Universality Classes of Hitting Probabilities of Jump Processes

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Quantifying the efficiency of random target search strategies is a key question of random walk theory, with applications in various fields. If many results do exist for recurrent processes, for which the probability of eventually finding a target in infinite space—so called hitting probability—is one, much less is known in the opposite case of transient processes, for which the hitting probability is strictly less than one. Here, we determine the universality classes of the large distance behavior of the hitting probability for general d-dimensional transient jump processes, which we show are parametrized by a transience exponent that is explicitly given.

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Quantifying the statistics of encounter events of a random walker with a target has become a central question of random walk theory [1,2], with applications ranging from chemical reaction kinetics to animal foraging behaviors [3–5]. In the simplest setting of a single target in an unbounded d-dimensional space, two radically different scenarios naturally emerge: either the target is eventually found with probability $\Pi = 1$ (compact or recurrent case), or with probability $\Pi < 1$ (noncompact or transient case). This defines the hitting probability Π [1,6].

In the recurrent case, a classical observable is the survival probability $S(t)$, i.e., the probability that the target has not been found until time t. This quantity typically decreases at long timescales as $t^{-\theta}$, where θ is the persistence exponent, which has been the focus of numerous studies [7]. In the transient case, the survival probability admits a nonzero large time limit, which is readily expressed in terms of the hitting probability $S \to_{t\to\infty} 1 - \Pi$. This makes the hitting probability a key quantifier of the search process in the transient case. The hitting probability is expected to decrease with the distance R from the starting position of the random walk and the target of radius a according to $\Pi \sim C(a/R)^{\psi}$. The transience exponent ψ recently introduced in [8] parallels the persistence exponent of recurrent processes and is an intrinsic characteristic of transient processes that has, however, been largely unexplored so far; in turn, the prefactor C depends on the process and is required for the full determination of the asymptotics of the hitting probability.

For jump processes [9,10], target capture events must be defined. Two conventions have been adopted [8,11,12]: the target can be found either when a jump *ends* inside the target—arrival convention, or when a jump *crosses* the target—crossing convention (see Fig. 1). Importantly, for $d = 1$, it has been shown that arrival and crossing conventions can lead to strikingly different first-passage properties [11,13] in the case of Levy processes. These can be defined as the continuous time limit of Levy flights, which are discrete time jump processes whose jump-length distribution has a power law tail $p(l) \propto 1/l^{1+\alpha}$ with index $\alpha \in [0, 2]$. For $d \geq 2$, exact results for the hitting probability are sparse and available only for the arrival convention, either for examples of jump distributions with finite variance [14–16], or for the specific case of Levy processes [17]. However, the hitting probability with the crossing convention—which is clearly larger than with the arrival convention and nontrivial for $d \geq 2$ only—has not been studied, despite its relevance to various examples of target search problems. In particular, the hitting probability of Levy flights with the crossing convention gives access to the hitting probability of Levy walks, which play an important role in the context of animal behavior [18].

In this Letter, we consider jump processes with a general distribution of jump length, determine the large R behavior of the hitting probability for both conventions, and reveal its universality classes that we show are parametrized by the transience exponent, which is explicitly given. Our results are summarized in Table I. More precisely, in the case of processes whose jump distribution has a finite variance we

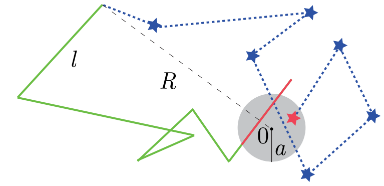

FIG. 1. Hitting probability of jump processes with crossing convention (plain line trajectory) or arrival convention (dashed line trajectory). The target is detected when the path turns red.

TABLE I. Universality classes of the asymptotic hitting probability of general jump processes (for arbitrary a, σ), for both arrival and crossing problem. α is defined by the decay of the jump distribution and short-range distribution is included in the $\alpha = 2$ case. Value of the prefactors $\mathcal{D}_\alpha$ and $\mathcal{E}$ are given in the text, and "cst" means that the prefactor value is not universal and depends on the full jump statistics. Note that the special case $d = 2$ and $\alpha = 2$ is not covered here, as the process becomes compact [$\Pi(R) = 1$]. Note that for $d = 1$, only the case $\alpha < 1$ with the arrival convention is transient; it is covered by the table.

	Arrival	Crossing
$\alpha < 1$	$\mathrm{cst}(a/R)^{d-\alpha}$	$D_\alpha(a/R)^{d-1}$
$\alpha = 1$	$\mathrm{cst}(a/R)^{d-1}$	$\mathcal{E}_1 \ln(R)(a/R)^{d-1}$
$1 < \alpha \leq 2$	$\mathrm{cst}(a/R)^{d-\alpha}$	$\mathrm{cst}(a/R)^{d-\alpha}$

find that for both conventions $\psi = d - 2$ (for the transient case $d \geq 3$) is universal, in the sense that it is independent of the jump distribution. In the case of processes whose jump distribution has an infinite variance, typically Levy flights [18–22], we establish that $\psi = d - \alpha$ for the arrival convention, in agreement with the specific example of stable Levy processes given in [17], which was the only example known so far, and that $\psi = d - \max(\alpha, 1)$ for the crossing convention. This shows that the transience exponent only depends on α, and not on the shape of the jump distribution at small length scales. Of note, for $\alpha > 1$ both conventions yield the same transience exponents, while for $\alpha < 1$ different values are obtained; this reflects the significant weight of long jumps overshooting the target for $\alpha < 1$ with the arrival convention. Moreover, we show that the prefactor C of the hitting probability depends on the short length scale behavior of the jump distribution for all α in the arrival convention, and for $\alpha > 1$ for the crossing convention. Remarkably, C is independent of the jump distribution at short length scales for $\alpha < 1$. In this case of superuniversal behavior, we fully compute the asymptotics of Π (i.e., C and ψ).

Definitions and integral equation.—We consider a jump process $\mathbf{X}(n)$ (discrete-time random walk) in a continuous d-dimensional space, starting from position $\mathbf{R}$, with a spherical target of radius a centered at $\mathbf{0}$. We assume that the increments $\mathbf{Y}(n) = \mathbf{X}(n+1) - \mathbf{X}(n)$ of the walk are independent, stationary, and isotropic; we denote $p(\mathbf{Y})$ the corresponding distribution. Following the usual classification of jump processes [23], we introduce the Fourier transform $\tilde{p}(\mathbf{k}) = \int d\mathbf{Y} e^{i\mathbf{k}\mathbf{Y}} p(\mathbf{Y})$ and write without loss of generality its small $\mathbf{k}$ expansion: $\tilde{p}(\mathbf{k}) = 1 - \sigma^\alpha |\mathbf{k}|^\alpha + \cdots$. Here, $\alpha = 2$ for all distributions with finite second moment and $\alpha < 2$ otherwise. In turn, σ provides a natural length scale that makes it possible to define the continuous limit of the process by taking $\sigma \to 0$; according to the generalized central limit theorem, this limit yields the Brownian motion for $\alpha = 2$ and α-stable Levy processes for $\alpha < 2$ [23]. We consider transient processes, such that $\Pi(R) \to 0$ for $R \to \infty$, and aim at determining this large R asymptotics of Π, which by dimensional analysis is a function of R/a, σ/a. Below, we first consider the case $\sigma/a \ll 1$ (continuous limit), and next the general case σ/a arbitrary; we provide the main steps of the derivation of the results and discuss their impact, while details can be found in Supplemental Material (SM) [24].

With the arrival convention, we define the trajectory of the walker as the set of points $\{\mathbf{X}(n)\}_{n\in\mathbb{N}}$ only; with the crossing convention, the trajectory is defined as the full set $\{[\mathbf{X}(n), \mathbf{X}(n+1)]\}_{n\in\mathbb{N}}$ of segments joining the successive $\mathbf{X}(k)$. With these definitions, with either convention, the target is found when the trajectory intercepts the target (see Fig. 1). Making a partition over the first step of the process, which brings the walker from a position $\mathbf{R}$ to a position $\mathbf{R} + \mathbf{Y}$, the probability $\Pi(\mathbf{R})$ that the walker eventually finds the target satisfies

$$\Pi(\mathbf{R}) = \int_{\mathbf{Y}\notin E} \Pi(\mathbf{R}+\mathbf{Y}) p(\mathbf{Y}) d\mathbf{Y} + \int_{\mathbf{Y}\in E} p(\mathbf{Y}) d\mathbf{Y}, \quad (1)$$

where E denotes the set of jumps that cross the target starting from $\mathbf{R}$. Note that E is empty for the arrival convention, but not for the crossing one. This linear integral equation, complemented by the boundary conditions $\Pi(R) = 1$ for $R < a$ and $\Pi(R) \to 0$ for $R \to \infty$ fully defines the problem at the core of this Letter.

Continuous limit ($\sigma/a \ll 1$).—In this limit, Eq. (1) becomes

$$(-\Delta)^{\alpha/2} \Pi(\mathbf{R}) = \lim_{\sigma\to 0} \frac{1}{\sigma^\alpha} \int_{\mathbf{Y}\in E} [1 - \Pi(\mathbf{R}+\mathbf{Y})] p(\mathbf{Y}) d\mathbf{Y}, \quad (2)$$

where the fractional Laplacian operator is defined by

$$-(-\Delta)^{\alpha/2} \Pi(\mathbf{R}) = \lim_{\sigma\to 0} \frac{1}{\sigma^\alpha} \mathbb{E}[\Pi(\mathbf{R}+\mathbf{Y}) - \Pi(\mathbf{R})] \quad (3)$$

and generalizes the classical Laplacian obtained for $\alpha = 2$.

Arrival convention.—In this case, $E = \varnothing$, and Eq. (2) reduces to

$$(-\Delta)^{\alpha/2} \Pi_{\mathrm{arr}}(\mathbf{R}) = 0 \quad \text{for } R > a, \quad (4)$$

which is complemented by the boundary conditions stated after Eq. (1). For $\alpha = 2$, the hitting probability of Brownian motion is readily recovered [1]:

$$\Pi_{\mathrm{arr}}(R) = (a/R)^{d-2} \quad \text{for } d > 2, \quad (5)$$

while for $d \leq 2$ the process is compact and $\Pi = 1$ for all R. We now assume $\alpha < 2$ and $d \geq 2$. Of note, this regime of the arrival case in the continuous limit has been studied in the mathematical literature [17]. A reminder is included here for the sake of self-consistency, and because it will be at the basis of the solution of the crossing problem presented below; details can be found in SM [24]. The main step of the derivation is to make use of the so-called Kelvin transform (inversion in the sphere of radius a) [27]; this amounts to introducing the new function

$$\tilde{\Pi}_{\rm arr}(\mathbf{R}) = \left(\frac{a}{R}\right)^{d-\alpha} \Pi_{\rm arr}(a^2\mathbf{R}/R^2), \tag{6}$$

which is now defined for $R < a$, i.e., inside the ball B_a of radius a. Note that this transform is an involution, so that (6) readily defines $\Pi_{\rm arr}$ from $\tilde{\Pi}_{\rm arr}$. We show in SM that $\tilde{\Pi}_{\rm arr}$ satisfies [24]

$$(-\Delta)^{\alpha/2}\tilde{\Pi}_{\rm arr}(\mathbf{R}) = 0 \quad \text{for } R < a, \tag{7}$$

with $\tilde{\Pi}_{\rm arr}(R) = (a/R)^{d-\alpha}$ for $R > a$. The original problem (4), defined outside B_a, is thus exactly mapped to the problem (7) defined inside B_a, for which the Green's function is known analytically. This gives access to the following explicit determination of $\tilde{\Pi}_{\rm arr}$ for $R < a$ (see SM [24])

$$\tilde{\Pi}_{\rm arr}(\mathbf{R}) = \int_{\mathbb{R}^d\setminus B_a} \left(\frac{a}{\mathbf{R}'}\right)^{d-\alpha} P(\mathbf{R}', \mathbf{R}) d\mathbf{R}', \tag{8}$$

where the Poisson kernel is given by [28]

$$P(\mathbf{R}', \mathbf{R}) = \frac{\Gamma(d/2)\sin(\pi\alpha/2)}{\pi^{d/2+1}} \left(\frac{a^2 - R^2}{R'^2 - a^2}\right)^{\alpha/2} \frac{a^d}{|\mathbf{R} - \mathbf{R}'|^d}. \tag{9}$$

After inversion of the Kelvin transform according to (6), the large R behavior of $\Pi_{\rm arr}$ can finally be determined explicitly (see SM [24])

$$\Pi_{\rm arr}(R) \sim \frac{2}{d-\alpha} \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{d-\alpha}{2})} \left(\frac{a}{R}\right)^{d-\alpha} \equiv F_\alpha \left(\frac{a}{R}\right)^{d-\alpha}. \tag{10}$$

Note that this result, obtained previously in [17] also holds for $d = 1$ and $\alpha < 1$ (which corresponds to transient processes).

Crossing convention.— In this case, $E \neq \emptyset$, and Eq. (2) can be written

$$(-\Delta)^{\alpha/2}\Pi_{\rm cross}(\mathbf{R}) = g(\mathbf{R}); \tag{11}$$

the boundary conditions stated after Eq. (1) are unchanged. Note that g, defined as the right-hand side of (2) depends on $\Pi_{\rm cross}$ itself, which makes the resolution for arbitrary $\mathbf{R}$ a complex problem. However, making use of $\Pi_{\rm cross}(\mathbf{R}) \to 0$ for $R \to \infty$, the asymptotic behavior of $g(\mathbf{R})$ is found to be independent of $\Pi_{\rm cross}$; more explicitly, it is shown in SM to satisfy for $R \to \infty$ [24]:

$$g(\mathbf{R}) \sim \frac{V_{d-1}}{S_d} \frac{2\Gamma(\alpha)\sin(\pi\alpha/2)}{\pi} \left(\frac{a}{R}\right)^{d-1} \frac{1}{R^\alpha} \tag{12}$$

$$\equiv c_0 \left(\frac{a}{R}\right)^{d-1} \frac{1}{R^\alpha}, \tag{13}$$

where S_n and V_n denote, respectively, the surface and the volume of the unit sphere of $\mathbb{R}^n$. This is in fact sufficient to derive the large R asymptotics of $\Pi_{\rm cross}$. It is convenient to write the solution to Eq. (11) as

$$\Pi_{\rm cross} = \Pi_{\rm arr} + \Pi_{\rm part}, \tag{14}$$

where $\Pi_{\rm arr}$ has been determined in Eq. (10) and $\Pi_{\rm part}$ satisfies the linear equation (11) with the boundary condition $\Pi_{\rm part}(\mathbf{R}) = 0$ for $R < a$. Making use again of the Kelvin transform, we find that $\tilde{\Pi}_{\rm part}(\mathbf{R})$ is solution of the following problem:

$$(-\Delta)^{\alpha/2}\tilde{\Pi}_{\rm part}(\mathbf{R}) = \tilde{g}(\mathbf{R}) \quad \text{for } R < a, \tag{15}$$

with $\tilde{\Pi}_{\rm part}(R) = 0$ for $R > a$, and $\tilde{g}(\mathbf{R}) = g(a^2\mathbf{R}/R^2)\cdot (a/R)^{d+\alpha}$. This auxiliary problem, defined inside B_a, can be solved exactly by using the Green's function [28]:

$$G(\mathbf{R}, \mathbf{R}') = C_\alpha |\mathbf{R} - \mathbf{R}'|^{\alpha-d} \int_0^{r_0(\mathbf{R},\mathbf{R}')} \frac{r^{\alpha/2-1}}{(r+1)^{d/2}} dr, \tag{16}$$

with $r_0(\mathbf{R}, \mathbf{R}') = [(a^2 - R^2)(a^2 - R'^2)]/a^2|\mathbf{R} - \mathbf{R}'|^2$ and $C_\alpha = \{[\Gamma(d/2)]/[2^\alpha \pi^{d/2}\Gamma(\alpha)^2]\}$. The solution, valid for any $0 < \alpha \leq 2$ reads

$$\tilde{\Pi}_{\rm part}(\mathbf{R}) = \int_{B_a} G(\mathbf{R}, \mathbf{R}')\tilde{g}(\mathbf{R}')d\mathbf{R}', \tag{17}$$

and $\Pi_{\rm part}(\mathbf{R})$ is finally deduced by inverse Kelvin transform (6), and $\Pi_{\rm cross}(\mathbf{R})$ from (14).

First, we consider the case $\alpha = 2$ (Brownian limit). Equations (12) and (13) then yield $g = 0$, so that $\Pi_{\rm part}(\mathbf{R}) = 0$. From (14), the solution with the crossing convention is thus identical to that of the arrival convention [Eq. (5)], as expected from the continuous nature of the trajectories in the Brownian limit.

In the case $0 < \alpha < 2$, in order to analyze the $R \to \infty$ asymptotics of $\Pi_{\rm cross}(\mathbf{R})$, we take the $R \to 0$ limit in Eq. (17). For $1 < \alpha < 2$, it is shown in SM that $\tilde{\Pi}_{\rm cross}(R = 0)$ is finite [24]. Inverting the Kelvin transform then yields

$$\Pi_{\rm cross}(R) \propto (a/R)^{d-\alpha}. \tag{18}$$

Note that, as opposed to the arrival convention, the value of the prefactor is not determined by our approach because the full function g is involved (and not only its large R asymptotics); however, the prefactor depends only on the asymptotics of the jump distribution $p(\mathbf{Y})$. For $\alpha < 1$, it is shown in SM that $\tilde{\Pi}_{\mathrm{part}} \underset{R\to 0}{\propto} R^{\alpha-1}$, so that $\tilde{\Pi}_{\mathrm{arr}} = O(1)$ can be neglected [24]. After inversion of the Kelvin transform, we finally get the full asymptotics (including the prefactor)

$$\Pi_{\mathrm{cross}}(\mathbf{R}) \sim C_\alpha c_0 \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{d}{2})} \left(\frac{a}{R}\right)^{d-1} \int_{\mathbb{R}^d} \frac{|\mathbf{e}-\mathbf{u}|^{\alpha-d}}{|\mathbf{u}|} d\mathbf{u} \equiv \mathcal{D}_\alpha \left(\frac{a}{R}\right)^{d-1}. \tag{19}$$

Note that in the ballistic limit $\alpha \to 0$, we get $\mathcal{D}_\alpha \to \lim_{\alpha\to 0} c_0 = V_{d-1}/S_d$. This is indeed as expected the probability that a random straight line intersects a spherical target at distance R.

For completeness, we finally provide the full asymptotics of the "marginal" case $\alpha = 1$, which is derived in SM [24]:

$$\Pi_{\mathrm{cross}}(\mathbf{R}) \sim \frac{2}{\pi(d-1)} \frac{\ln(R) a^{d-1}}{R^{d-1}} \equiv \frac{\mathcal{E}_1 \ln(R) a^{d-1}}{R^{d-1}}. \tag{20}$$

These results in the continuous limit are confirmed by numerical simulations in Fig. 2, and allow us to quantify the relative impact of the crossing and arrival conventions on the hitting probability. For $1 < \alpha < 2$, we find that the transience exponent $\psi = d - \alpha$ is the same with both conventions with, however, a larger prefactor with the crossing convention. The prefactor is determined explicitly for the arrival convention, but not for the crossing one. For $\alpha < 1$, the weight of long jumps crossing the target in the crossing convention becomes large enough to impact the transience exponent itself, which is found to be $\psi = d - 1$ in the crossing convention and $\psi = d - \alpha$ in the arrival convention. In this case, for both conventions the prefactor is determined explicitly.

General jump processes (σ/a arbitrary).—In the general case, the hitting probability Π *a priori* depends on the full jump distribution $p(\mathbf{Y})$. This has been highlighted in particular in the case of jump processes with finite variance ($\alpha = 2$) with the arrival convention in $3d$ [16]. While the exponent $\psi = d - 2$ obtained in the continuous limit is recovered in this case (for $d \geq 3$), the determination of the prefactor is nontrivial and leads to an explicit dependence on $p(\mathbf{Y})$. We now analyze the case of general jump processes with $0 < \alpha \leq 2$, and determine the large R asymptotics of $\Pi(R)$.

Let us first focus on the arrival problem. We show in SM that exact bounds for Π_{arr} can be obtained [24], which allows us to show that

$$\Pi_{\mathrm{arr}} \sim K_{\mathrm{arr}} \left(\frac{a}{R}\right)^{d-\alpha} \tag{21}$$

for large R. Of note the prefactor K_{arr} depends on the full distribution $p(\mathbf{Y})$, but the transience exponent $\psi = d - \alpha$ is the same as in the continuous limit considered above and depends only on α.

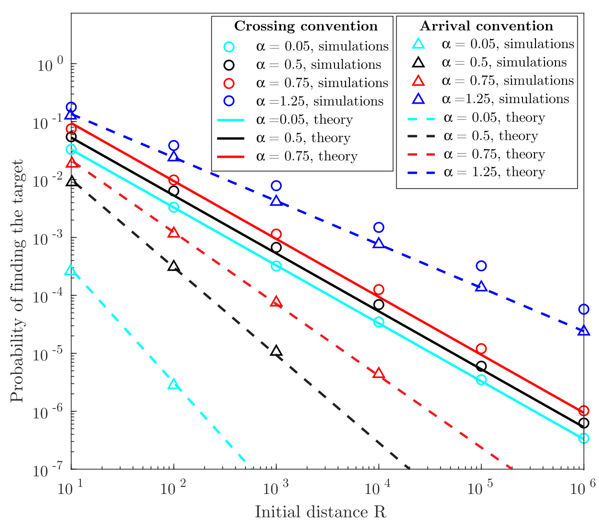

FIG. 2. Asymptotic behavior of $\Pi(R)$ in the continuous limit. Circles are simulations for various α with the crossing convention. Plain lines are the theoretical expression (19) for $\alpha < 1$. Note that there is no fitting parameter. Dashed lines show the theoretical result for the arrival problem, Eq. (10). We see that the decay as $R^{d-\alpha}$ matches the one of the crossing problem with $\alpha > 1$, although with a different prefactor. The unit of length is fixed by the radius of the target.

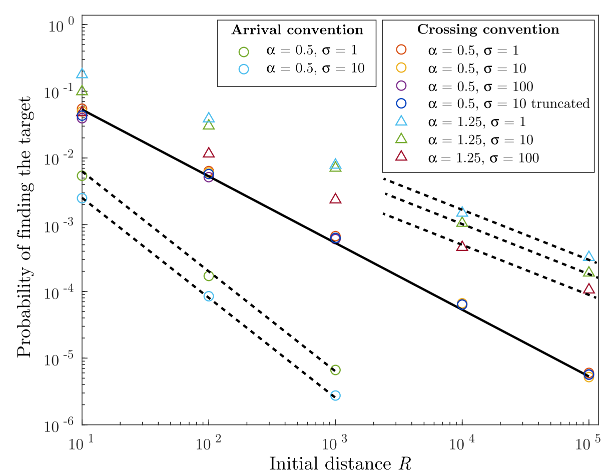

FIG. 3. Universality and superuniversality of the hitting probability for general jump processes (for arbitrary a, σ). Hitting probability of Levy walks for $\alpha = 0.5$ and $\alpha = 1.25$ and various scale parameter σ for both arrival and crossing conventions. Symbols stand for simulations of Levy walks. The "truncated" Levy walk has a jump length l conditioned by $l > \sigma$. Plain lines refer to Eq. (19); dashed lines are power laws of expected exponent ψ with fitted prefactor. The value of the prefactor is clearly independent of the details of the jump distribution (σ, truncation …) only for the crossing convention with $\alpha \leq 1$. The unit of length is fixed by the radius of the target.

We focus now on the crossing problem. We show in SM that exact bounds for Π_{cross} can again be obtained [24], which allows us to show that

$$\Pi_{\mathrm{cross}} \sim K_{\mathrm{cross}} \left(\frac{a}{R}\right)^{d-\max(1,\alpha)}, \tag{22}$$

where K_{cross} depends on $p(\mathbf{Y})$ for $1 < \alpha \leq 2$ and $K_{\mathrm{cross}} = \mathcal{D}_\alpha$ for $\alpha < 1$.

Finally, two important results are obtained. First, the transience exponent is given by $\psi = d - \max(1, \alpha)$, as in the continuum limit. Second, for $\alpha < 1$, one has $\Pi_{\mathrm{cross}} \sim \mathcal{D}_\alpha (a/R)^{d-1}$, which corresponds to a superuniversal regime: the transience exponent is fully independent of the jump distribution, and the prefactor depends on the jump distribution only through the Levy exponent α—it is in particular independent of the scale parameter σ. These results are summarized in Table I, and confirmed by numerical simulations (see Fig. 3).

To conclude, we have analyzed the asymptotic behavior of the hitting probability for general transient jump processes, which is a key, although largely unexplored, observable to quantify the efficiency of random search strategies. We have shown that the result is strongly dependent on the ability of the walker to detect the target, so that the transience exponent ψ itself is different for arrival and crossing problems for $\alpha < 1$. Moreover, we have determined explicitly the universality classes of the hitting probability, and unveiled a superuniversal regime where both the transient exponent and the prefactor are independent of the microscopic details of the process.

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