption is not always valid. We show that, owing to its integrability, the 1D Bose gas with contact interactions can have a contribution to its $1/|p|^4$ tail of different origin, so that $C > C_c$. This happens in a peculiar class of stationary states, which we characterize.

Importantly, such peculiar stationary states are generated by atom losses. That makes them ubiquitous in modern cold atoms experiments in 1D [23,24], which always suffer from losses [25–27]. We stress that those states are stationary with respect to Hamiltonian dynamics, so even if losses are no longer present at long times, the breakdown of Tan's relation persists. Therefore, an important implication of our findings is that Tan's relation will most probably be violated experimentally in 1D Bose gases.

The essence of the breakdown of Tan's relation for a gas submitted to losses is as follows. Immediately after a loss event, the wave function has a singularity at the position of the lost atoms, in addition to the singularities when two of the remaining particles meet. In the momentum distribution, this additional singularity is reflected as a $1/p^4$ term that adds to the usual contact term. If the gas were chaotic, then it would relax to a new thermal equilibrium state. The effect on the momentum distribution would therefore be observable only at a short time after the loss, since thermal states belong to the class of states that fulfill Tan's relation. However, the 1D Bose gas is not chaotic and the effect remains present even after relaxation to a stationary state.

The results presented in this Letter are twofold. First, we characterize the class of states for which Tan's relation is violated, and we provide a formula that supersedes it [Eq. (4)]. Second, we demonstrate that losses bring the gas to such a state. Our results on losses are supported by exact analyses in the hard-core and quasicondensate regimes, for which we can exploit recent results of Refs. [28–30]. In both regimes, we find that the amplitude of the tail of the momentum distribution C becomes substantially larger than the value C_r predicted by Tan's relation.

The contact in the 1D Bose gas.—We consider bosons with contact repulsion in a periodic system of size L. The Hamiltonian is, with $[\Psi(z), \Psi^+(z')] = \delta(z - z')$,

$$H = \int_0^L dz \Psi^+(z)\left(-\frac{\hbar^2\partial_z^2}{2m} + \frac{g}{2}\Psi^+(z)\Psi(z)\right)\Psi(z). \qquad (2)$$

We start by recalling the effects of the contact interaction on the tails of the momentum distribution, following Ref. [3]. Because of the contact interaction, the many-body wave function $\psi(z_1, \ldots, z_N) = \langle 0|\Psi(z_1)\ldots\Psi(z_N)|\psi\rangle$ has a cusp singularity whenever two positions coincide [31]: $\partial_{z_i}\psi|_{z_i\to z_j^-} - \partial_{z_i}\psi|_{z_i\to z_j^+} = (mg/\hbar^2)\psi(\ldots, z_i = z_j, \ldots)$. When one takes the Fourier transform, those cusps become $1/p^2$ tails, which give a $\sim 1/p^4$ contribution to the momentum distribution after taking the squared modulus of the wave function. When this calculation is done carefully (as in Ref. [3]), it shows that the contact interaction contributes to the tail of the momentum distribution $w(p)$ as C_c/p^4 with

$$C_c = \frac{m^2}{2\pi\hbar} g^2 n^2 g^{(2)}(0). \qquad (3)$$

Here $n = N/L$ is the atom density and $g^{(j)}(0) = \langle\Psi(z)^{+j}\Psi(z)^j\rangle/n^j$, where $j \in \mathbb{N}$, is the normalized zero-distance j-body correlation function, independent of z in a translation-invariant system. Equation (3) is an alternative, more general, definition of the contact density C_c in 1D, which works for all states including nonstationary ones. For stationary states (diagonal density matrices), it is equivalent to the one in Eq. (1). Indeed, if $|\psi\rangle$ is an eigenstate, a straightforward application of the Hellmann-Feynman theorem leads to $n^2 g^{(2)}(0) = 2\langle\psi|\partial H/\partial g|\psi\rangle/L = 2\partial(E/L)/\partial g$.

We now argue that there exist peculiar states, not considered in Ref. [3], where the equality $C = C_c$ breaks down.

The rapidity distribution, its tails, and tails of the momentum distribution.—Because of the extensive number of its conserved quantities, the 1D Bose gas typically relaxes to a generalized Gibbs ensemble (see, e.g., [32]) that is parametrized by its rapidity distribution [33–35]. The rapidities are conserved by the Hamiltonian dynamics: they characterize the eigenstates of the Hamiltonian (2), which take the form of Bethe states [36,37]. The rapidities are the asymptotic momenta of the atoms if one lets the gas expand freely in 1D [38–42]. They are conveniently thought of as the momenta of quasiparticles with infinite lifetime [43,44], dubbed "Bethe quasiparticles" in this Letter. After relaxation to a generalized Gibbs ensemble, expectation values of local observables are functionals of the rapidity distribution $\rho(q)$ [33–35]. In the following, we normalize the rapidity distribution as $\int dq\rho(q) = N/L$.

We stress that the rapidity distribution is not equal to the momentum distribution of the atoms. This is well illustrated by the ground state of the system: its rapidity distribution $\rho(k)$ vanishes outside a finite interval [36,37], while its momentum distribution $w(p)$ presents the aforementioned $1/p^4$ tails that extend to infinity [3].

Nevertheless, for large rapidities the momentum distribution may reflect features of the rapidity distribution and vice versa. To be more precise, let us imagine that the rapidity distribution of the gas has tails decaying as $1/q^4$ (we will argue below that atom losses naturally produce such tails), and let $C_r \coloneqq \lim_{q\to\infty} q^4\rho(q)$ be their amplitude. Then we argue below that

$$C \coloneqq \lim_{p\to\infty} p^4 w(p) = C_c + C_r. \qquad (4)$$

This formula, which generalizes Eq. (1), is our first main result. In states in which $C_r = 0$, which include single eigenstates of H in finite size, thermal states, and states produced by merging two thermal clouds with different temperatures [45], Tan's relation (1) is recovered. On the other hand, a nonvanishing C_r results in its breakdown. We note that Eq. (4) can also be applied to nonstationary ones [46] if one uses Eq. (3) to define C_c.

Derivation of Eq. (4).—We develop separate arguments for the hard-core regime $g \to \infty$ and for finite g. When $g \to \infty$, exact formulas are available [47–49] for the correlation function $g^{(1)}(z) = \langle\Psi^+(z)\Psi(0)\rangle/n$, which allow us to infer its short-distance behavior. For a rapidity distribution $\rho(q)$ with a C_r/q^4 tail, we find [49]

$$g^{(1)}(z) \underset{z\to 0}{=} 1 - i\frac{q_1}{n}z - \frac{q_2}{n}z^2 + i\frac{q_3}{n}z^3 + \frac{\pi(C_r + C_c)}{6\hbar^3 n}|z|^3 + O(z^4), \qquad (5)$$

where $q_j = (1/\hbar^j j!)\int q^j\rho(q)dq$. We arrive at this result by studying a lattice regularization of the Bose gas, for which we use an exact finite-distance formula for the two-point correlation function, and then by taking the continuum limit [49]. Equation (5) generalizes known formulas for the short-z expansion of $g^{(1)}(z)$ in the $g \to \infty$ limit [3,48,50] to the case of arbitrary rapidity distributions, including those with a C_r/q^4 tail. We then use the fact that the Fourier transform of a cusp singularity in $|z|^j$ has tails decaying as $1/p^{j+1}$. Evaluating that Fourier transform, we obtain $w(p) = (n/2\pi\hbar)\int e^{ipz/\hbar} g^{(1)}(z) dz \underset{|p|\to\infty}{\simeq} (C_\mathrm{r} + C_\mathrm{p})/p^4$. Thus we arrive at Eq. (4).

For finite g and arbitrary rapidity distributions, a direct computation of the momentum distribution or of its Fourier transform $g^{(1)}(z)$ is much more difficult, even numerically (see, e.g., Refs. [41,51]). Instead, we turn to a different argument, which formalizes the physical intuition that Bethe quasiparticles with large rapidities q must correspond to atoms with large momenta $p \simeq q$. We give a brief sketch of the argument here, in order to convey the main physical idea. Details are deferred to the Supplemental Material [49].

Let us introduce a cutoff Λ, large enough so that $\rho(q) \simeq C_\mathrm{r}/q^4$ as soon as $q > \Lambda$. We split the rapidity distribution into two terms, $\rho_{<\Lambda}(q) = \theta(\Lambda^2 - q^2)\rho(k)$ and $\rho_{>\Lambda}(q) = \theta(q^2 - \Lambda^2)\rho(q)$, where $\theta(\cdot)$ is the Heaviside step function. Then one can think of the gas as a two-component fluid. The idea is to take Λ large enough so that $\Lambda \gg \max[(mg/\hbar), (C_\mathrm{r}\xi mg/\hbar)^{1/4}]$, where ξ is the correlation length of the gas.

We focus first on the component with rapidity distribution $\rho_{>\Lambda}$. Within a cell of size $\gtrsim \xi$, large enough so that the particles it contains are not correlated with the rest of the system, the typical number of rapidities in an interval $[q, q+dq]$ is $\xi\rho_{>\Lambda}(q)dq$. This implies that the typical spacing between neighbor rapidities is of order $\Delta q \sim 1/(\xi\rho_{>\Lambda}) \sim 1/(\xi C_\mathrm{r}/\Lambda^4) \gg mg/\hbar$. This ensures that this fluid component behaves as an ideal Bose gas. In particular, its momentum distribution equals its rapidity distribution: $w_{>\Lambda}(p) \simeq \rho_{>\Lambda}(p) \simeq \theta(p^2 - \Lambda^2) C_r/p^4$. Moreover, the condition $\Lambda \gg \max[(mg/\hbar), (C_\mathrm{r}\xi mg/\hbar)^{1/4}]$ also ensures that the two fluid components do not interact with each other.

The other fluid component is characterized by a rapidity distribution $\rho_{<\Lambda}$ with no tails, so it satisfies Tan's relation. Thus, its momentum distribution $w_{<\Lambda}(p)$ decays as C_c/p^4 at large p.

The total momentum distribution $w(p)$ of the gas is the sum of the momentum distributions of both components, which leads to Eq. (4).

Having established the key formula (4), we now turn to the question: Is there a physical process that produces such peculiar states with $1/q^4$ tails in their rapidity distribution? We are aware of only one such example in the literature so far: a sudden quench of the interaction strength g, which relaxes to a state with $C_r > 0$ [52]. In the rest of this Letter we argue that atom losses, which are ubiquitous in experiments, always generate these peculiar states.

Losses and $1/q^4$ tails of the rapidity distribution.—We consider the general case of local K-body losses, where $K = 1, 2, 3, \ldots$ is the number of atoms lost in each loss event. Depending on the experiment, losses are typically dominated by $K = 1$, $K = 2$ [53,54], or $K = 3$ processes [26,27], but it is convenient to keep K arbitrary. The atom density then decays as $dn/dt = -KGg^{(K)}(0)n^K$, where G is a constant with units of $\mathrm{length}^{K-1}\cdot\mathrm{time}^{-1}$ that characterizes the loss rate. Following Ref. [28] (see also Refs. [55,56]), we assume that the loss rate Gn^{K-1} is much smaller than the relaxation time, so that the gas relaxes to a generalized Gibbs ensemble after each loss event. This allows to represent the evolution of the gas under losses by its time-dependent rapidity distribution [28].

Let us assume that, at $t = 0$ the rapidity distribution of the gas has no $1/q^4$ tails; i.e., $C_\mathrm{r}(t = 0) = 0$. For instance, the gas could be in a thermal state. We want to show that at $t = 0$, $dC_\mathrm{r}/dt > 0$, implying that the rapidity distribution will develop nonvanishing $1/q^4$ tails.

To do this, we elaborate on the microscopic mechanism presented in the introduction. Consider the many-body wave function $\psi_{t=t_l^-}(z_1, \ldots, z_N)$ just before a loss event occurring at time t_l and position z_l. Right after the loss, the wave function of the remaining $N - K$ atoms is $\tilde{\psi}_{t=t_l^+}(z_1, \ldots, z_{N-K}) = L^{K/2}\psi_{t=t_l^-}(z_1, \ldots, z_{N-K}, z_{N-K+1} = z_l, \ldots, z_N = z_l)$. As a reminiscence of its cusp singularities before the loss, the wave function $\tilde{\psi}_{t=t_l^+}$ still has a cusp at $z_j = z_l$ $(j = 1, \ldots, N - K)$. Following the calculation of Ref. [3], we find that it results in a contribution $C^{(1\,\mathrm{loss})}/p^4$ to the momentum distribution, with the amplitude

$$C^{(1\,\mathrm{loss})} = \frac{\hbar^3}{2\pi} L^{K-1}(N-K)\int dz_2 \ldots dz_{N-K} \left|\partial_{z_1}\psi_{|_{z_1\to z_l^+}} - \partial_{z_1}\psi_{|_{z_1\to z_l^-}}\right|^2, \tag{6}$$

where the variables $z_{N-K+1}, \ldots, z_N$ in the integrand are taken equal to z_l. The boundary condition imposed by the contact interaction gives $\partial_{z_1}\psi_{|_{z_1\to z_l^-}} - \partial_{z_1}\psi_{|_{z_1\to z_l^+}} = Kmg/\hbar^2\psi(z_1 = z_l, z_2, \ldots, z_{N-K+1} = z_l, \ldots z_N = z_l)$. Then, using the expression of $g^{(K+1)}(0)$ in the first quantization, we get

$$C^{(1\,\mathrm{loss})} = \frac{m^2}{2\pi\hbar}\frac{nK^2}{L} g^2 g^{(K+1)}(0). \tag{7}$$

Here we have used the fact that, as $N \to \infty$, $N - K \simeq N$ and $N \ldots (N-K) \simeq N^{K+1}$.

Next, we rely on formula (4), and argue that the contribution (7) of one loss event to the momentum distribution translates into the same contribution to the rapidity distribution. Indeed, the contribution (7) is not taken into account in the contact density C_c at time $t = t_l^+$;

therefore, according to formula (4) it must appear in the tail of the rapidity distribution:

$$C_r|_{t=t_l^+} - C_r|_{t=t_l^-} = C^{(1\,\mathrm{loss})}. \quad (8)$$

Like $\rho(k)$, C_r is conserved by the Hamiltonian dynamics, so this increase of C_r remains after relaxation to a generalized Gibbs ensemble. Finally, we multiply this result by $LGn^K g^{(K)}(0)dt$, the number of loss events occurring in the system during a short time interval dt. This leads to the initial growth rate

$$\frac{dC_r}{dt}(t=0) = \frac{m^2}{2\pi\hbar} Gn^{K+1} K^2 g^2 g^{(K)}(0) g^{(K+1)}(0). \quad (9)$$

This equation is the second main formula of this Letter. It shows that $dC_r/dt|_{t=0} > 0$, such that C_r becomes nonzero. Together with Eq. (4), it implies that the momentum distribution develops tails that are larger than what is expected from Tan's relation.

We stress that Eq. (9) gives only the initial growth rate of the tail of the rapidity distribution. At later times, its evolution will also involve additional damping effects. Indeed, under atom losses the gas ultimately evolves to the vacuum; therefore the whole rapidity distribution—including its tails—will go to zero at very long times. The calculation of the damping of C_r at longer times is not obvious. Below we obtain further results in the hard-core and quasicondensate regimes.

Exact results in the hard-core regime.—In the hard-core regime ($g \to \infty$), only one-body losses are relevant, since $g^{(K)}(0) = 0$ for $K > 1$. Thus, in this paragraph we fix $K = 1$. The evolution of the rapidity distribution $\rho(t, q)$ under losses has been computed recently in Ref. [28], for an arbitrary initial distribution $\rho(t = 0, q)$; see in particular Eq. (14) in that reference. Here we exploit that general result to study the evolution of the $1/q^4$ tail.

Expanding Eq. (14) of Ref. [28] for large q, we find that $\rho(t, q) = C_r(t)/q^4 + o(1/q^4)$, with

$$C_r(t) = \frac{4\hbar m}{\pi}[n(0)e(0) - j(0)^2/(2m)]e^{-Gt}(1 - e^{-Gt}). \quad (10)$$

Here $j(t) = \int q\rho(t, q)dq$ and $e(t) = \int q^2/(2m)\rho(t, q)dq$ are the momentum and energy density, respectively [57]. The right-hand side of Eq. (10) involves these quantities at time $t = 0$. Using the fact that, under losses, the particle, momentum, and energy densities evolve as $n(t) = n(0)e^{-Gt}$, $j(t) = j(0)e^{-Gt}$, and $e(t) = e(0)e^{-Gt}$, respectively, in the $g \to \infty$ limit [49], the right-hand side can also be written as $(4\hbar m/\pi)[n(t)e(t) - j(t)^2/(2m)](e^{Gt} - 1)$.

We note that formula (10) provides a nontrivial check of our general prediction (9) for the initial growth rate: using the standard identity $\lim_{g\to\infty} n^2 g^2 g^{(2)}(0) = (8\hbar^2/m)[ne - j^2/(2m)]$ [49], one sees that Eqs. (9) and (10) agree.

Importantly, Eq. (10) also allows us to compare the amplitude $C_r(t)$ with the contact density at time t. Using again the standard identity above, together with Eq. (3), we find

$$C_r(t)/C_c(t) = \exp(Gt) - 1. \quad (11)$$

We see that the ratio of the amplitude C_r to the contact density C_c grows exponentially as time increases. This is our third main result: not only does the term C_r/p^4 contribute to the momentum distribution, but it can also become dominant compared to the contact term. Numerical calculations of $w(p)$ [49] show that, for an initial degenerate gas, $w(p) \simeq (C_r + C_c)/p^4$ as soon as $p \gtrsim 7\hbar n_0$.

We now investigate the ratio $C_r(t)/C_c(t)$ for weak repulsion.

Results for the quasicondensate.—In the quasicondensate regime, correlations between atoms are weak and $g^{(j)}(0) \simeq 1$ for all j. An effective description of the gas is obtained by a phase-density representation [58]: in Eq. (2), one writes the atomic field Ψ as $\sqrt{n + \delta n}e^{i\theta}$, where θ and δn are phase and density fluctuation fields (with $\delta n, \partial\theta/\partial z \ll n$), which satisfy the commutation relation $[\delta n(z), \theta(z')] = i\delta(z - z')$. The Bogoliubov approximation then leads to a collection of independent harmonic modes. The Hamiltonian for each mode is of the form $H_k = \varepsilon_k b_k^+ b_k$ (up to the additive constant), where b_k^+ ($k \in (2\pi\hbar/L)\mathbb{Z}$) is a linear combination of the Fourier modes δn_k and θ_k [49,58] and $\varepsilon_q = \sqrt{(k^2/2m)[(k^2/2m) + 2gn]}$. The Bogoliubov creation and annihilation operators satisfy $[b_k, b_{k'}^+] = \delta_{k,k'}$, and the occupation of each mode is $\alpha_k = \langle b_k^+ b_k\rangle$.

The effect of slow losses on the Bogoliubov mode occupations α_k has been analyzed in Refs. [29,30,59,60]. In Ref. [29], the effect of K-body losses on α_k was computed for small k. In Refs. [30,59], the evolution of α_k was studied for any k, but only $K = 1$ was considered. Combining these results, we are able to compute $d\alpha_k/dt$ for any K and k [49]. The result reads

$$\frac{d\alpha_k}{dt} = K^2 G n^{K-1}\left[-\alpha_k - \frac{1}{2} + \frac{1}{4}\left(\frac{\varepsilon_k}{k^2/(2m)} + \frac{k^2/(2m)}{\varepsilon_k}\right)\right]. \quad (12)$$

The precise link between Bogoliubov excitations and Bethe quasiparticles is not obvious. However, it has been discussed by Lieb [61] (see also Ref. [62]), who identifies, for states close to the ground state, the large-k Bogoliubov excitations to Bethe quasiparticles with rapidities $q \simeq k$. Therefore a C_r/q^4 tail in the rapidity distribution translates to Bogoliubov mode occupations decaying as $\alpha_k \simeq 2\pi\hbar C_r/k^4$ for large k [63]. We have checked [49] that this identification $q \simeq k$, together with the known exact expression for $g^{(1)}(z)$ [58], is compatible with our Eq. (4) within the framework of Bogoliubov theory, as it should.

Using the large-k expansion of ε_k in Eq. (12), we find that the amplitude of the $1/q^4$ tails of $\rho(q)$ evolves according to

$$\frac{dC_r}{dt} = K^2 G n^{K-1}\left(-C_r + \frac{m^2}{2\pi\hbar} g^2 n^2\right). \tag{13}$$

This differential equation can be easily solved [49], which allows us to obtain $C_r(t)$ at all times. In particular, at long times, we find that the ratio of $C_r(t)$ to the contact density $C_c(t) = (m^2/2\pi\hbar) g^2 n(t)^2$ [Eq. (3), with $g^{(j)}(0) = 1$] behaves as

$$\frac{C_r(t)}{C_c(t)} \underset{t\to\infty}{=} \begin{cases} \exp(Gt) & \text{if } K = 1, \\ 2\log(G n_0^{K-1} t) & \text{if } K = 2, \\ K/(K-2) & \text{if } K \geq 3. \end{cases} \tag{14}$$

This is the fourth main result of this Letter. For $K = 1$, one finds the same behavior as in the hard-core regime. For $K \geq 3$, the ratio takes an asymptotic value. For instance, the ratio C_r/C_c goes to 3 for three-body losses, so the tail of the momentum distribution C/p^4 is four times larger than its value predicted by Tan's relation (1).

Experimental prospects.—An experimental test of the predictions of this Letter is within reach in current cold atom setups. There exist different ways of measuring the momentum distribution of 1D gases [64–67]. Because of the small amplitude of the tails, such a measurement requires a high dynamical range, which can be achieved, for instance, using metastable atoms [68]. Usually, gases in experiments are nonuniform. Within a local density approximation, our results are straightforwardly generalized to include a trapping potential [49].

Conclusion.—On the theory side, our results open several research lines. First, for quantitative comparison with experiment, one should compute the evolution of the rapidity tails in intermediate regimes of the 1D gas. For this, one can in principle rely on the method presented in Ref. [28], although an improvement of the numerical efficiency of that method would be required (see also the recent analytical progress in Ref. [69]). Second, our results can probably be extended to integrable 1D Fermi gases [70]. Third, it would be interesting to investigate the effects of losses in higher dimension. The singularity of the wave function at the position of the lost atoms is also expected to have a effect that remains to be elucidated. Finally, it would be interesting to study loss processes that are not purely local or not purely Markovian. How would this impact the development of the momentum tails?

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