Incorporating Heisenberg's Uncertainty Principle into Quantum Multiparameter Estimation

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The quantum multiparameter estimation is very different from the classical multiparameter estimation due to Heisenberg's uncertainty principle in quantum mechanics. When the optimal measurements for different parameters are incompatible, they cannot be jointly performed. We find a correspondence relationship between the inaccuracy of a measurement for estimating the unknown parameter with the measurement error in the context of measurement uncertainty relations. Taking this correspondence relationship as a bridge, we incorporate Heisenberg's uncertainty principle into quantum multiparameter estimation by giving a trade-off relation between the measurement inaccuracies for estimating different parameters. For pure quantum states, this trade-off relation is tight, so it can reveal the true quantum limits on individual estimation errors in such cases. We apply our approach to derive the trade-off between attainable errors of estimating the real and imaginary parts of a complex signal encoded in coherent states and obtain the joint measurements attaining the trade-off relation. We also show that our approach can be readily used to derive the trade-off between the errors of jointly estimating the phase shift and phase diffusion without explicitly parametrizing quantum measurements.

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The random nature of quantum measurement imposes ultimate limits on the precision of estimating unknown parameters with quantum systems. Quantum parameter estimation theory has been developing for more than half a century to reveal and pursue the quantum-limited measurement [1–9]. In classical parameter estimation theory, the Cramér-Rao bound (CRB) together with the asymptotic normality of the maximum likelihood estimator give a satisfactory approach to derive the asymptotically attainable accuracy of estimation, where the Fisher information matrix (FIM) plays a pivotal role [10–17]. The CRB and the FIM have been extended to the quantum regime [1–6], where not only estimators—data processing—but also quantum measurements are taken into consideration in optimization.

For single-parameter estimation, Helstrom's version of quantum CRB can be attained at large samples due to the asymptotic efficiency of adaptive measurements [8,18–20]. However, unlike the classical parameter estimation, the quantum CRB does not possess the asymptotic attainability in general for multiparameter estimation. This can be understood as a consequence of the fact that the optimal measurements for different parameters may be incompatible in quantum mechanics so that they cannot be jointly performed according to Heisenberg's uncertainty principle (HUP) [21,22]. Many application scenarios, e.g., superresolution imaging [23,24], quantum enhanced estimation of a magnetic field [25,26], and joint estimation of phase shift and phase diffusion [27], essentially belong to quantum multiparameter estimation problems. Therefore, the characterization of the quantum-limited bound on the estimation errors is of great importance to many practical applications of quantum estimation. Nevertheless, it is still challenging to derive, characterize, and understand the quantum limit on accuracies of the multiparameter estimation [9,28–48].

Because of the difficulty in identifying the boundary between the forbidden and permissible regions of error composition, as a compromise, many prior error bounds are formulated in terms of the weighted mean errors of estimation [1–3,6,9,28–48]. The most powerful lower bound on weighted mean errors up to now is the Holevo bound [6,30], which is asymptotically attainable by collective measurements on a large number of identical samples [49–53]. However, the Holevo bound contains an optimization over a set of special operators, so that the evaluation of the Holevo bound is difficult [30]. Remarkably, Carollo *et al.* derived an upper bound on the discrepancy ratio between the Holevo bound and Helstrom's version of quantum CRB through a quantity measuring the incompatibility regarding different parameters [28]. With these lower and upper bounds, we can reveal the quantum limits on weighted mean errors; nevertheless, it is still difficult to completely identify the trade-off curve or surface regarding the attainable errors for estimating different parameters [34,42]. It is still unclear how the HUP affects the boundary of the attainable errors.

In this work, we tackle the problem of completely identifying the boundary of the attainable errors of estimating multiple parameters by *directly* incorporating the HUP into quantum multiparameter estimation. We define the regret of Fisher information for a quantum measurement that is used to estimate an unknown parameter and shall derive the following correspondence relation:

$$\text{information regret} \leftrightarrow \text{measurement error}.$$

Taking this relationship as a bridge, we obtain trade-off relations between the information regrets for different parameters through Branciard's and Ozawa's versions of measurement uncertainty relations in terms of the state-dependent measurement error defined by Ozawa [54–59]. This trade-off relation is tight for pure quantum states, so it can faithfully reveal the quantum limits on multiparameter estimation errors with pure quantum states. We shall apply the regret trade-off relation to the coherent state estimation and the joint estimation of phase shift and phase diffusion.

Let us start with a brief introduction on quantum multiparameter estimation. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n$ be an unknown vector parameter, which can be estimated via observing a quantum system. The state of the quantum system depends on the true value of θ and is described by a parametric density operator ρ_θ. The quantum measurement can be characterized by a positive-operator-valued measure (POVM) $M = \{M_x | M_x \geq 0, \sum_x M_x = \mathbb{1}\}$, where x denotes the outcome and $\mathbb{1}$ is the identity operator. Denote the estimator for θ by $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n)$, which is a map from the observation data to the estimates. The estimation error can be characterized by the error-covariance matrix defined by its entries $\mathcal{E}_{jk} = \mathbb{E}_\theta[(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)]$, where the expectation $\mathbb{E}_\theta[\bullet]$ is taken with respect to the observation data with the joint probability mass function $p_\theta(x_1, x_2, \ldots, x_\nu) = \prod_{j=1}^{\nu} \mathrm{tr}(M_{x_\nu} \rho_\theta)$, with ν being the number of experimental runs with independent and identically distributed samples. The error-covariance matrix of any unbiased estimator $\hat{\theta}$ obeys the CRB $\mathcal{E} \geq \nu^{-1} F^{-1}$ in the sense that the matrix $\mathcal{E} - \nu^{-1} F^{-1}$ is positive semidefinite [14–17], where F is the (classical) FIM for a single experimental run and defined by

$$F_{jk} = \mathbb{E}_\theta\left[\frac{\partial \ln p_\theta(x)}{\partial \theta_j} \frac{\partial \ln p_\theta(x)}{\partial \theta_k}\right], \quad (1)$$

with $p_\theta(x) = \mathrm{tr}(M_x \rho_\theta)$. The CRB is asymptotically attainable by the maximum likelihood estimator [10,11], whose distribution at a large ν is approximate to a multivariate normal distribution with the mean being the true value of θ and the covariance matrix being $\nu^{-1} F^{-1}$, according to the central limit theorem (Theorem 9.27 [15]).

The FIM depends on the quantum measurement via $p_\theta(x) = \mathrm{tr}(M_x \rho_\theta)$, and so does the CRB. We use $F(M)$ to explicitly indicate the dependence of F on a POVM M. Quantum parameter estimation takes into consideration the optimization over quantum measurements. For any quantum measurement, the FIM is bounded by the following matrix inequality [18,60]:

$$F(M) \leq \mathcal{F}, \quad (2)$$

where $\mathcal{F}$ is the so-called quantum FIM, also known as the Helstrom information matrix [1,2]. The quantum FIM is the real part of a Hermitian matrix $\mathcal{Q}$ (i.e., $\mathcal{F} = \mathrm{Re}\mathcal{Q}$) defined by

$$\mathcal{Q}_{jk} = \mathrm{tr}(L_j L_k \rho_\theta), \quad (3)$$

where L_j, the symmetric logarithmic derivative (SLD) operator for θ_j, is a Hermitian operator satisfying $(L_j \rho_\theta + \rho_\theta L_j)/2 = \partial \rho_\theta / \partial \theta_j$. Combining Eq. (2) with the CRB yields the quantum CRB $\mathcal{E} \geq \nu^{-1} \mathcal{F}^{-1}$ for any quantum measurement and any unbiased estimator. This quantum CRB was first obtained by Helstrom with a different method [1,2].

To characterize the inaccuracy of a quantum measurement for multiparameter estimation, we here define the regret of Fisher information by

$$R(M) = \mathcal{F} - F(M). \quad (4)$$

This matrix $R(M)$ is positive semidefinite due to Eq. (2) and real symmetric as both the quantum and classical FIMs are real symmetric according to their definitions. For single-parameter estimation, Braunstein and Caves proved that the classical Fisher information can equal the quantum Fisher information with an optimal quantum measurement [18], and thus the regret $R(M)$ thereof vanishes. In the multiparameter setting, for any column vector $v \in \mathbb{R}^n$, there exists a quantum measurement M such that $v^\top R(M) v = 0$, where $\top$ denotes matrix transpose. This is because $v^\top F(M) v$ and $v^\top \mathcal{F} v$ can be interpreted as the classical and quantum Fisher information, respectively, about a parameter φ satisfying $\partial / \partial \varphi = \sum_j v_j \partial / \partial \theta_j$. The POVM M making $v R(M) v^\top$ vanish can be considered as an optimal measurement for estimating φ and in general depends on v. For different parameters, the optimal measurement may be different and even incompatible. Consequently, the entries of $R(M)$ in general cannot simultaneously vanish, which is a manifestation of HUP. In what follows, we give a quantitative characterization of the mechanism in which the HUP affects the regret matrix of Fisher information.

Define by $\Delta_j = \sqrt{R_{jj}/\mathcal{F}_{jj}}$ the normalized-square-root regret of Fisher information with respect to θ_j. Note that Δ_j takes value in the interval [0, 1]. Our main result is the following trade-off relation:

$$\Delta_j^2 + \Delta_k^2 + 2\sqrt{1 - c_{jk}^2} \Delta_j \Delta_k \geq c_{jk}^2, \quad (5)$$

where c_{jk} is a real number given by

$$c_{jk} = \frac{|\mathrm{Im}\mathcal{Q}_{jk}|}{\sqrt{\mathrm{Re}\mathcal{Q}_{jj}\mathrm{Re}\mathcal{Q}_{kk}}} = \frac{|\mathrm{Im}\mathcal{Q}_{jk}|}{\sqrt{\mathcal{F}_{jj}\mathcal{F}_{kk}}}, \tag{6}$$

with $\mathcal{Q}_{jk}$ being given by Eq. (3). For nonzero c_{jk}, Eq. (5) describes the trade-off between the regrets of Fisher information with respect to different parameters. For a family ρ_θ of pure states, the inequality Eq. (5) is tight, in the sense that there exists a quantum measurement M such that the equality in Eq. (5) holds; in such a case, our result fully reflects the trade-off between different regrets of Fisher information. For mixed states ρ_θ, the inequality Eq. (5) can be tightened by replacing c_{jk} thereof by its variant,

$$\tilde{c}_{jk} = \frac{\mathrm{tr}|\sqrt{\rho_\theta}[L_j, L_k]\sqrt{\rho_\theta}|}{2\sqrt{\mathcal{F}_{jj}\mathcal{F}_{kk}}}, \tag{7}$$

where $|X| = \sqrt{X^\dagger X}$ for an operator X. Note that the coefficient $\tilde{c}_{jk}$ is not less than c_{jk} for all quantum states and equal to c_{jk} for all pure states. We also give the second form of the trade-off relation in terms of the estimation errors:

$$\gamma_j + \gamma_k - 2\sqrt{1-\tilde{c}_{jk}^2}\sqrt{(1-\gamma_j)(1-\gamma_k)} \le 2 - \tilde{c}_{jk}^2, \tag{8}$$

where we have defined $\gamma_j = 1/(\nu\mathcal{E}_{jj}\mathcal{F}_{jj})$ for simplicity. The above inequality is a result of combining Eq. (5) with the classical CRB $\mathcal{E}_{jj} \ge \nu^{-1}(F^{-1})_{jj} \ge 1/(\nu F_{jj})$.

We here outline the proof of Eq. (5) and provide the details in the Supplemental Material [61]. Denote by $\mathcal{H}_s$ the Hilbert space associated with the underlying quantum system. For a given POVM M on $\mathcal{H}_s$, we define a measurement channel $\Phi(\rho) = \sum_x \mathrm{tr}(M_x\rho)|x\rangle\langle x|$, where $\{|x\rangle\}$ is an orthonormal basis associated with the measurement outcomes x's and span another Hilbert space $\mathcal{H}_r$. Note that the density operators $\Phi(\rho_\theta)$ are always diagonal with the basis $\{|x\rangle\}$. As a result, the SLD operators of $\Phi(\rho_\theta)$ are also diagonal with the basis $\{|x\rangle\}$ and can be represented as

$$\tilde{L}_j = \sum_x \frac{\partial \ln \mathrm{tr}(M_x\rho_\theta)}{\partial\theta_j}|x\rangle\langle x|. \tag{9}$$

The measurement channel Φ can be implemented by a unitary operation U acting on $\mathcal{H}_s \otimes \mathcal{H}_r \otimes \mathcal{H}_r$ such that

$$\Phi(\rho) = \mathrm{tr}_{1,3}[U(\rho \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|)U^\dagger] \tag{10}$$

for all density operators ρ on $\mathcal{H}_s$, where $\mathrm{tr}_{1,3}$ denotes the partial trace over the first and third tensor factors of the Hilbert space and $|0\rangle$ can be an arbitrary initial state (Ref. [65], Chap. 2) (see Fig. 1 for a schematic illustration).

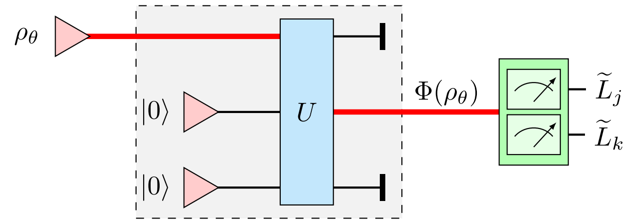

FIG. 1. Unitary implementation (the dashed box) of the measurement channel. The thick red lines stand for the input and output ports. The commuting observables $\tilde{L}_j$ and $\tilde{L}_k$ can be jointly measured in the output state $\Phi(\rho_\theta)$, which in the Heisenberg picture is equivalent to the joint measurement of a pair of commuting observables $\mathcal{L}_j = U^\dagger(\mathbb{1}_s \otimes \tilde{L}_j \otimes \mathbb{1}_r)U$ and $\mathcal{L}_k = U^\dagger(\mathbb{1}_s \otimes \tilde{L}_k \otimes \mathbb{1}_r)U$ in the initial state $\rho_{\mathrm{total}} = \rho_\theta \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|$ of the entirety.

Using the techniques developed in Ref. [66], we show that [61]

$$R_{jj} = \mathrm{tr}[(\mathcal{L}_j - L_j \otimes \mathbb{1}_r \otimes \mathbb{1}_r)^2\rho_{\mathrm{total}}], \tag{11}$$

where $\mathcal{L}_j = U^\dagger(\mathbb{1}_s \otimes \tilde{L}_j \otimes \mathbb{1}_r)U$ with $\mathbb{1}_s$ and $\mathbb{1}_r$ being the identity operators on $\mathcal{H}_s$ and $\mathcal{H}_r$, respectively, and $\rho_{\mathrm{total}} = \rho_\theta \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|$.

We observe that R_{jj} expressed in Eq. (11) is of the same form as the square of Ozawa's definition of measurement error [54,55,67], by taking L_j as the ideal observable we intend to measure and L_j as the observable actually measured. We list in Table I the correspondence relation between the parameter estimation scenario and the measurement error scenario. Note that the Hermitian operators $\mathcal{L}_j$ and $\mathcal{L}_k$ always commute, as both $\tilde{L}_j$ and $\tilde{L}_k$ are diagonal with the basis $\{|x\rangle\}$. Therefore, the observables $\mathcal{L}_j$ and $\mathcal{L}_k$ can be jointly measured in quantum mechanics. When two ideal observables L_j and L_k do not commute, it may be impossible to make their measurement errors, which equals the regrets R_{jj} and R_{kk} in our context, simultaneously vanish. By invoking the measurement uncertainty relations [54–56,58,59,68] in terms of Ozawa's definition of measurement error, we can derive the trade-off relation between the regrets of Fisher information with respect to different parameters. Concretely, the inequality Eq. (5) follows from Branciard's version of measurement uncertainty relation, which is tight for pure states [58]. Using Ozawa's work on strengthening Branciard's inequality for mixed states [68],

TABLE I. Correspondence relation.

Estimation-regret scenario	Measurement-error scenario
Regret of Fisher information	Measurement error
SLD L_j of ρ_θ	Ideal observable L_j on ρ_θ
SLD $\tilde{L}_j$ of $\Phi(\rho_\theta)$	Approximate observable on $\Phi(\rho_\theta)$
$\mathcal{L}_j = U^\dagger(\mathbb{1}_s \otimes \tilde{L}_j \otimes \mathbb{1}_r)U$	Approximate observable on ρ_θ

the inequality Eq. (5) can be tightened through replacing c_{jk} by $\tilde{c}_{jk}$.

It is worth pointing out that we do not designate the SLD operator as the ideal observable in reality to optimally estimate an individual parameter. Although the eigenstates of the SLD operator, which possibly depend on the true value of the parameter, in principle constitute a measurement basis extracting the maximum Fisher information at a parameter point [18], it is possible for some models to find a global optimal measurement that is independent of the parameter [6,19]; a global optimal measurement is often more ideal than a local one for estimating the unknown parameter.

We can give an operational significance to the coefficients $\tilde{c}_{jk}$ through the trade-off relation Eq. (5) as follows. If the quantum Fisher information about a parameter θ_j is exhaustively extracted by a quantum measurement M, i.e., $\Delta_j = 0$, then it follows from Eq. (5) that the regret for any other parameter θ_k obeys $\Delta_k \geq \tilde{c}_{jk}$. That is, $\tilde{c}_{jk}$ is the lower bound on the residual regret for θ_k when there is no regret for θ_j. For pure states, this lower bound becomes $\Delta_k \geq c_{jk}$, which can be attained as Eq. (5) is tight in such cases.

Let us now consider as an example the estimation of a complex number α encoded in a coherent state [69] $|\alpha\rangle$. The parameters of interest are the real and imaginary parts of α; i.e., $\theta_1 = \mathrm{Re}\alpha$ and $\theta_2 = \mathrm{Im}\alpha$. After some algebra, we get $\mathcal{Q} = 4\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$ and thus $c_{12} = 1$. The regret trade-off Eq. (5) then becomes $\Delta_1^2 + \Delta_2^2 \geq 1$, which is equivalent to $F_{11} + F_{22} \leq 4$ in terms of Fisher information or

$$\frac{1}{\nu\mathcal{E}_{11}} + \frac{1}{\nu\mathcal{E}_{22}} \leq 4 \tag{12}$$

in terms of estimation errors. As shown in Fig. 2, Eq. (12) gives the most informative lower bound on the estimation error, compared with the error bounds that were previously investigated [3,5,6,34].

For this example, there exists a family of optimal (single-copy) measurements extracting the Fisher information such that the regret trade-off relation in the above example, $\Delta_1^2 + \Delta_2^2 \geq 1$, is saturated. As a result, the error bound in Eq. (12) can be asymptotically attained. We construct the optimal measurement as follows. Denote by a the annihilation operators for the mode for which the coherent state is defined. The measurements of the quadrature components $Q = (a + a^\dagger)/2$ and $P = (a - a^\dagger)/(2i)$ are natural for estimating the coherent signal, as $\langle\alpha|Q|\alpha\rangle = \mathrm{Re}\alpha$ and $\langle\alpha|P|\alpha\rangle = \mathrm{Im}\alpha$. Indeed, the maximum Fisher information about θ_1 and θ_2 can be obtained by measuring Q and P, respectively, corresponding to either $F_{11} = 4$ or $F_{22} = 4$. However, Q and P are not commuting so that they cannot be jointly measured. It is known that we can jointly measure the commuting operators $Q - Q'$ and $P + P'$, where Q' and P' are the quadrature components of an ancillary mode (whose annihilation operator is denoted by a') in the vacuum state, to estimate the real and imaginary parts of α; see Refs. [3,5,6]. This measurement strategy attains the minimum unweighted arithmetic-mean error of estimation with $F_{11} = F_{22} = 2$; see the blue circle in Fig. 2. We show in the Supplemental Material [61] that other error combinations on the bound Eq. (12) can be asymptotically attained if we prepare the ancillary mode in a squeezed vacuum state $\exp[\frac{1}{2}(ra'^2 - ra'^{\dagger 2})]|0\rangle$ with $r \in \mathbb{R}$. In such case, the extracted FIM can be tuned by changing r as $F_{11} = 4/(e^{2r} + 1)$, $F_{22} = 4e^{2r}/(e^{2r} + 1)$, and $F_{12} = F_{21} = 0$. Moreover, the joint probability density function of outcomes of $Q - Q'$ and $P + P'$ are both Gaussian, so taking their sample means as the estimates for θ_1 and θ_2 asymptotically attains the classical CRB.

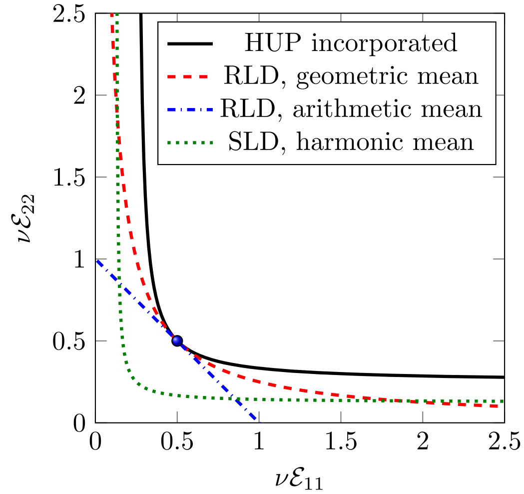

FIG. 2. Mean-square errors of estimating the real and imaginary parts of a complex number α encoded in a coherent state $|\alpha\rangle$. The regions below the curves are forbidden by the corresponding inequalities. The black solid curve stands for the inequality Eq. (12) from the regret trade-off relation. The other three curves correspond to the generalized-mean CRBs based on the SLD and the right logarithmic derivative (RLD) obtained in Ref. [34]. Specifically, the red dashed curve stands for the RLD-based geometric-mean quantum CRB given by $\sqrt{\mathcal{E}_{11}\mathcal{E}_{22}} \geq 1/(2\nu)$, the blue dash-dotted curve stands for the RLD-based arithmetic-mean quantum CRB given by $(\mathcal{E}_{11} + \mathcal{E}_{22})/2 \geq 1/(2\nu)$, and the green dotted curve stands for the SLD-based harmonic-mean quantum CRB given by $2/(\mathcal{E}_{11}^{-1} + \mathcal{E}_{22}^{-1}) \geq 1/(4\nu)$.

In our second example, we consider the joint estimation of phase shift and phase diffusion [27]. For a two-mode probe state, the parametric density operator can be effectively simplified as $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| + e^{-i\theta_1 - \theta_2^2}|0\rangle \times \langle 1| + e^{i\theta_1 - \theta_2^2}|1\rangle\langle 0|)$, where θ_1 stands for the phase shift and θ_2 the phase diffusion. In Ref. [27], Vidrighin *et al.* obtained the trade-off relation $F_{11}/\mathcal{F}_{11} + F_{22}/\mathcal{F}_{22} \leq 1$ by explicitly parametrizing the rank-1 POVMs and then taking optimization. We here show that the trade-off relation of Vidrighin *et al.* follows from our regret trade-off relation Eq. (5) in a very easy way. We only need to show $\tilde{c}_{12} = 1$ by a straightforward calculation according to its definition (see the Supplemental Material [61] for the details). As a result, we get $\Delta_1^2 + \Delta_2^2 \geq 1$, which is equivalent to the trade-off relation of Vidrighin *et al.*, by recognizing $\Delta_j^2 = 1 - F_{jj}/\mathcal{F}_{jj}$.

In conclusion, we have directly incorporated the HUP into quantum multiparameter estimation by deriving a trade-off relation between the regrets of Fisher information about different parameters. Unlike the quantum CRBs on scalar mean errors, the regrets trade-off quantitatively characterizes how the HUP affects the combinations of estimation errors for multiple parameters. The correspondence relationship we found between information regret and measurement error also, as a bonus, supplies an operational meaning to Ozawa's definition of the state-dependent measurement error, on which a controversy has existed for a long time [67,70,71].

Our approach also opens a new perspective on quantum geometry. The matrix $\mathcal{Q}$ defined by Eq. (3) is known as the quantum geometric tensor on the manifold of physical quantum state, up to an insignificant constant factor [72,73]. The real part of $\mathcal{Q}$—the quantum FIM—gives a Riemannian metric on the manifolds of quantum states. The imaginary part of $\mathcal{Q}$ gives a curvature form of Berry's connection [73], which has relations to the quantum FIM [28,74] and the density of quantum states [75]. It is known that a zero curvature is necessary for the simultaneous vanishing of the regrets of Fisher information about different parameters [8,33,36,48]. Note that in our trade-off relation, c_{jk} is the curvature divided by a scalar related to the entries of the quantum FIM. So our trade-off relation quantitatively characterizes the intricate mechanism in which the simultaneous reduction of the regrets of Fisher information about different parameters is restricted by a nonzero quantum curvature, which is indicated as

$$\text{information regret} \leftarrow \text{quantum curvature}.$$

Carollo *et al.* has proposed an incompatibility index, which is similar to c_{jk}, based on the ratio between the curvature and the quantum FIM as a figure of merit for the quantumness of a quantum multiparameter estimation model [28]. In addition, since $\tilde{c}_{jk}$ is better than c_{jk} to manifest the regrets trade-off for mixed states, it may be possible to take the quantity $\mathrm{tr}|\sqrt{\rho_\theta}[L_j, L_k]\sqrt{\rho_\theta}|$ as an alternative form of quantum curvature.

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[1] C. Helstrom, Minimum mean-squared error of estimates in quantum statistics, Phys. Lett. **25A**, 101 (1967).
[2] C. Helstrom, The minimum variance of estimates in quantum signal detection, IEEE Trans. Inf. Theory **14**, 234 (1968).
[3] H. Yuen and M. Lax, Multiple-parameter quantum estimation and measurement of nonselfadjoint observables, IEEE Trans. Inf. Theory **19**, 740 (1973).
[4] V. P. Belavkin, Generalized uncertainty relations and efficient measurements in quantum systems, Theor. Math. Phys. **26**, 213 (1976).
[5] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[6] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[7] S. Personick, Application of quantum estimation theory to analog communication over quantum channels, IEEE Trans. Inf. Theory **17**, 240 (1971).
[8] *Asymptotic Theory of Quantum Statistical Inference: Selected Papers*, edited by M. Hayashi (World Scientific, Singapore, 2005).
[9] M. Tsang, F. Albarelli, and A. Datta, Quantum Semiparametric Estimation, Phys. Rev. X **10**, 031023 (2020).
[10] R. A. Fisher, On the mathematical foundations of theoretical statistics, Phil. Trans. R. Soc. Lond. A **222**, 309 (1922).
[11] R. A. Fisher, Theory of statistical estimation, Math. Proc. Cambridge Philos. Soc. **22**, 700 (1925).
[12] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946).
[13] C. R. Rao, Information and the accuracy attainable in the estimation of statistical parameters, Bull. Calcutta Math. Soc. **37**, 81 (1945).
[14] S. M. Kay, *Fundamentals of Statistical Signal Processing, Volume I: Estimation Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1993).
[15] L. Wasserman, *All of Statistics: A Concise Course in Statistical Inference* (Springer, New York, 2010).
[16] G. Casella and R. L. Berger, *Statistical Inference*, 2nd ed. (Duxbury Press, Pacific Grove, CA, 2002).
[17] E. Lehmann and G. Casella, *Theory of Point Estimation* (Springer-Verlag, New York, 1998).
[18] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, Phys. Rev. Lett. **72**, 3439 (1994).
[19] S. L. Braunstein, C. M. Caves, and G. Milburn, Generalized uncertainty relations: Theory, examples, and Lorentz invariance, Ann. Phys. (N.Y.) **247**, 135 (1996).
[20] A. Fujiwara, Strong consistency and asymptotic efficiency for adaptive quantum estimation problems, J. Phys. A **39**, 12489 (2006).
[21] W. Heisenberg, The physical content of quantum kinematics and mechanics, Z. Phys. **43**, 172 (1927); English translation in *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton, NJ, 1984), p. 62.
[22] P. Busch, T. Heinonen, and P. Lahti, Heisenberg's uncertainty principle, Phys. Rep. **452**, 155 (2007).

[23] M. Tsang, R. Nair, and X.-M. Lu, Quantum Theory of Superresolution for Two Incoherent Optical Point Sources, Phys. Rev. X **6**, 031033 (2016).

[24] M. Tsang, Resolving starlight: A quantum perspective, Contemp. Phys. **60**, 279 (2019).

[25] T. Baumgratz and A. Datta, Quantum Enhanced Estimation of a Multidimensional Field, Phys. Rev. Lett. **116**, 030801 (2016).

[26] Z. Hou, Z. Zhang, G.-Y. Xiang, C.-F. Li, G.-C. Guo, H. Chen, L. Liu, and H. Yuan, Minimal Tradeoff and Ultimate Precision Limit of Multiparameter Quantum Magnetometry under the Parallel Scheme, Phys. Rev. Lett. **125**, 020501 (2020).

[27] M. D. Vidrighin, G. Donati, M. G. Genoni, X.-M. Jin, W. S. Kolthammer, M. Kim, A. Datta, M. Barbieri, and I. A. Walmsley, Joint estimation of phase and phase diffusion for quantum metrology, Nat. Commun. **5**, 3532 (2014).

[28] A. Carollo, B. Spagnolo, A. A. Dubkov, and D. Valenti, On quantumness in multi-parameter quantum estimation, J. Stat. Mech. (2019) 094010.

[29] J. Rubio, P. Knott, and J. Dunningham, Non-asymptotic analysis of quantum metrology protocols beyond the Cramér–Rao bound, J. Phys. Commun. **2**, 015027 (2018).

[30] F. Albarelli, J. F. Friel, and A. Datta, Evaluating the Holevo Cramér-Rao Bound for Multiparameter Quantum Metrology, Phys. Rev. Lett. **123**, 200503 (2019).

[31] M. Tsang, The Holevo Cramér-Rao bound is at most thrice the Helstrom version, arXiv:1911.08359.

[32] F. Albarelli, M. Tsang, and A. Datta, Upper bounds on the Holevo Cramér-Rao bound for multiparameter quantum parametric and semiparametric estimation, arXiv:1911.11036v1.

[33] J. S. Sidhu and P. Kok, Geometric perspective on quantum parameter estimation, AVS Quantum Sci. **2**, 014701 (2020).

[34] X.-M. Lu, Z. Ma, and C. Zhang, Generalized-mean Cramér-Rao bounds for multiparameter quantum metrology, Phys. Rev. A **101**, 022303 (2020).

[35] J. S. Sidhu, Y. Ouyang, E. T. Campbell, and P. Kok, Tight Bounds on the Simultaneous Estimation of Incompatible Parameters, Phys. Rev. X **11**, 011028 (2021)

[36] S. Ragy, M. Jarzyna, and R. Demkowicz-Dobrzański, Compatibility in multiparameter quantum metrology, Phys. Rev. A **94**, 052108 (2016).

[37] N. Li, C. Ferrie, J. A. Gross, A. Kalev, and C. M. Caves, Fisher-Symmetric Informationally Complete Measurements for Pure States, Phys. Rev. Lett. **116**, 180402 (2016).

[38] H. Zhu and M. Hayashi, Universally Fisher-Symmetric Informationally Complete Measurements, Phys. Rev. Lett. **120**, 030404 (2018).

[39] J. Suzuki, Explicit formula for the Holevo bound for two-parameter qubit-state estimation problem, J. Math. Phys. (N.Y.) **57**, 042201 (2016).

[40] J. Suzuki, Information geometrical characterization of quantum statistical models in quantum estimation theory, Entropy **21**, 703 (2019).

[41] J. Suzuki, Y. Yang, and M. Hayashi, Quantum state estimation with nuisance parameters, J. Phys. A **53**, 453001 (2020).

[42] I. Kull, P. A. Guérin, and F. Verstraete, Uncertainty and trade-offs in quantum multiparameter estimation, J. Phys. A **53**, 244001 (2020).

[43] A. Carollo, D. Valenti, and B. Spagnolo, Geometry of quantum phase transitions, Phys. Rep. **838**, 1 (2020).

[44] N. Li and S. Luo, Fisher concord: Efficiency of quantum measurement, Quantum Meas. Quantum Metrol. **3**, 44 (2016).

[45] J. Liu, H. Yuan, X.-M. Lu, and X. Wang, Quantum Fisher information matrix and multiparameter estimation, J. Phys. A **53**, 023001 (2020).

[46] R. D. Gill and S. Massar, State estimation for large ensembles, Phys. Rev. A **61**, 042312 (2000).

[47] H. Nagaoka, A new approach to Cramér-Rao bounds for quantum state estimation, in *Asymptotic Theory of Quantum Statistical Inference*, edited by M. Hayashi (World Scientific, Singapore, 2005), pp. 100–112.

[48] K. Matsumoto, A new approach to the Cramér-Rao-type bound of the pure-state model, J. Phys. A **35**, 3111 (2002).

[49] R. D. Gill and M. I. Gu, On asymptotic quantum statistical inference, in *From Probability to Statistics and Back: High-Dimensional Models and Processes—A Festschrift in Honor of Jon A. Wellner*, edited by M. Banerjee, F. Bunea, J. Huang, V. Koltchinskii, and M. H. Maathuis, IMS Collections Vol. 9 (Institute of Mathematical Statistics, Beachwood, OH, 2013), pp. 105–127.

[50] M. Hayashi and K. Matsumoto, Asymptotic performance of optimal state estimation in qubit system, J. Math. Phys. (N.Y.) **49**, 102101 (2008).

[51] J. Kahn and M. Gu, Local asymptotic normality for finite dimensional quantum systems, Commun. Math. Phys. **289**, 597 (2009).

[52] K. Yamagata, A. Fujiwara, and R. D. Gill, Quantum local asymptotic normality based on a new quantum likelihood ratio, Ann. Stat. **41**, 2197 (2013).

[53] M. Guţă and J. Kahn, Local asymptotic normality for qubit states, Phys. Rev. A **73**, 052108 (2006).

[54] M. Ozawa, Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement, Phys. Rev. A **67**, 042105 (2003).

[55] M. Ozawa, Uncertainty relations for joint measurements of noncommuting observables, Phys. Lett. A **320**, 367 (2004).

[56] M. J. W. Hall, Prior information: How to circumvent the standard joint-measurement uncertainty relation, Phys. Rev. A **69**, 052113 (2004).

[57] M. M. Weston, M. J. W. Hall, M. S. Palsson, H. M. Wiseman, and G. J. Pryde, Experimental Test of Universal Complementarity Relations, Phys. Rev. Lett. **110**, 220402 (2013).

[58] C. Branciard, Error-tradeoff and error-disturbance relations for incompatible quantum measurements, Proc. Natl. Acad. Sci. U.S.A. **110**, 6742 (2013).

[59] X.-M. Lu, S. Yu, K. Fujikawa, and C. H. Oh, Improved error-tradeoff and error-disturbance relations in terms of measurement error components, Phys. Rev. A **90**, 042113 (2014).

[60] F. Hiai and D. Petz, *Introduction to Matrix Analysis and Applications*, 1st ed., Universitext (Springer International Publishing, Cham, 2014).

[61] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.120503 for detailed derivations, which includes Refs. [62–64].

[62] E. Arthurs and J. L. Kelly, On the simultaneous measurement of a pair of conjugate observables, Bell Syst. Technol. J. **44**, 725 (1965).

[63] C. W. Helstrom, Cramér-Rao inequalities for operator-valued measures in quantum mechanics, Int. J. Theor. Phys. **8**, 361 (1973).

[64] J. C. Garrison and R. Y. Chiao, *Quantum Optics* (Oxford University Press, New York, 2008).

[65] M. M. Wolf, Quantum channels & operations—Guided tour, http://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf.

[66] X.-M. Lu, S. Yu, and C. H. Oh, Robust quantum metrological schemes based on protection of quantum Fisher information, Nat. Commun. **6**, 7282 (2015).

[67] M. Ozawa, Soundness and completeness of quantum root-mean-square errors, npj Quantum Inf. **5**, 1 (2019).

[68] M. Ozawa, Error-disturbance relations in mixed states, arXiv:1404.3388.

[69] R. J. Glauber, Coherent and incoherent states of the radiation field, Phys. Rev. **131**, 2766 (1963).

[70] P. Busch, P. Lahti, and R. F. Werner, Colloquium: Quantum root-mean-square error and measurement uncertainty relations, Rev. Mod. Phys. **86**, 1261 (2014).

[71] D. M. Appleby, Quantum errors and disturbances: Response to Busch, Lahti and Werner, Entropy **18**, 174 (2016).

[72] J. Provost and G. Vallee, Riemannian structure on manifolds of quantum states, Commun. Math. Phys. **76**, 289 (1980).

[73] M. Berry, The quantum phase, five years after, in *Geometric Phases in Physics*, edited by A. Shapere and F. Wilczek (World Scientific, Singapore, 1989), Chap. 1.1, pp. 7–28.

[74] W. Guo, W. Zhong, X.-X. Jing, L.-B. Fu, and X. Wang, Berry curvature as a lower bound for multiparameter estimation, Phys. Rev. A **93**, 042115 (2016).

[75] H. Xing and L. Fu, Measure of the density of quantum states in information geometry and quantum multiparameter estimation, Phys. Rev. A **102**, 062613 (2020).