Test of the Eigenstate Thermalization Hypothesis Based on Local Random Matrix Theory

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We verify that the eigenstate thermalization hypothesis (ETH) holds universally for locally interacting quantum many-body systems. Introducing random matrix ensembles with interactions, we numerically obtain a distribution of maximum fluctuations of eigenstate expectation values for different realizations of interactions. This distribution, which cannot be obtained from the conventional random matrix theory involving nonlocal correlations, demonstrates that an overwhelming majority of pairs of local Hamiltonians and observables satisfy the ETH with exponentially small fluctuations. The ergodicity of our random matrix ensembles breaks down because of locality.

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Introduction.—Deriving statistical mechanics from unitary dynamics of isolated quantum systems has been a holy grail since von Neumann [1]. The last two decades have witnessed a resurgence of interest in this problem [2–6], partly motivated by experiments in ultracold atoms [7–13] and ions [14,15].

The eigenstate thermalization hypothesis (ETH) is widely accepted as the main scenario for thermalization in isolated quantum systems [16–18]. The ETH states that every energy eigenstate is thermal and ensures that any initial state relaxes to thermal equilibrium. Despite considerable efforts [19–44], the rigorous proof of this hypothesis has remained elusive.

A popular approach to understanding the universal validity of the ETH is to invoke the typicality argument [1], which gives a mathematically rigorous bound on the probability weight of the ETH-breaking Hamiltonians, thereby demonstrating that an overwhelming majority of Hamiltonians satisfy the ETH [1,25–27,45]. It is tempting to argue that most realistic Hamiltonians satisfy the ETH because most Hamiltonians do. However, almost all Hamiltonians considered in Ref. [45] involve nonlocal and many-body operators. In fact, the typicality argument has recently been demonstrated to be inapplicable to a set of local Hamiltonians and local observables [46].

Another approach is to numerically test the ETH for physically realistic models involving local interactions between spins [21,22,30,31,33,34,36,37,41,42], fermions [24,42,43,47], and bosons [20,24,28,29,38,47]. This approach cannot clarify how generally the ETH applies to physical systems. Indeed, recent studies have revealed exceptional systems for which the ETH breaks down: examples include systems with an extensive number of local conserved quantities [7,48–56], many-body localization (MBL) [9,13,15,57–63], and quantum many-body scars [64–73].

In this Letter, we present the first evidence that the ETH *universally* holds true for *locally* interacting quantum many-body systems. We introduce random matrix ensembles constructed from local interactions and investigate their generic properties. In particular, we evaluate the weight of the ETH-breaking Hamiltonians by numerically obtaining distributions of fluctuations of eigenstate expectation values [74]. We find that the ETH with exponentially small fluctuations is satisfied for an overwhelming majority of ensembles with local interactions. The obtained distribution shows that the fraction of exceptions is less suppressed for local ensembles than the conventional random matrix ensemble, which involves nonlocal interactions and many-body interactions. Here, by many body, we mean that the number of particles involved is comparable with the total number of particles. If we allow less local interactions, the distribution rapidly approaches that predicted by the conventional random matrix theory. We find that the ergodicity of our random matrix ensembles breaks down because of locality.

Local random matrix ensembles.—We consider N spins on a one-dimensional lattice with the periodic boundary condition and ensembles of Hamiltonians with local interactions. We denote the local Hilbert space on each site as $\mathcal{H}_{\rm loc}$ and the total Hilbert space as $\mathcal{H}_N \coloneqq \mathcal{H}_{\rm loc}^{\otimes N}$. We choose an arbitrary orthonormal basis $\mathcal{B}_{\rm loc} = \{|\sigma\rangle\}$ of $\mathcal{H}_{\rm loc}$ and define the corresponding basis of $\mathcal{H}_N$ as $\mathcal{B}_N = \{|\sigma_1,\ldots,\sigma_N\rangle| \ \forall \ j, |\sigma_j\rangle \in \mathcal{B}_{\rm loc}\}$. The translation operator $\hat{T}_N$ acting on $\mathcal{H}_N$ satisfies $\hat{T}_N|\sigma_1\sigma_2,\ldots,\sigma_N\rangle \coloneqq |\sigma_{2,}\ldots,\sigma_N\sigma_1\rangle$ for all $|\sigma_1\sigma_{2,}\ldots,\sigma_N\rangle \in \mathcal{B}_N$.

Let $\mathcal{L}(\mathcal{H})$ be the space of all Hermitian operators acting on a Hilbert space $\mathcal{H}$. For a given Hamiltonian $\hat{H}$, an energy shell $\mathcal{H}_{E,\delta E}$ centered at energy E with width $2\delta E$ is defined as $\mathcal{H}_{E,\delta E} \coloneqq \mathrm{span}\{|E_\alpha\rangle \,|\, |E_\alpha - E| \le \delta E\}$, where $|E_\alpha\rangle$ is an eigenstate of the Hamiltonian $\hat{H}$ with eigenenergy E_α. We randomly choose a local Hamiltonian $\hat{h}^{(l)}$ from the space $\mathcal{L}(\mathcal{H}_{\mathrm{loc}}^{\otimes l})$ with respect to the Gaussian unitary ensemble. We call an element of $\mathcal{L}(\mathcal{H}_{\mathrm{loc}}^{\otimes l})$ an l-local operator and an integer $l \in \mathbb{N}$ the locality of an interaction. We define the range of the spectrum of an operator $\hat{O}$ as $\eta_O \coloneqq \max_\alpha a_\alpha - \min_\alpha a_\alpha$, where a_α's are eigenvalues of $\hat{O}$. We consider Hamiltonians of the form

$$\hat{H}_N \coloneqq \sum_{j=0}^{N-1} \hat{T}_N^j \hat{h}_j^{(l)} \hat{T}_N^{-j} \tag{1}$$

and introduce the following three types of ensembles [76]: Case 1: $h_j^{(l)} = h^{(l)}$ for all j. Case 2: $h_j^{(l)}$ is normalized so that $\eta_{h_j^{(l)}} = \eta$ for all j. Case 3: No restrictions.

The number of parameters needed to characterize a single Hamiltonian increases from case 1 to case 3 [79]. We randomly choose an l-local observable $\hat{o}^{(l)} \in \mathcal{L}(\mathcal{H}_{\mathrm{loc}}^{\otimes l})$ from the Gaussian unitary ensemble and construct an extensive observable $\hat{O}_N \in \mathcal{L}(\mathcal{H}_N)$ as in Eq. (1) with $\hat{o}_j^{(l)} = \hat{o}^{(l)}$ for all j.

Measure of the strong ETH.—We focus on the strong ETH, which asserts that all eigenstates should be thermal. While several definitions for a measure of the ETH have been proposed [19–23,45,46], we consider a measure applicable to generic local systems. We require that the measure be (i) invariant under linear transformations: $\hat{H} \mapsto a\hat{H} + b, \hat{O} \mapsto a'\hat{O} + b'$, (ii) dimensionless, (iii) thermodynamically intensive, and (iv) applicable to eigenstate expectation values after subtraction of weak energy dependences. Here, (i) is needed because the measure of the strong ETH should be invariant against a change of physical units and translation, (ii) is needed because we compare quantities with different physical dimensions, and (iii) is needed because we admit subextensive fluctuations from a macroscopic point of view. Finally, (iv) is important because the energy dependence generically appears in the presence of local interactions. Such a dependence invalidates the typicality argument based on a unitary Haar measure unless the energy width is exponentially small [46]. Since this energy dependence of a macroscopic observable can be observed, it should not be considered to be a part of fluctuations of eigenstate expectation values. To be concrete, consider a measure of the strong ETH as $\tilde{\Delta}_\infty \coloneqq \max_{\alpha: |E_\alpha\rangle \in \mathcal{H}_{E,\delta E}} |O_{\alpha\alpha} - \langle \hat{O} \rangle_{\delta E}^{\mathrm{mc}}(E)| / \eta_O$, where $\langle \cdots \rangle_{\delta E}^{\mathrm{mc}}(E)$ is the microcanonical average within $\mathcal{H}_{E,\delta E}$. This is essentially the same quantity as that used in Ref. [45]. The scaling behavior of this measure depends on the scaling of the energy width δE, which is inappropriate as the measure of the strong ETH. Such an energy dependence is removed if we consider an eigenstate-dependent microcanonical energy shell and introduce the following measure:

$$\Delta_\infty \coloneqq \frac{\max_\alpha |O_{\alpha\alpha} - \langle \hat{O} \rangle_{\delta E}^{\mathrm{mc}}(E_\alpha)|}{\eta_O}, \tag{2}$$

where the maximum is taken from the middle 10% of the energy spectrum to avoid finite-size effects at both edges of the spectrum where the density of states is small. The strong ETH implies that $\Delta_\infty \to 0$ in the thermodynamic limit.

We employ the exact diagonalization method to investigate the universality of the ETH. For case 1, we restrict ourselves to the zero-momentum sector. An analytical method based on a uniform-random-vector method over the Haar measure [45] can no longer be applied to our Hamiltonians because of their local structures [46].

Strong ETH for almost all local random matrices.—We numerically obtain the distributions of Δ_∞ for several system sizes N and locality l. We first demonstrate that the ETH holds true for almost all local random matrices on the basis of Markov's inequality,

$$\mathrm{Prob}_N^{(l)}[\Delta_\infty \ge \epsilon] \le \frac{\mathbb{E}_N^{(l)}[\Delta_\infty]}{\epsilon}. \tag{3}$$

Here, Prob and $\mathbb{E}$ denote the probability and the expectation value with respect to random realizations of $\hat{H}$ and $\hat{O}$. The vanishing of $\mathbb{E}_N^{(l)}[\Delta_\infty]$ in the thermodynamic limit is a sufficient condition for the strong ETH with an arbitrary constant $\epsilon > 0$ for almost all sets of local Hamiltonians and observables.

We compare our numerical results with the prediction of conventional random matrix theory, whose asymptotic N dependence is obtained as [see discussions after Eq. (6)]

$$\mathbb{E}_N[\Delta_\infty] = m_0 N e^{-N/N_m} \sqrt{1 - \frac{N_m}{2}\frac{\log N}{N} - \frac{N_0}{N}} \tag{4}$$

for case 1 and

$$\mathbb{E}_N[\Delta_\infty] = m_0 N^{1/2} e^{-N/N_m} \sqrt{1 - \frac{N_0}{N}} \tag{5}$$

for cases 2 and 3, where m_0, N_0, and N_m are constants. As shown in Fig. 1, these formulas fit well to our numerical data for all the ensembles irrespective of locality l. While Eqs. (4) and (5) are expected to apply to a less local case (i.e., l is large) with not too small system sizes, where $\mathrm{Prob}_N^{(l)}[\Delta_\infty \ge \epsilon]$ itself is close to that for the conventional random matrix theory, they fit well to the cases with strong

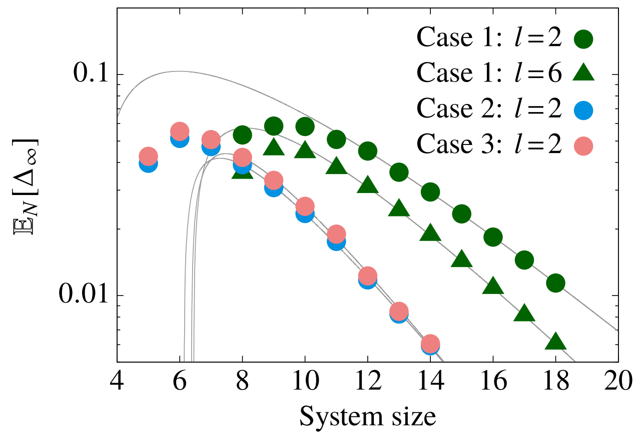

FIG. 1. Mean value of the measure in Eq. (2) for various ensembles. The solid curve is the fitting function in Eq. (4) for case 1 or Eq. (5) for cases 2 and 3 (see Supplemental Material [80]). The values of the fitting parameters (N_m, N_0, m_0) are (3.20,2.90,0.21) for case 1 with $l=2$, (2.71,5.26,0.33) for case 1 with $l=6$, (2.29,6.13,0.94) for case 2 with $l=2$, and (2.20,6.37,1.25) for case 3 with $l=2$. The values N_m for cases 2 and 3 are smaller than the expected value $2/\log 2$ owing to a finite-size effect. The number of samples lies between 7980 and 947 770 for all data points.

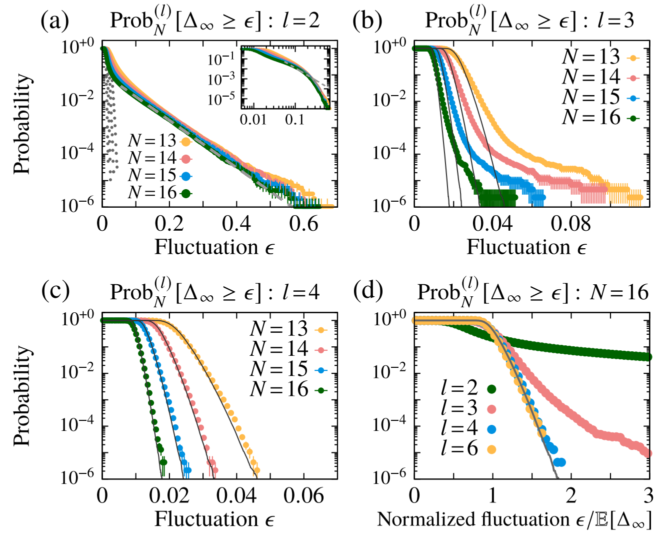

FIG. 2. (a) $\mathrm{Prob}_N^{(l)}[\Delta_\infty \geq \epsilon]$ with $l=2$ as a function of ϵ for various system sizes (colored dots). The distributions for $l=6$ (gray dots) with $N = 13 \sim 16$ are shown for comparison. The gray dashed lines show exponential functions of the form $C\exp(-\epsilon/\epsilon_0)$. The dashed line in the inset (log-log plot) shows a polynomial function of the form $(\epsilon_1/\epsilon)^a$. The tail fittings are performed in the region $\epsilon > 3\mathbb{E}_N[\Delta_\infty]$. The number of samples is 947 770. (b),(c) $\mathrm{Prob}_N^{(l)}[\Delta_\infty \geq \epsilon]$ with $l=3$ and 4, respectively. (d) Distribution of Δ_∞ normalized with $\mathbb{E}_N^{(l)}[\Delta_\infty]$ for case 1 with $l=2$, 3, 4, 6 and $N=16$. The gray line is a maximum value distribution predicted from the conventional random matrix theory, which is rescaled so that its mean becomes unity.

locality ($l=2$), where $\mathrm{Prob}_N^{(l)}[\Delta_\infty \geq \epsilon]$ is distinct from that of conventional random matrix theory as discussed later.

The exponential decay of $\mathbb{E}_N^{(l)}[\Delta_\infty]$ allows one to make both ϵ and the right-hand side of Eq. (3) exponentially small by taking $\epsilon_N \propto \exp(-N/N_1)$ with $N_1 > N_m$. This means that the strong ETH with exponentially small fluctuations [$\sim\exp(-N/N_1)$] holds for an overwhelming majority of the ensemble, where the fraction of exceptional cases is exponentially small [83]. We also note that N_m is close to $2/\log 2$, since the standard deviation of $O_{\alpha\alpha}$ decays as $1/\sqrt{d_{\rm sh}}$ [84] and $d_{\rm sh} \propto \dim \mathcal{H}_N = 2^N/N$ or 2^N, where $d_{\rm sh} \coloneqq \dim \mathcal{H}_{E,\delta E}$ irrespective of l unless δE decreases exponentially with N (see Supplemental Material [80]).

Notably, the ETH universally holds even for cases 2 and 3. Our results show that MBL rarely occurs for these types of spatial disorder. This is similar to the many-body chaos found in random unitary circuits [85,86]. Our results further suggest that the randomness does not prevent thermalization even with energy conservation owing to continuous-time evolution. We also examine the argument that localization may occur when the magnitude of the sum of off-diagonal elements of a Hamiltonian exceeds that of a diagonal one. Assuming independent Gaussian elements, we estimate the probability that a sample may show MBL to be $\sim\exp[-\mathcal{O}(Nd_{\rm loc}^l)]$ [80]. However, since off-diagonal elements of our Hamiltonians are highly correlated because of the spatial locality, the relevance of the above estimate remains unclear [87].

Distribution of the maximum fluctuation.—Since Markov's inequality gives only a loose upper bound, we directly obtain distributions of Δ_∞ for several values of N. Below, we focus on case 1 (see Supplemental Material for cases 2 and 3 [80]). The results are shown in Fig. 2, where the distribution with $l=2$ [Fig. 2(a)] is distinct from the prediction of the conventional random matrix theory involving nonlocal operators. We find that its tail decays single exponentially or slightly slower than a single exponential $\exp(-\epsilon/\epsilon_1)$, unlike the conventional random matrix theory, which predicts a much faster decay of the tail as $\exp[-\mathcal{O}(\epsilon^2)]$. This suggests that locality favors Hamiltonians with relatively large Δ_∞ because of the closeness to those Hamiltonians that are integrable or host scars. The distribution of Δ_∞ for case 1 with $l=3$ shows a crossover from a rapid decay in the region $\mathrm{Prob}_N[\Delta_\infty \geq \epsilon] \gtrsim P_c \sim 1.0\times 10^{-3}$ followed by a slower decay in the region $\mathrm{Prob}_N[\Delta_\infty \geq \epsilon] \lesssim P_c \sim 1.0\times 10^{-3}$ [Fig. 2(b)]. This behavior is similar to the case with $l=2$, but P_c is much smaller ($P_c \sim 1.0\times 10^{-2}$ for $l=2$). Our finite-size scaling analysis suggests that these deviations from the random matrix theory prediction persist in the thermodynamic limit.

As the locality l increases, the distributions of Δ_∞ rapidly approach the prediction of the conventional random matrix theory (RMT), where the fluctuations of eigenstates distribute according to the Gaussian distribution with zero mean and the identical variance s_N^2 for each sample. Indeed, as shown in Figs. 2(c) and 2(d), even for l as small as 4, $\mathrm{Prob}_N^{(l)}[\Delta_\infty \geq \epsilon]$ is well fitted by the cumulative function of the maximum absolute value of d_{sh}-independent and identically distributed Gaussian variables,

$$\mathrm{Prob}_N^{(\mathrm{RMT})}[\Delta_\infty \geq \epsilon] = 1 - \mathrm{CDF}(\epsilon) = 1 - \left[\mathrm{erf}\left(\epsilon/\sqrt{2s_N^2}\right)\right]^{d_{\mathrm{sh}}}, \quad (6)$$

where $\mathrm{erf}(x)$ is the error function and $d_{\mathrm{sh}} \coloneqq \dim \mathcal{H}_{E,\delta E}$.

The extreme value theory [91] allows us to obtain the asymptotic form of the cumulative distribution function (CDF): if we set $b_N \sim s_N\sqrt{2\log d_{\mathrm{sh}}}$ and $a_N = s_N^2/b_N$, the right-hand side in Eq. (6) converges to the Gumbel distribution $\mathrm{Prob}_N^{(\mathrm{RMT})}[\Delta_\infty \geq \epsilon] \sim 1 - \exp[-e^{-(\pi/\sqrt{6})y-\gamma}]$ for large d_{sh}, where $y \coloneqq (\epsilon - b_N)/a_N$ is a rescaled random variable, and $\gamma \simeq 0.577$ is the Euler-Mascheroni constant [80]. This fact implies that $\mathbb{E}_N[\Delta_\infty] \simeq b_N \sim s_N\sqrt{2\log d_{\mathrm{sh}}}$ and $\mathbb{S}_N[\Delta_\infty] \simeq a_N \sim s_N/\sqrt{2\log d_{\mathrm{sh}}}$, where $\mathbb{S}_N$ denotes the standard deviation. This distribution is applicable in the range $\epsilon = \mathbb{E}_N[\Delta_\infty] + c\mathbb{S}_N[\Delta_\infty]$, where c is a constant of $\mathcal{O}(1)$ with respect to N.

These formulas lead to the asymptotic N dependence of $\mathbb{E}_N[\Delta_\infty]$ in the conventional random matrix regime. Since $d_{\mathrm{sh}} = e^{-2N_0/N_m} \dim \mathcal{H}_N$ and $s_N \propto (d_{\mathrm{sh}})^{-1/2}$ for sufficiently large N, we obtain the asymptotic formulas in Eqs. (4) and (5) by inserting $\dim \mathcal{H}_N = d_{\mathrm{loc}}^N/N$ for case 1 and $\dim \mathcal{H}_N = d_{\mathrm{loc}}^N$ for cases 2 and 3 in $\mathbb{E}_N[\Delta_\infty] \sim s_N\sqrt{2\log d_{\mathrm{sh}}}$, and by setting $d_{\mathrm{loc}}^N = e^{2N/N_m}$.

Ergodicity breaking for local random matrices.—Let us now discuss the structure of the expectation values over eigenstates for random realizations of sets of local Hamiltonians and observables.

Srednicki conjectured [84] that the fluctuations of expectation values can be expressed as $\delta(O_N)_{\alpha\alpha} \sim e^{-(S_N(E)/2)} f_O(E)\tilde{R}_{\alpha\alpha}$. Here, $S_N(E)$ is the thermodynamic entropy of the system that depends only on the Hamiltonian $\hat{H}_N$, $f_O(E)$ is a smooth function of energy E that depends on $\hat{H}_N$ and $\hat{O}_N$, and $\tilde{R}_{\alpha\alpha}$ distributes according to the normal Gaussian distribution.

We test the above conjecture for our three ensembles for $l = 2$ as a local case and $l = 8$ as a nonlocal case. We find that a majority of systems satisfy Srednicki's conjecture irrespective of the locality; that is, the standard deviation of $\delta(O_N)_{\alpha\alpha}$ inside an energy shell $\mathcal{H}_{E,\delta E}$ scales as $\propto (\sqrt{d_{\mathrm{sh}}})^{-a} \sim e^{-a(S_N(E)/2)}$ with $a \simeq 1$ (Fig. 3), and $\tilde{R}_{\alpha\alpha}$ distributes according to the normal Gaussian [80]. Since the probability density peaks more sharply around unity as we increase the system size, we expect that Srednicki's conjecture holds typically in the thermodynamic limit. However, the rate of decrease is relatively slow, especially at small a for the distribution with $l = 2$ [Fig. 3(a)].

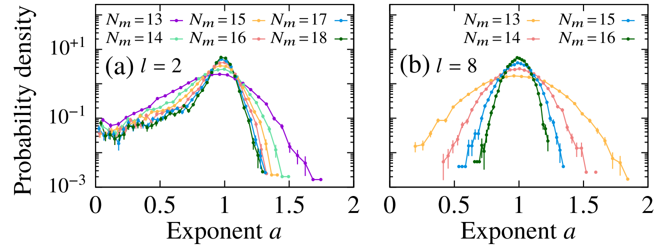

FIG. 3. Distribution of the value of a in the fitting of the standard deviation $\mathcal{S}_\gamma^{(E,\delta E)}[\delta(O_N)_{\gamma\gamma}] \propto (\sqrt{d_{\mathrm{sh}}})^{-a}$ in the shell $\mathcal{H}_{E,\delta E}$ for the case-1 ensemble with (a) $l = 2$ and (b) $l = 8$, where E is chosen to be the center of the spectrum and δE is 5% of the spectral range. The fittings are performed in the region $8 \leq N \leq N_m$. The number of samples is 10 000 for each panel.

We also find that sample-to-sample fluctuations become large for local random matrices. The ergodicity of a random matrix ensemble [92], which means that the spectral average equals the ensemble average, does not apply to situations with locality. We observe its signature in Fig. 3(a), where samples with small a exist for $l = 2$, while the distribution concentrates around $a = 1$ for a less local case with $l = 8$. Atypical samples with small a have multifractal eigenstates even in the middle of the spectrum [80].

Figure 4 shows the mean of the $L2$ norm δ defined by

$$\delta \coloneqq \left[\frac{1}{N_{\mathrm{bin}}}\sum_{\mathrm{bin}=1}^{N_{\mathrm{bin}}} \left(\langle X(E_\alpha)\rangle_{\mathrm{bin}} - \mathbb{E}_N[\langle X(E_\alpha)\rangle_{\mathrm{bin}}]\right)^2\right]^{1/2}, \quad (7)$$

where $\langle\cdots\rangle_{\mathrm{bin}}$ denotes the average inside each bin and $\mathbb{E}_N[\cdots]$ denotes the ensemble average. Figures 4(a) and 4(b) show the N dependence of $X(E) = f_O(E)$ and the normalized density of states $X(E) = \rho(E)$ (i.e., the density of states divided by $\dim \mathcal{H}_N$), respectively, for $N_{\mathrm{bin}} = 100$. The mean of the $L2$ norm for $f_O(E)$ or $\rho(E)$ decreases with N for case 1 with $l = 8$, which is a manifestation of ergodicity in the conventional random matrix ensemble [92]. It converges to a finite value [for $\rho(E)$] or even increasing [for $f_O(E)$] for local ensembles with $l = 2$ for

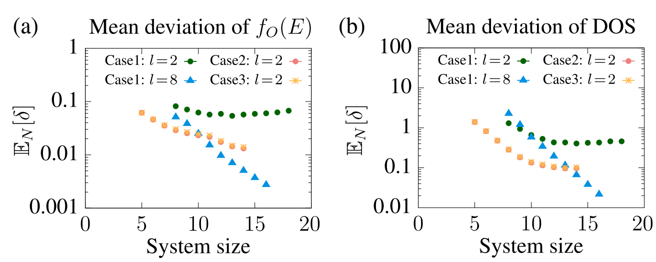

FIG. 4. Mean $\mathbb{E}_N[\delta]$ of the $L2$ norm from the ensemble averages for (a) $f_O(E)$ in Srednicki's conjecture and (b) the normalized density of states $\rho(E)$. The number of samples is 48 800 for case 1 with $l = 2$, 43 293 for case 1 with $l = 8$, and 10 000 for cases 2 and 3.

case 1, which indicates the breakdown of ergodicity. The breakdown of ergodicity for local ensembles can also be seen in the density of states $\rho(E)$ for cases 2 and 3. The mean deviation for $f_O(E)$ seems to level off once around $N = 11$ but then continues to decrease for larger N for these cases. We thus cannot conclusively judge whether ergodicity breaks down with currently achievable system sizes for f_O in cases 2 and 3.

Conclusion.—We find that the locality of interactions on an ensemble of Hamiltonians makes distributions of local observables significantly different from those of the conventional random matrix theory. However, the strong ETH with exponentially small fluctuations holds true for an overwhelming majority of the ensemble, where the fraction of exceptions is exponentially small. We also find that ergodicity of random matrix ensembles breaks down because of locality. We expect that the universality of the ETH still holds true for higher dimensions, since integrability such as Bethe-ansatz solvability is unique to 1D and MBL seems unstable in higher dimensions [93].

While the universality of the ETH is confirmed in all three ensembles studied here, it is of fundamental interest to investigate whether imposing additional conserved quantities can prevent the universality. It is of interest to examine whether the ETH-MBL transition occurs if we implement more structured randomness than the case-2 and case-3 ensembles, such as ensembles where the strengths of one- and two-site disorder are different. Our ensembles can provide a relaxation timescale of generic interacting Hamiltonians with locality, which is not taken into account in related works [94–96].

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