Detecting Many-Body Bell Nonlocality by Solving Ising Models

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(Received 21 May 2020; revised 29 November 2020; accepted 5 March 2021; published 9 April 2021)

Bell nonlocality represents the ultimate consequence of quantum entanglement, fundamentally undermining the classical tenet that spatially separated degrees of freedom possess objective attributes independently of the act of their measurement. Despite its importance, probing Bell nonlocality in many-body systems is considered to be a formidable challenge, with a computational cost scaling exponentially with system size. Here we propose and validate an efficient variational scheme, based on the solution of inverse classical Ising problems, which in polynomial time can probe whether an arbitrary set of quantum data is compatible with a local theory; and, if not, it delivers the many-body Bell inequality most strongly violated by the quantum data. We use our approach to unveil new many-body Bell inequalities, violated by suitable measurements on paradigmatic quantum states (the low-energy states of Heisenberg antiferromagnets), paving the way to systematic Bell tests in the many-body realm.

DOI: 10.1103/PhysRevLett.126.140504

Introduction: Bell tests and quantum certification.— Quantum correlations, such as entanglement [1], Einstein-Podolsky-Rosen correlations [2,3], and Bell correlations [4,5], are common features of microscopic ensembles of quantum degrees of freedom (d.o.f.), such as the electronic and nuclear spins in atoms and molecules [6], or pairs of photons produced by parametric down-conversion [7]. Their persistence in many-body systems is a central issue: an obstruction to the scalability of quantum correlations would be the core feature of a putative quantum-to-classical transition [8,9]; and, in parallel, they are the essential resource for most quantum technologies of second generation [10]. In view of all this, the robust certification of quantum correlations in many-body systems stands as a central problem for theoretical as well as experimental quantum physics.

The most robust certification scheme is undoubtedly offered by the *device-independent* (DI) approach, relying on the violation of a Bell inequality—a so-called Bell test—which does not assume anything about the quantum system except what can be assessed experimentally. Specifically, we assume that a many-body system is composed of N spatially separated d.o.f.—that we imagine as arranged over a lattice—on which k different observables (*inputs*) can be experimentally measured; and each of them can deliver p results (*outputs*). We indicate with $\sigma_a^{(i)}$ the p possible values of the ath observable ($a = 0, \ldots, k-1$) on the ith d.o.f. ($i = 1, \ldots, N$)—these choices define a (N, k, p) scenario for a Bell test [Fig. 1(a)]. Moreover, we indicate with $\langle f(\boldsymbol{\sigma})\rangle_{\rm Q}$ (where $\boldsymbol{\sigma} = \{\sigma_a^{(i)}\}$) the average value of any function f of the measurement outputs—hereafter denoted as quantum data. The DI approach certifies the strongest form of quantum correlations—Bell nonlocality—when the quantum data violate a Bell inequality [4,5], a constraint for all local-variable (LV) models, designed to capture the most general form of classical correlations. First envisioned by Bell [11], such models are defined by a joint probability distribution $P_{\rm LV}(\boldsymbol{\sigma})$ for all measurement outcomes, treated as classical variables [12]. If the dataset involves the outcomes of incompatible measurements, such a joint probability distribution is not admitted by quantum mechanics, creating a fundamental tension with LV models. In the following we indicate with $\langle \ldots \rangle_{\rm LV}$ an average over the $P_{\rm LV}$ distribution. The simplest Bell inequalities are linear combinations of few-body expectation values:

$$\sum_{i=1}^{N}\sum_{a=1}^{k}\alpha_a^{(i)}\langle\sigma_a^{(i)}\rangle_{\rm LV} + \sum_{i<j}\sum_{a,b=1}^{k}\beta_{a,b}^{(i,j)}\langle\sigma_a^{(i)}\sigma_b^{(j)}\rangle_{\rm LV} + \cdots \geq -B_c, \tag{1}$$

where $-B_c$ is the so-called classical bound. Geometrically, every such inequality defines a hyperplane in the space of correlations, separating two half-spaces, one of which contains all datasets compatible with LV models. The intersection of these half-spaces defines the so-called local polytope [Fig. 1(b)]. Certifying Bell nonlocality corresponds then to assessing that the quantum data of interest lie outside the local polytope [Fig. 1(c)].

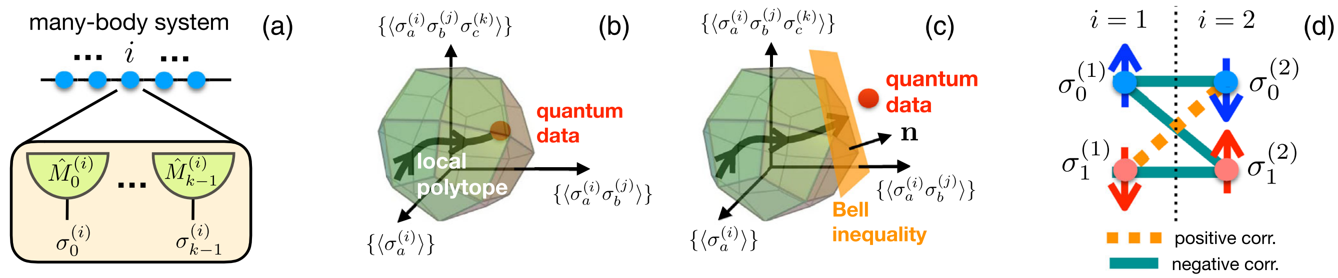

FIG. 1. Variational search of local-variable models. (a) Sketch of the generic (N, k, p) setting for Bell tests: each d.o.f. of a quantum many-body system is subject to the measurement of k different operators $\hat{M}_a$, with p different outcomes σ_a for each measurement. (b) Our Bell test of a set of quantum data—comprising arbitrary moments ($\langle\sigma_a^{(i)}\rangle$, $\langle\sigma_a^{(i)}\sigma_b^{(j)}\rangle$, etc.) of the statistics of the measurement outcomes—consists of generating a family of local-variable (LV) models which approximate the quantum data at best, describing a trajectory (black line) within the local polytope bounding all predictions of LV models; (c) if the LV predictions maintain a finite distance from the quantum data, this reveals the existence of a Bell inequality (corresponding to the closest polytope facet) which the quantum data violate. (d) Frustrated correlation pattern among local variables in a (2,2,2) setting, which an LV model should reproduce in order to realize the correlations of a Bell pair $(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle)/\sqrt{2}$.

The quantum membership problem.—The search for Bell inequalities violated by quantum many-body data for systems with $N \gg 1$ represents a formidable task. Indeed, given a quantum dataset $\{\langle f_r(\boldsymbol{\sigma})\rangle_{\mathrm{Q}}; r = 1, \ldots, R\}$—where the $f_r(\boldsymbol{\sigma})$'s are terms such as $\sigma_a^{(i)}$ or $\sigma_a^{(i)}\sigma_b^{(j)}$ in Eq. (1)—the local polytope has p^{kN} vertices, and its full reconstruction has a prohibitive (exponential) cost [4]. Many-body Bell inequalities have been successfully identified in the past [13–15], but they are violated only by selected quantum states [16]. More systematic strategies have been devised recently that either restrict the search to Bell inequalities which are fully symmetric under exchange of lattice-site indices [namely, with $\alpha_a^{(i)} = \alpha_a, \beta_{a,b}^{(i,j)} = \beta_{a,b}$, etc., in Eq. (1)] [17,18], or to inequalities which only involve a restricted range of correlations under translational invariance [19], circumventing the exponential cost but losing in generality; an alternative strategy is that of approximating the local polytope from the outside [20], with an exponential cost for the approximation to converge to the exact polytope. Hence the *quantum membership problem* ("Does a set of quantum data belong to the local polytope?") is considered to be an exponentially hard one. Our main result is to exhibit an algorithm solving this problem in polynomial time under very general assumptions; and to validate such an approach by discovering new Bell inequalities violated by relevant quantum many-body states in the thermodynamic limit.

Solving the membership problem by inverse statistical methods.—Our approach to the above problem consists of trying to explicitly build an LV model P_{LV} which reproduces the quantum data, namely, such that $\langle f_r(\boldsymbol{\sigma})\rangle_{\mathrm{LV}} = \langle f_r(\boldsymbol{\sigma})\rangle_{\mathrm{Q}}$ for all $r = 1, \ldots, R$ (within the error bar of the quantum data). In a realistic scenario, R scales polynomially with N; therefore, if such a distribution exists, it is certainly not unique, because it can be parametrized by many more parameters ($p^{kN} - 1$ independent values) than the number R of constraints. Yet, if multiple distributions exist, there is one of them which is least biased, parametrized by the *minimal* number of parameters. This distribution maximizes Shannon entropy under the constraints [21,22], or equivalently it minimizes the "free-energy" functional

$$F[P_{\mathrm{LV}}] = \sum_{\boldsymbol{\sigma}} P_{\mathrm{LV}} \log P_{\mathrm{LV}} - \sum_r K_r(\langle f_r\rangle_{\mathrm{LV}} - \langle f_r\rangle_{\mathrm{Q}}). \quad (2)$$

The solution to the minimization problem takes the form of a Boltzmann distribution [21]

$$P_{\mathrm{LV}}(\boldsymbol{\sigma}) = \exp[\sum_r K_r f_r(\boldsymbol{\sigma})]/\mathcal{Z} \quad (3)$$

in which the Lagrange multipliers K_r (forming the vector $\boldsymbol{K} = \{K_r\}$) play the role of coupling constants defining an effective Hamiltonian $\mathcal{H}(\boldsymbol{\sigma}; \boldsymbol{K}) = -\sum_r K_r f_r(\boldsymbol{\sigma})$, and $\mathcal{Z}$ is the corresponding partition function. Therefore, our central observation is the following: if a LV model reproducing the quantum data exists, it can be found in the form of Eq. (3) upon adjusting the coupling constants. In the case of binary outcomes ($p = 2$), to which we hereafter specialize, the σ's are classical Ising variables ($\sigma = \pm 1$), and therefore the LV model represents the equilibrium Boltzmann distribution of a generalized classical Ising model with Hamiltonian $\mathcal{H}$.

In summary, without loss of generality, the problem is reduced to adjusting the coupling constants of a classical Ising model so as to fit the quantum data. This, however, is a well-known problem in statistical inference, namely, an *inverse Ising problem* [23], which has the remarkable feature of being a convex optimization problem upon introducing the following cost function:

$$\mathcal{L}(\boldsymbol{K}) = \log \mathcal{Z}(\boldsymbol{K}) - \sum_r K_r \langle f_r\rangle_{\mathrm{Q}}, \quad (4)$$

where $\mathcal{L}$ is related to (minus) the log likelihood. Indeed, the Hessian of the cost function,

$$H_{rs} = \frac{\partial^2 \mathcal{L}}{\partial K_r \partial K_s} = \langle f_r f_s \rangle_{\mathrm{LV}} - \langle f_r \rangle_{\mathrm{LV}} \langle f_s \rangle_{\mathrm{LV}}, \qquad (5)$$

is the covariance matrix of the f_r functions, and it is therefore semidefinite positive. The convexity of the cost function implies that a simple gradient-descent algorithm, following the gradient $\boldsymbol{G} = \{G_r\}$ of the cost function:

$$G_r = \frac{\partial \mathcal{L}}{\partial K_r} = \langle f_r \rangle_{\mathrm{LV}} - \langle f_r \rangle_{\mathrm{Q}}, \qquad (6)$$

is guaranteed to converge to the global minimum [24].

Building a data-tailored Bell inequality.—Our algorithm presents then two possible behaviors: (i) if the quantum data are reproducible by an LV model, it converges to well-defined couplings $\boldsymbol{K}$ which lead to the vanishing of the gradient $\boldsymbol{G}$ [Eq. (6)], namely, of the distance vector between the quantum data and the LV predictions [Fig. 1(b)]; (ii) otherwise, the quantum data lie outside of the local polytope, so that $\boldsymbol{G}$ remains necessarily finite, leading to a runaway to infinity of the coupling constants as updated by the gradient-descent algorithm: $K'_r = K_r - \epsilon G_r$ (with $\epsilon \ll 1$ the step variable in the numerical implementation of the gradient descent). In this case, the algorithm converges in practice when the minimal distance $|\boldsymbol{G}_\infty|^2 = \min_{\mathrm{LV}} \sum_r \langle (f_r)_{\mathrm{LV}} - \langle f_r \rangle_{\mathrm{Q}})^2$ between the LV predictions and the classical data is attained numerically. This convergence criterion marks the fact that the variational search of the LV model has hit from the inside a facet of the polytope [Fig. 1(c)], defining a Bell inequality violated by the quantum data. The latter inequality stems from a simple rewriting of the condition $|\boldsymbol{G}_\infty|^2 > 0$, namely,

$$\sum_r G_{r,\infty} \langle f_r \rangle_{\mathrm{Q}} < \min_{\mathrm{LV}} \sum_r G_{r,\infty} \langle f_r \rangle_{\mathrm{LV}} = -B_c. \qquad (7)$$

The minimization of the right-hand side of Eq. (7)—defining the classical bound B_c—is attained as the ground-state energy of the classical Hamiltonian $\mathcal{K}$ (not to be confused with $\mathcal{H}$): $\mathcal{K}(\boldsymbol{\sigma}) = \sum_r G_{r,\infty} f_r(\boldsymbol{\sigma})$. Interestingly, we observe that $\mathcal{K}(\boldsymbol{\sigma})$ is necessarily a frustrated Hamiltonian, namely, a function whose minimum is not obtained by minimizing each term $G_{r,\infty} f_r(\boldsymbol{\sigma})$ individually. Indeed, in the absence of frustration, the quantum data has no chance of being strictly lower than the classical bound defined in Eq. (7).

Before demonstrating the practical use of our approach, we would like to point out its computational efficiency. Its strength relies fundamentally upon its data-driven nature: instead of trying to reconstruct the whole local polytope (potentially producing a large number of unviolated Bell inequalities), it directly tests for the nonlocality of a particular dataset; and it delivers the Bell inequality most strongly violated by the available quantum data. Its main computational cost is imposed by the calculation of the statistical averages $\langle f_r(\boldsymbol{\sigma}) \rangle_{\mathrm{LV}}$: such a calculation is generically efficient and scalable to arbitrary N by using classical Monte Carlo, unless the classical Ising Hamiltonian $\mathcal{H}(\boldsymbol{\sigma}; \boldsymbol{K})$ happens to be a spin-glass model—something which is categorically avoided if the quantum data have elementary spatial symmetries, and if the local observables are not chosen randomly. Otherwise, the computational cost to reach a relative precision of ε scales at worst as $\mathcal{O}(N^{n+z/d} \times \varepsilon^{-2})$ if the f_r's are correlation functions involving up to n points—here z is the dynamical critical exponent, which is nonzero [and $\mathcal{O}(1)$] only if the classical Ising model sits exactly at a critical point (see Supplemental Material for further discussion [25]).

As explained above, our approach starts from a thoughtfully chosen set of quantum data: in the following we illustrate it in three paradigmatic cases, in which the input quantum data are offered by the spin expectation values and 2-point correlation functions of (i) a Bell pair, (ii) the quantum critical point of the $d = 2$ transverse-field Ising model, and (iii) the low-energy states of the Heisenberg antiferromagnet on hypercubic lattices.

Bell pair: failure of LV theories from frustration.—We first explain the conceptual significance of our approach in the paradigmatic case of a Bell pair $(|\uparrow_1 \downarrow_2\rangle - |\downarrow_1 \uparrow_2\rangle)/\sqrt{2}$ of two $S = 1/2$ spins. In the case of a (2,2,2) scenario, choosing the measurements $\hat{\sigma}_0^{(1)} = \hat{\sigma}_x$, $\hat{\sigma}_1^{(1)} = \hat{\sigma}_y$; $\hat{\sigma}_0^{(2)} = \cos\theta \hat{\sigma}_x - \sin\theta \hat{\sigma}_y$, $\hat{\sigma}_1^{(2)} = \cos\theta \hat{\sigma}_x + \sin\theta \hat{\sigma}_y$, one obtains the following quantum data for the correlation functions, $\langle \sigma_0^{(1)} \sigma_0^{(2)} \rangle_{\mathrm{Q}} = \langle \sigma_1^{(1)} \sigma_1^{(2)} \rangle_{\mathrm{Q}} = -\cos\theta$, and $\langle \sigma_0^{(1)} \sigma_1^{(2)} \rangle_{\mathrm{Q}} = -\langle \sigma_1^{(1)} \sigma_0^{(2)} \rangle_{\mathrm{Q}} = -\sin\theta$. Notice that $\hat{\sigma}_a^{(i)}$'s are quantum operators, while $\sigma_a^{(i)}$'s are classical Ising variables representing the binary outcomes of their measurement. Choosing the optimal angle $\theta = \pi/4$ leads to correlation functions which take the common absolute value $1/\sqrt{2}$, but realize a fully frustrated correlation loop [three negative correlations and a positive one—see Fig. 1(d)]. When trying to reproduce this correlation pattern with the equilibrium state of a classical Ising model $\mathcal{H} = -\sum_{a,b \in \{0,1\}} K_{ab} \sigma_a^{(1)} \sigma_b^{(2)}$, one can easily realize that the optimal choice is to take $K_{ab} = \beta J_{ab}$ with $\beta \to \infty$ (restricting the phase space to the ground state manifold of the Hamiltonian) and $J_{00} = J_{11} = J_{01} = -J_{10}$, defining a fully frustrated square (3 antiferromagnetic couplings and a ferromagnetic one), such that $\langle \sigma_0^{(1)} \sigma_0^{(2)} \rangle_{\mathrm{LV}} = \langle \sigma_1^{(1)} \sigma_1^{(2)} \rangle_{\mathrm{LV}} = \langle \sigma_0^{(1)} \sigma_1^{(2)} \rangle_{\mathrm{LV}} = -\langle \sigma_1^{(1)} \sigma_0^{(2)} \rangle_{\mathrm{LV}} = -1/4$ (since $\mathcal{H}$ has 8 degenerate ground states, in which there is always one correlation function out of 4 with the wrong sign). As a consequence, one obtains for the gradient vector the components $G_{00} = G_{11} = G_{01} = -G_{10} = (2\sqrt{2} - 1)/4$ defining an effective Hamiltonian $\mathcal{K}$ which has the same form as $\mathcal{H}$, and which reconstructs the celebrated Clauser-Horne-Shimony-Holt (CHSH) inequality [37] $\langle \sigma_0^{(1)}\sigma_0^{(2)} + \sigma_1^{(1)}\sigma_1^{(2)} + \sigma_0^{(1)}\sigma_1^{(2)} - \sigma_0^{(1)}\sigma_1^{(2)}\rangle_{\rm LV} \geq -B_c = -2$ (while the quantum data achieve the value $-2\sqrt{2}$).

Bell inequality for the quantum Ising model at its quantum-critical point.—Moving on to many-body systems, we consider the $2d$ transverse-field Ising model at its quantum-critical (QC) point. Here, the quantum data consist of the net magnetization and pair correlation functions, and our approach reconstructs a permutationally invariant Bell inequality violated by the quantum data [25]. The relevant inequality, first identified in Ref. [17], is violated by strongly squeezed states [38–40], and squeezing is also a property of the QC point in question [41]. Yet our approach allows us to make a stronger statement, namely, that—given the $(N, 2, 2)$ scheme with measurement bases suggested by Ref. [38]—a symmetric Bell inequality is the optimal one, namely, the one which is most strongly violated, for quantum data that are not at all symmetric (unlike those produced in the experiments of Refs. [38,39]), because of the spatial decay of correlations functions at criticality.

Bell inequality for Heisenberg antiferromagnets.—We conclude our article by focusing on the equilibrium states of a paradigmatic quantum spin-lattice model, namely, the quantum Heisenberg antiferromagnet (QHAF) with Hamiltonian $\hat{\mathcal{H}} = J\sum_{\langle ij\rangle}\hat{\boldsymbol{S}}^{(i)}\cdot\hat{\boldsymbol{S}}^{(j)}$, where $\hat{\boldsymbol{S}}^{(i)}$ are quantum $S = 1/2$ operators, and the sum runs over pairs of nearest neighbors on a hypercubic lattice with an even number of sites. The ground state of this model realizes a global singlet, namely, a many-body generalization of the Bell pair considered above. We focus on a $(N, k, 2)$ scenario, with $k \geq 3$ measurements per spin along axes $\boldsymbol{n}_a$ $(a = 0, 1, \ldots, k-1)$, and we consider a uniform measurement strategy in which the axes are coplanar and form an angle $a\pi/k$ with a given reference axis (this choice turns out to be optimal [25])—see Fig. 2(a). Feeding our algorithm with the two-point correlation function of the $2d$ QHAF with $k = 3$ measurements as quantum data, we discover that the latter violate the following symmetric Bell inequality:

$$\langle\mathcal{B}\rangle_{\rm LV} = \sum_{a=0}^{k-1}\mathcal{S}_{aa} + \sum_{a=0}^{k-2}\mathcal{S}_{a,a+1} - \mathcal{S}_{k-1,0} \geq -B_c = -2N(k-1), \tag{8}$$

where $\mathcal{S}_{ab} = \sum_{i\neq j}\langle\sigma_a^{(i)}\sigma_b^{(j)}\rangle_{\rm LV}$. This inequality (proven in the Supplemental Material [25]) turns out to be a many-body extension of the Pearle-Braunstein-Caves inequality [42,43] proposed for nonlocality detection in two-spin states. Similarly to the above-cited example of the QC point of the $2d$ quantum Ising model, it is remarkable to notice that quantum data with spatial structure—such as the correlation function of the $2d$QHAF—are found to most strongly violate a Bell inequality in which the spatial structure is washed out by the symmetrization procedure.

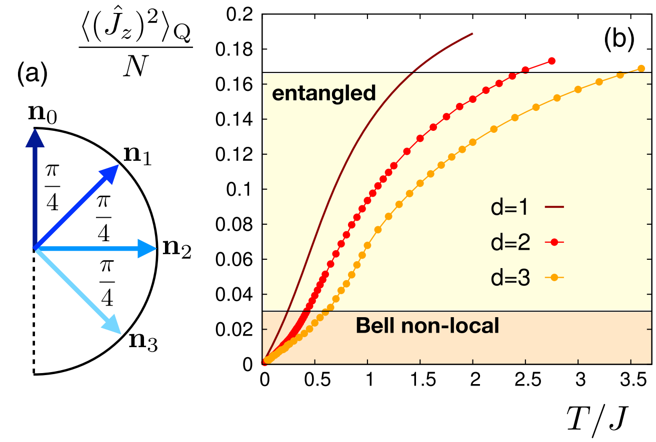

FIG. 2. Many-body Bell nonlocality of quantum Heisenberg antiferromagnets. (a) Measurement basis ($k = 4$) providing the strongest violation of the Bell inequality in Eq. (8) by the low-temperature data of quantum Heisenberg antiferromagnets. (b) Normalized fluctuations of a collective spin component $\langle(\hat{J}^z)^2\rangle/N$ in Heisenberg antiferromagnets on the linear chain ($d = 1$), the square lattice ($d = 2$) and the cubic lattice ($d = 3$). The data shown are obtained via the Bethe-Ansatz prediction [44] for $d = 1$, and by quantum Monte Carlo ($d = 2, 3$) on lattices of size 30^2 and 12^3, respectively—the thermodynamic limit values are essentially reached for these sizes. When the fluctuations become smaller than the classical bound β_4 (see text) they witness the appearance of Bell nonlocality; in the figure we report as well the known bound for witnessing entanglement [45,46].

To see explicitly that the ground state of the QHAF violates the inequality of Eq. (8), we make use of the SU(2) invariance to rewrite the Bell operator $\hat{\mathcal{B}}$ in the form [25]: $\hat{\mathcal{B}} = 4k[1+\cos(\pi/k)]\hat{J}_z^2 - Nk\cos(\pi/k) - Nk$, where $\hat{J}_z = \sum_i \hat{S}_z^{(i)}$ is the collective spin along z. Therefore, the classical bound $-B_c$ is violated by the quantum data whenever

$$\frac{\langle\hat{J}_z^2\rangle_{\rm Q}}{N} < \beta_k = \frac{1}{4} - \frac{k-1}{2k(1+\cos\frac{\pi}{k})}, \tag{9}$$

where the largest value of the right-hand side is found for $k = 4$, and it reads $\beta_4 = 1/(16+12\sqrt{2}) = 0.030330\ldots$. Equation (9) states that a sufficiently low value of the variance of one collective spin component (below the β_4 bound) is a witness [38] of Bell nonlocality ["witness" because, in order to derive it, we explicitly used the spin algebra as well as the hypothesis of SU(2) invariance of the state]. The ground state of all Heisenberg antiferromagnets with even N (regardless of the geometry of the underlying lattice) are total spin singlets (such that $\langle\hat{J}_z^2\rangle_{\rm Q} = 0$). Hence they satisfy the above criterion and violate the Bell inequality of Eq. (8). Moreover, in the Supplemental

Material [25] we show that the quantum violation of the inequality Eq. (8) offered by total spin singlets, $\langle\hat{\mathcal{B}}\rangle_{\mathrm{Q}} = -Nk[1+\cos(\pi/k)]$, is the maximal violation authorized by quantum mechanics (namely, regardless of the dimension of the Hilbert space of the system, of its quantum state, and of the chosen measurements). Figure 2(b) shows that the condition Eq. (9) is also met by thermal equilibrium states of the QHAF in $d = 1$, 2, and 3 up to very sizable temperatures (the higher the larger d is, as nonlocality is clearly protected by the strength of antiferromagnetic correlations). The condition of Eq. (9) is to be contrasted with the much looser one, $\langle\hat{J}_z^2\rangle_{\mathrm{Q}}/N < 1/6$ required to witness entanglement between the individual spins [45,46]—namely, to exclude the possibility of writing the state of the system as $\hat{\rho} = \sum_s p_s \otimes_i \hat{\rho}_s^{(i)}$, where $\hat{\rho}_s^{(i)}$ are arbitrary (pure or mixed) states of individual spins. This reflects the fact that, for mixed states, Bell nonlocality is a much stronger form of quantum correlations than entanglement. Moreover the fundamental connection between the collective spin variance and the spin susceptibility at thermal equilibrium $\chi_z = \langle\hat{J}_z^2\rangle_{\mathrm{Q}}/(k_B T N)$ (where T is the temperature) makes the above witness of nonlocality experimentally accessible to magnetometry experiments on quantum magnets at realistic temperatures.

*Conclusions.—*We have demonstrated a variational approach which can assess whether an arbitrary set of quantum data, coming from scalable many-body systems, exhibits quantum nonlocality; and which reconstructs the Bell inequality most strongly violated by the data at hand. The computational cost of the algorithm is polynomial in system size whenever the quantum data are not obtained from systems governed by random Hamiltonians, and are not obtained by using a random measurement basis for each d.o.f.—and it may still remain polynomial even if the above conditions are not met. Therefore, our approach opens the door to scalable and systematic certification of entanglement in synthetic quantum matter (quantum simulators [47,48], quantum processors [49,50]). When the violated Bell inequalities have a symmetric structure under the exchange of d.o.f. (as in the case of the Heisenberg antiferromagnets reported in this work), a witness of Bell nonlocality can be formulated by using collective observables only [38], and the latter is therefore accessible in the broader context of quantum materials in condensed matter physics.

We warmly thank A. Acín, A. Aloy, A. Aspect, F. Baccari, C. Branciard, M. Lewenstein, P. Ronceray, L. Tagliacozzo, and J. Tura for insightful discussions and encouragement. This work is supported by ANR (“EELS” project), QuantERA (“MAQS” project), the Spanish MINECO (Severo Ochoa SEV-2015-0522), the fundacio Mir-Puig and Cellex through an ICFO-MPQ postdoctoral fellowship, and the Generalitat de Catalunya (SGR 1381, QuantumCAT and CERCA Programme).

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[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.140504 for a discussion of: (i) the implementation of the variational approach; (ii) the reconstruction of the Bell inequality violated by the quantum critical point of the 2d quantum Ising model; (iii) the proof of the many-body Pearle-Braunstein-Caves inequality; (iv) a detailed discussion of how quantum data coming from Heisenberg antiferromagnets can violate the above inequality and (v) the proof that this violation is the maximal one admitted by quantum mechanics, which includes the Refs. [26–36].

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