Kolmogorov-Arnold-Moser Stability for Conserved Quantities in Finite-Dimensional Quantum Systems

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We show that for any finite-dimensional quantum systems the conserved quantities can be characterized by their robustness to small perturbations: for fragile symmetries, small perturbations can lead to large deviations over long times, while for robust symmetries, their expectation values remain close to their initial values for all times. This is in analogy with the celebrated Kolmogorov-Arnold-Moser theorem in classical mechanics. To prove this result, we introduce a resummation of a perturbation series, which generalizes the Hamiltonian of the quantum Zeno dynamics.

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Symmetries and conserved quantities are the cornerstones of modern theoretical physics [1]. In quantum mechanics, it is well known that conserved quantities are characterized by observables that commute with the system Hamiltonian. Here, we show that this characterization is incomplete, because some symmetries in quantum mechanics are more conserved than others.

More precisely, we can consider the *robustness* of symmetries. Some fundamental symmetries (such as those related to superselection rules [2]) are considered almost unbreakable in nonrelativistic quantum mechanics, while other, accidental [3], symmetries are easily perturbed.

We now introduce such distinction into fundamental, robust symmetries and accidental, fragile ones in an analogous, but much more applied context, namely, one provided by a time-independent Hamiltonian on a finite-dimensional quantum system. This Hamiltonian H acts as a reference for its symmetries S, characterized by $[H,S]=0$, and with respect to which we define their robust component S_{robust} as the part almost conserved [up to a term $O(\varepsilon)$] for *all* times and for *any* small time-independent perturbation εV, while we define their *fragile* component S_{fragile} as the part for which there are perturbations that will accumulate large amounts of change over time. As an alternative view, for any robust symmetry there is a slightly modified observable that is conserved in the perturbed system $S_{\mathrm{robust}}\to S_{\mathrm{robust}}^{\varepsilon}$, while for fragile symmetries there is not. Such conserved quantities were constructed recently in many-body systems for specific perturbations [4], while we provide a general construction and characterization, and show a natural decomposition of any symmetry S

$$S=S_{\mathrm{robust}}+S_{\mathrm{fragile}}. \tag{1}$$

The importance of robust observables is exemplified by analog quantum simulations [5], where the aim is to run a complex Hamiltonian long enough such that observable quantities are no longer easily computable by classical computers. The problem is, however, that small perturbations in the lab are not under control and can destroy the reliability of the simulation [6,7]. On the other hand, as we show below, the expectation values of robust observables remain reliable even in the long term.

More fundamentally, our result is in close analogy to the Kolmogorov-Arnold-Moser (KAM) perturbation theory in classical mechanics [8,9], which proved the long-time stability of planetary orbits, despite accumulating perturbations. Quantum mechanical versions of KAM perturbation have been considered previously by Scherer [10] to mimic a superconvergent series. In the context of many-body systems, Nekhoroshev estimates were used to show a robustness of certain observables for intermediate times [11–15]. Our focus, instead, is an algebraic approach based on the adiabatic theorem, enabling us to provide nonperturbative bounds valid for arbitrarily long times, with no structural assumptions on the generators and observables, and generalizations to open systems (Lindbladians). This way, we prove a result analogous to the KAM stability in finite-dimensional quantum systems.

How can we characterize which observables are fragile and which are robust? Under which conditions are there robust ones, and just *how* robust are they? In the unperturbed system, the conserved quantities are the observables commuting with H, given by all Hermitian matrices that are block diagonal with respect to the eigenspaces of H. They may share the degenerate eigenspaces of H or they may lift their degeneracy. In this Letter, we will show that this precisely distinguishes robust and fragile symmetries. Moreover, unless H is the identity, there always exist nontrivial robust symmetries.

Fragile symmetries.—First, consider a symmetry M that breaks degeneracy in an eigenspace of H. We show that such a conserved quantity is not robust against perturbation. For instance, take two simultaneous eigenstates of H and M, say $|e_1\rangle$ and $|e_2\rangle$, belonging to the same eigenspace of H but belonging to different eigenspaces of M; i.e., $H|e_1\rangle = e|e_1\rangle$ and $H|e_2\rangle = e|e_2\rangle$, while $M|e_1\rangle = m_1|e_1\rangle$ and $M|e_2\rangle = m_2|e_2\rangle$, with $\Delta = m_1 - m_2 > 0$. Let us take $V = |e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|$ as a perturbation and consider $H + \varepsilon V$. If we focus on initial states $|\psi\rangle$ in the subspace spanned by $\{|e_1\rangle, |e_2\rangle\}$, the problem is reduced to a two-dimensional problem. Take, for instance, $|e_1\rangle$ as an initial state. We find

$$\langle M\rangle_t^\varepsilon - \langle M\rangle_t = \langle M\rangle_t^\varepsilon - \langle M\rangle_0 = -\Delta \sin^2 \varepsilon t, \qquad (2)$$

where the expectations $\langle\cdot\rangle_t$ and $\langle\cdot\rangle_t^\varepsilon$ are taken with respect to states evolved under the free and the perturbed evolution, $e^{-itH}|\psi\rangle$ and $e^{-it(H+\varepsilon V)}|\psi\rangle$, respectively. At time $t = \pi/(2\varepsilon)$ the error is Δ, which is independent of ε. This kind of example can be constructed for any M that is nondegenerate within a subspace of H, and we conclude that such conserved observables are fragile.

Robust symmetries.—Second, consider a conserved observable that acts uniformly within each eigenspace of H. We may write $M = \sum m_k P_k$, where $\{P_k\}$ are the spectral projections of $H = \sum_k e_k P_k$ (with $e_k \neq e_\ell$ for $k \neq \ell$ and $P_k P_\ell = \delta_{k\ell} P_\ell$). Using results on the quantum Zeno dynamics [16–19], one can show that such observables are endowed with some intrinsic robustness with respect to small perturbations εV, with $\|V\| = 1$. Indeed, we have a bound [20]

$$\delta_Z(t) = \|e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon V_Z)}\| \leq 2\sqrt{d}\varepsilon(1 + \varepsilon t)/\eta, \qquad (3)$$

where d is the number of distinct eigenvalues of the Hamiltonian H, $\eta = \min_{k\neq\ell}|e_k - e_\ell|$ is the spectral gap of H (strictly positive for any nontrivial H), and $V_Z = \sum_k P_k V P_k$ is the "Zeno Hamiltonian" [16,17]. By construction, $[M, V_Z] = 0$, and we obtain [19,20]

$$\|M_t^\varepsilon - M\| \leq 2\|M\|\delta_Z(t) \leq 4\sqrt{d}\|M\|\varepsilon(1 + \varepsilon t)/\eta, \qquad (4)$$

where $M_t^\varepsilon = e^{it(H+\varepsilon V)} M e^{-it(H+\varepsilon V)}$ is the perturbed evolution of observable M. This bound, however, is informational as far as it is less than the trivial bound $2\|M\|$, which is not for sufficiently large times t.

This is, anyway, just an upper bound, and it might be a loose bound. Let us look more carefully at a two-dimensional example again and show that there are indeed perturbations V such that $\delta_Z(t)$ in (3) saturates the trivial bound 2, for every ε, however small. Consider $H = \sigma_z$ and $V = \sigma_x$, the third and first Pauli matrices, respectively. In this case, we have $V_Z = 0$, and $\delta_Z(t) = \|e^{it(\sigma_z+\varepsilon\sigma_x)} - e^{it\sigma_z}\|$. This is a complicated quasiperiodic function, with $\sup_t \delta_Z(t) = 2$ [31], proving that, in general, the Zeno Hamiltonian V_Z is not a good approximation for long times.

However, notwithstanding the negative result about the smallness of the distance $\delta_Z(t)$ in (3), the conserved quantity $M = \sum m_k P_k$ considered above is actually stable for all times, *eternally*. The key idea behind the above phenomenon is to choose an (ε-dependent) approximation of V that has the same block structure as H and is therefore commutative with M. The Zeno Hamiltonian V_Z is not a good choice. To make the point, consider again the above two-dimensional example with $H = \sigma_z$ and $V = \sigma_x$, and now choose, in place of V_Z, the operator $V_H(\varepsilon) = \varepsilon^{-1}(\sqrt{1+\varepsilon^2} - 1)\sigma_z$ as an approximation of V. Obviously, $[V_H, H] = 0$. Moreover, $V_H(\varepsilon) = V_Z + O(\varepsilon)$. With this choice, we get

$$\begin{aligned}\delta(t) &= \|e^{it(\sigma_z+\varepsilon\sigma_x)} - e^{it[\sigma_z+\varepsilon V_H(\varepsilon)]}\| \\ &= \sqrt{2\left(1 - \frac{1}{\sqrt{1+\varepsilon^2}}\right)}|\sin(t\sqrt{1/\varepsilon^2 + 1})| \leq \varepsilon. \end{aligned} \qquad (5)$$

This bound is independent of time t and implies that [20]

$$\|M_t^\varepsilon - M\| \leq 2\|M\|\delta(t) \leq 2\|M\|\varepsilon, \qquad (6)$$

for all times and for any observable of the form $M = \mathrm{diag}(m_1, m_2)$. Such observables are robust.

General result.—Of course, we did not just guess $V_H(\varepsilon)$ arbitrarily. We discovered a way of constructing such eternal block-diagonal approximations for any finite-dimensional quantum systems, including noisy systems with Lindbladians. They can be seen as resummation of a perturbative series, whose zeroth-order term is the Zeno Hamiltonian $V_H(0) = V_Z$. Its theory, proof, and generalizations are discussed in great detail in Ref. [32].

The crucial ingredient is that the block-diagonal approximation $H + \varepsilon V_H(\varepsilon)$, unlike $H + \varepsilon V_Z$, can be chosen to have the *same* spectrum of $H + \varepsilon V$ and thus to be unitarily equivalent to it: $H + \varepsilon V_H(\varepsilon) = W_\varepsilon^\dagger(H + \varepsilon V)W_\varepsilon$, with a unitary $W_\varepsilon = \mathbb{1} + O(\varepsilon)$ [20]. This is a necessary condition, since geometrically the evolution of a Hamiltonian with d distinct eigenvalues yields a (quasi-)periodic motion of a point on a torus. Two motions with different frequencies, however small the differences may be, will eventually accumulate a divergence of $O(1)$. The only way to avoid this slow drift is that the two motions be isochronous, that is, the two Hamiltonians be isospectral. In such a case, we get

$$\begin{aligned}\delta_\infty &= \sup_t \| e^{it(H+\varepsilon V)} - e^{it[H+\varepsilon V_H(\varepsilon)]} \| \\ &= \sup_t \| e^{it(H+\varepsilon V)} - W_\varepsilon^\dagger e^{it(H+\varepsilon V)} W_\varepsilon \| < 7\sqrt{d}\varepsilon/\eta \quad (7)\end{aligned}$$

(see the Supplemental Material [20] for a perturbative proof and Ref. [32] for explicit bounds). It follows that any quantum system has robust conserved quantities $S = S_{\text{robust}}$, with $[S, H] = 0$, and hence $S = S_t$, such that for every perturbation εV,

$$\| S_t^\varepsilon - S \| \leq 2 \| S \| \delta_\infty = O(\varepsilon), \quad (8)$$

for all times [20], where $S_t^\varepsilon = e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)}$, and they are precisely of the form $S = S_{\text{robust}}$, where

$$S_{\text{robust}} = \sum_k s_k P_k, \quad (9)$$

with $H = \sum_k e_k P_k$. All other conserved quantities are fragile, as the distance (8) becomes $O(1)$.

While this is a complete characterization of robust conserved quantities, the representation in terms of spectral projections requires diagonalization of the Hamiltonian H and is impractical for high-dimensional systems. However, given $H^n = \sum_k e_k^n P_k$, one can invoke the invertibility of the Vandermonde matrix (e_k^{j-1}) to see that $S = \sum_{k=0}^{d-1} c_k H^k$. This means that any primary matrix function $f(H)$ of the Hamiltonian H is robust. If the original Hamiltonian is sparse, for instance, low-order polynomials can be constructed efficiently. In particular, we obtain that for any state the energy expectation value and the variance

$$\langle H \rangle_t^\varepsilon - \langle H \rangle_0 = O(\varepsilon), \qquad \langle \Delta H^2 \rangle_t^\varepsilon - \langle \Delta H^2 \rangle_0 = O(\varepsilon) \quad (10)$$

remain close to their unperturbed values forever. This is easily generalized to higher moments.

We can also rephrase the fact that any robust observable is a polynomial function of H in terms of the symmetries of H. That is, S is robust if and only if it shares all symmetries of H: for any C such that $[H, C] = 0$, we also have $[S, C] = 0$ [33]. For more details on the algebraic structure, see the Supplemental Material [20].

Finally, let us emphasize that the above characterization provides a natural decomposition of any observable M into three parts: one dynamical part that is not conserved by H, one that is conserved but is fragile to perturbations, and one that is robust. The nonconserved part is off diagonal with respect to the spectral projections of H,

$$M_{\text{noncons}} = M - M_{\text{cons}} = M - \sum_k P_k M P_k, \quad (11)$$

the robust component of the conserved part M_{cons} acts trivially within the eigenspaces of H,

$$M_{\text{robust}} = \sum_k d_k^{-1} \mathrm{tr}(P_k M P_k) P_k, \quad (12)$$

with d_k being the dimension of the kth eigenspace, and the fragile part is the remaining symmetry

$$M_{\text{fragile}} = \sum_k P_k [M - d_k^{-1} \mathrm{tr}(P_k M P_k)] P_k. \quad (13)$$

Integrable example.—While an unambiguous and universal definition of integrability for quantum systems is still lacking [34,35], we take the Heisenberg chain as a typical example of a system which we think of as integrable. The Hamiltonian acts on N qubits and is given by $H = -J \sum_{n=1}^N \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1}$, where $\boldsymbol{\sigma}_n = (\sigma_{n,x}, \sigma_{n,y}, \sigma_{n,z})$ is the vector of Pauli matrices acting on the nth qubit, and we impose the periodic boundary conditions $\boldsymbol{\sigma}_{N+1} = \boldsymbol{\sigma}_1$. The Heisenberg chain can be solved analytically by the algebraic Bethe ansatz. The corresponding conserved charges $Q_2, \ldots, Q_N$ can be generated using the boost operator $B = \frac{1}{2} \sum_{n=1}^N n \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1}$ as $Q_{n+1} = -i[B, Q_n]$ with $Q_2 = H$, and Q_n acts nontrivially on sets of n neighbors on the chain only [36]. Combined with the total magnetization $Q_1 = \sum_{n=1}^N \sigma_{n,z}$, they provide a maximal Abelian algebra. These conserved charges are the pinnacle of integrability. However, they are fragile: because the charges are algebraically independent, except for $Q_2 = H$, none are robust. Incidentally, this shows that the findings in Ref. [4] are restricted to specific perturbation classes. A simple example is given by the total magnetization Q_1: due to the rotational invariance of H, we could have equally chosen the magnetization in another direction, say $\tilde{Q}_1 = \sum_{n=1}^N \sigma_{n,x}$. As a perturbation, however, $\tilde{Q}_1$ causes the expectation value of Q_1 to oscillate and deviate vastly from its original value. For instance, if we start with a z-polarized state, we obtain $\langle Q_1 \rangle_t^\varepsilon = \cos(\omega t) N$ for some ω depending on the perturbation strength ε. We show numerical examples of the evolution of a randomly chosen observable (Fig. 1) as well as physical ones (Fig. 2).

Our bound (7) scales with the number d of distinct eigenvalues and the inverse of the spectral gap η of the Hamiltonian H. Therefore, in the context of many-body physics, it is useful only for particular systems as system size grows. In many-body setups, weaker types of robustness of observables were shown, assuming locality of the Hamiltonian, observable, and perturbation [11–15]. In the context of KAM, such bounds are analogous to Nekhoroshev estimates, showing stability for an

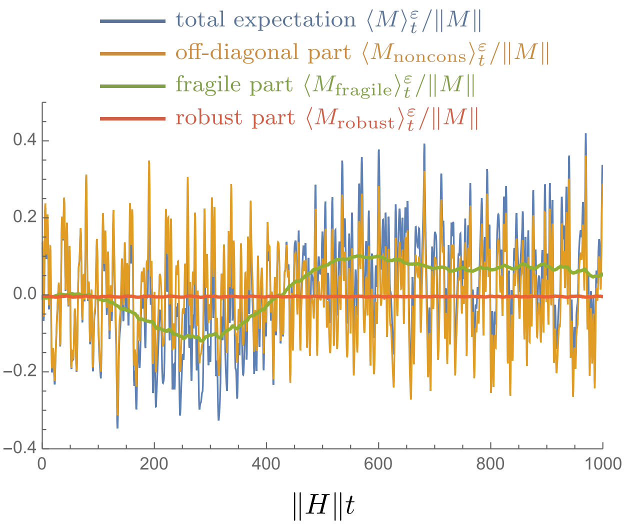

FIG. 1. Dynamics of the expectation of a randomly picked observable M in a Heisenberg chain with $N = 4$ as a function of time. We show its decomposition into nonconserved, robust, and fragile parts [Eqs. (11)–(13)]. The initial state and the perturbation V are chosen randomly with strength $\varepsilon\|V\| = 0.02\|H\|$. Shown is one realization.

exponentially long time. While the spectral gap is important for our bound, our result is valid for arbitrarily long times, requiring no structural assumptions on the Hamiltonians, observables, and perturbations.

Thermalization.—It is one of the most celebrated results in mathematical quantum statistical mechanics that the Kubo-Martin-Schwinger state [the Gibbs state $\propto\exp(-\beta H)$] is the unique state that maximizes entropy, stationary under the time evolution of the Hamiltonian H, and robust under perturbations [37]. However, to date, this was only considered for short times. Our characterization of robust observables implies that

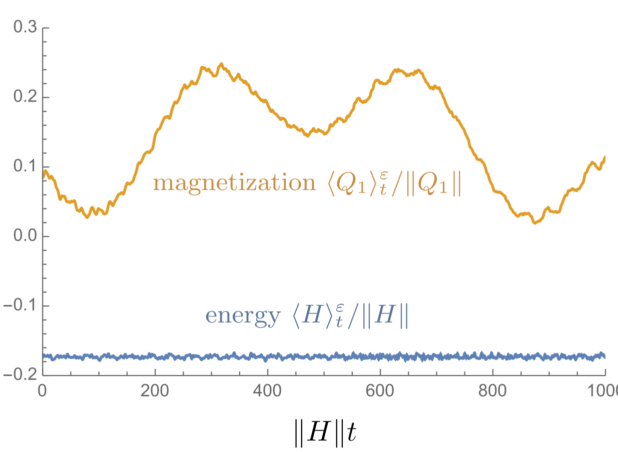

FIG. 2. Same setup and realization as in Fig. 1, but now showing the dynamics of the expectation of $Q_2 = H$ (robust) and $Q_1 = \sum_{n=1}^N \sigma_{n,z}$ (fragile).

$$e^{-it(H+\varepsilon V)}\exp(-\beta H)e^{it(H+\varepsilon V)} = \exp(-\beta H) + O(\varepsilon), \quad (14)$$

uniformly in time for any finite-dimensional system. Note, on the other hand, that generalized Gibbs ensembles [38–44] such as $\exp(-\sum_j \beta_j Q_j)$ for integrable charges are *not* robust.

Open systems.—How do we generalize this to Lindbladian dynamics? For a Lindbladian $\mathcal{L}$, it would be natural to consider $\mathcal{M} = \mathcal{M}_{\mathrm{robust}} = \sum_k m_k \mathcal{P}_k$, with $\{\mathcal{P}_k\}$ the spectral projections of $\mathcal{L}$, as a candidate for a robust symmetry. However, it is easy to see that the trace preservation of $\mathcal{L}$ implies that $\mathrm{tr}\mathcal{M}(\rho) = m_0 \mathrm{tr}\rho = m_0$, where $\mathcal{P}_0$ is the projection for the zero eigenvalue of $\mathcal{L}$. Therefore, this quantity is trivial. This is related to the fact that Noether's theorem breaks down for Lindbladian systems [45] and to the fact that we are talking about a superoperator structure on top of the usual observable space. Very recently, however, Styliaris and Zanardi showed [46] that for each conserved superoperator $\mathcal{M}$ satisfying $[\mathcal{M}, \mathcal{L}] = 0$ one can define a monotone function

$$f_{\mathcal{M}}(\rho) = \mathrm{tr}[\mathcal{M}(\rho)^\dagger(\mathbf{L}_\rho + \lambda\mathbf{R}_\rho)^{-1}(\mathcal{M}(\rho))], \quad (15)$$

with $\lambda \geq 0$, where $\mathbf{L}_\rho(X) = \rho X$ and $\mathbf{R}_\rho(X) = X\rho$ are the superoperators of left and right multiplication by ρ, respectively, and the inverse is well defined for strictly positive ρ. They showed that such a monotone, as complicated as it might look at first glance, is well motivated from entropic distances and is decreasing under the evolution $e^{t\mathcal{L}}$,

$$f_{\mathcal{M}}(\rho_t) \leq f_{\mathcal{M}}(\rho), \quad \text{for all } t \geq 0, \quad (16)$$

where $\rho_t = e^{t\mathcal{L}}\rho$. Using our generalized eternal block-diagonal approximation $\mathcal{V}_{\mathcal{L}}(\varepsilon)$ to a perturbation $\mathcal{V}$ for open systems [32], we can write the perturbed dynamics

$$e^{t(\mathcal{L}+\varepsilon\mathcal{V})} = e^{t[\mathcal{L}+\varepsilon\mathcal{V}_{\mathcal{L}}(\varepsilon)]} + O(\varepsilon) \quad (17)$$

for all times and see that, for any robust symmetry $\mathcal{M} = \mathcal{M}_{\mathrm{robust}}$ and for any perturbation $\varepsilon\mathcal{V}$, the monotone $f_{\mathcal{M}}(\rho)$ remains approximately monotonic [20],

$$f_{\mathcal{M}}(\rho_t^\varepsilon) \leq f_{\mathcal{M}}(\rho) + O(\varepsilon), \quad (18)$$

under the perturbed evolution $\rho_t^\varepsilon = e^{t(\mathcal{L}+\varepsilon\mathcal{V})}\rho$. In this sense, the monotone is robust against perturbation. See Fig. 3 for a couple of examples for a qubit dephasing evolution. The monotone defined with $\mathcal{M}_{\mathrm{robust}}$ is robust against perturbation: it is perturbed and becomes nonmonotonic, but the nonmonotonicity is small. On the other hand, the monotone defined with $\mathcal{M}_{\mathrm{fragile}}$ that lifts the degeneracy in $\mathcal{L}$ is fragile.

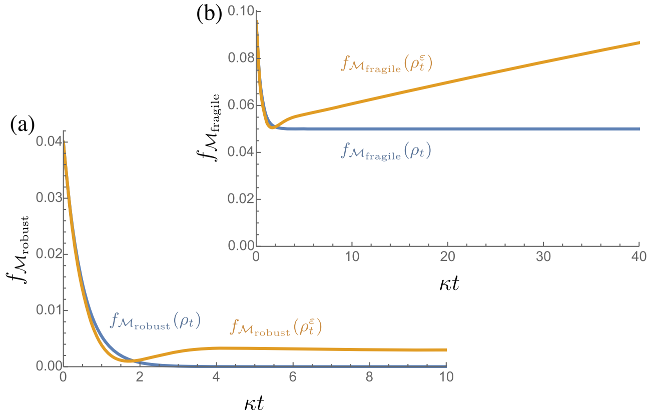

FIG. 3. The free and perturbed evolutions for a qubit of (a) the robust monotone $f_{\mathcal{M}_{\text{robust}}}(\rho) = |\langle 0|\rho|1\rangle|^2$, with $\mathcal{M}_{\text{robust}} = -(i/\sqrt{2})[\sigma_z, \cdot]$ and $\lambda = 1$, and of (b) the fragile monotone $f_{\mathcal{M}_{\text{fragile}}}(\rho)$, with $\mathcal{M}_{\text{fragile}} = -(i/\sqrt{2})[\sigma_z, \cdot] + |0\rangle\langle 0| \cdot |0\rangle\langle 0|$ and $\lambda = 1$, where $\mathcal{L} = -(i/2)\omega[\sigma_z, \cdot] - \frac{1}{2}\kappa(1 - \sigma_z \cdot \sigma_z)$ and $\varepsilon\mathcal{V} = -(i/2)\varepsilon g[\sigma_x, \cdot]$. The perturbation creates coherence between the eigenstates of σ_z, $|0\rangle$ and $|1\rangle$. The initial state of the qubit is given by the coherence vector $(r_x, r_y, r_z) = (0.2, 0, 0.8)$, and the parameters are set at $g = \kappa$ and $\varepsilon = 0.1$ (ω is irrelevant).

Conclusions.—While our results in spirit reproduce a lot of features one would hope a quantum KAM theory to feature—long-term stability of certain observables with respect to perturbations, in analogy with the KAM theory in classical mechanics [8,9]—there are also some perhaps surprising aspects. Conserved charges and generalized Gibbs states from quantum integrable models are not robust, while randomly chosen Hamiltonians (thus without degeneracies) have the property that all conserved quantities are robust.

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