Editors' Suggestion

Rigorous Results on the Ground State of the Attractive SU(N) Hubbard Model

Hironobu Yoshida[1,*] and Hosho Katsura[1,2,3]

[1]*Department of Physics, Graduate School of Science, The University of Tokyo, 7-3-1 Hongo, Tokyo 113-0033, Japan*
[2]*Institute for Physics of Intelligence, The University of Tokyo, 7-3-1 Hongo, Tokyo 113-0033, Japan*
[3]*Trans-scale Quantum Science Institute, The University of Tokyo, 7-3-1, Hongo, Tokyo 113-0033, Japan*

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We study the attractive SU(N) Hubbard model with particle-hole symmetry. The model is defined on a bipartite lattice with the number of sites N_A (N_B) in the A (B) sublattice. We prove three theorems that allow us to identify the basic ground-state properties: the degeneracy, the fermion number, and the SU(N) quantum number. We also show that the ground state exhibits charge density wave order when $|N_A - N_B|$ is macroscopically large. The theorems hold for a bipartite lattice in any dimension, even without translation invariance.

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Introduction.—The (fermionic) Hubbard model [1–3] is one of the most important models for describing strongly correlated fermions. Despite its apparent simplicity, the model has proved to be notoriously difficult to analyze analytically, and rigorous results are few and far between [4–7].

Recently, the SU(N) generalization of the Hubbard model has attracted much attention since it was realized with ultracold atoms in optical lattices [8–21]. In particular, attractive SU(N) Hubbard models are predicted to host a variety of exotic phases that do not appear in the SU(2) counterpart, including color superfluid and trion phases with $N = 3$ [22–25].

In the SU(2) case, the spin-reflection positivity method invented by Lieb [26] is a powerful tool to establish rigorous results. It exploits the symmetry between up-spin and down-spin electrons. When the interaction is attractive and the number of electrons is even, the ground state was shown to be unique and a spin singlet [26]. When the lattice is bipartite and the difference in the number of sites in the two sublattices is macroscopically large, the coexistence of superconductivity and charge density wave was proved [27–30]. This method has also been used to study the ground state of other strongly correlated electron systems [30–40], such as the periodic Anderson model and the Kondo lattice model.

However, the method in its original form is not applicable to the SU(N) Hubbard model with $N \geq 3$. Thus, a new approach has to be developed. Here, we use a method based on the Majorana representation of fermions called Majorana reflection positivity [41]. While the spin-reflection positivity method uses the symmetry between up-spin and down-spin electrons, the Majorana reflection positivity method relies on the symmetry between two species of Majorana fermions, $\gamma^{(1)}$ and $\gamma^{(2)}$. It has been used to solve the fermion sign problem in quantum Monte Carlo simulations [42–45]. For example, the SU(3) attractive Hubbard model on the honeycomb lattice was numerically studied, and a quantum phase transition from a semimetal to a charge density wave phase was observed [46]. The method was also used to discuss the ground-state degeneracy of interacting spinless fermions [47].

In this Letter, we extend the method of Majorana reflection positivity and prove three theorems on the attractive SU(N) Hubbard model with $N \geq 3$. First, we will identify the degeneracy, the fermion number, and the SU(N) quantum number of the ground state (Theorem 1). This is a natural generalization of Lieb's theorem on the SU(2) Hubbard model [26]. Next, we will prove an inequality for a correlation function, which is a measure of the charge density wave order (Theorem 2). Finally, combining Theorems 1 and 2, we will show that the system exhibits the charge density wave order when $|N_A - N_B|$ is macroscopically large, where N_A (N_B) is the number of sites in the A (B) sublattice (Theorem 3). This is a natural generalization of Tian's theorem on the SU(2) Hubbard model [29].

The model and main results.—We consider the attractive SU(N) Hubbard model on a finite bipartite lattice Λ. Bipartiteness means that the lattice Λ can be divided into two sublattices, A and B, and if two sites $x, y \in \Lambda$ belong to the same sublattice, the hopping matrix element $t_{x,y}$ is zero. Let us write the number of sites in the whole lattice Λ as N_s and the number of sites in the A (B) sublattice as N_A (N_B). For each site $x \in \Lambda$, we denote by $c^\dagger_{x,\sigma}$ and $c_{x,\sigma}$ the creation and annihilation operators, respectively, of a fermion with flavor $\sigma = 1, \ldots, N$. We define the number operators by $n_{x,\sigma} = c^\dagger_{x,\sigma} c_{x,\sigma}$ and $n_x = \sum_{\sigma=1}^{N} n_{x,\sigma}$. Let us consider the standard Hamiltonian of the attractive SU(N) Hubbard model

$$H = H_{\mathrm{hop}} + H_{\mathrm{int}}, \tag{1}$$

$$H_{\rm hop} = \sum_{x\in A, y\in B}\sum_{\sigma=1}^{N} t_{x,y}(c^\dagger_{x,\sigma}c_{y,\sigma} + c^\dagger_{y,\sigma}c_{x,\sigma}), \qquad (2)$$

$$H_{\rm int} = \sum_{x\in\Lambda} U_x\left(n_x - \frac{N}{2}\right)^2. \qquad (3)$$

The on-site interactions may depend on sites, as long as $U_x < 0$. We assume that the hopping matrix elements $t_{x,y}$ are real. We also assume that the lattice is connected via nonvanishing hopping matrix elements; i.e., for any $x, y \in \Lambda$ such that $x \neq y$, there exists a finite sequence $z_1, \ldots, z_n \in \Lambda$ with $z_1 = x, z_n = y$, where $t_{z_j,z_{j+1}}$ are nonvanishing for all $j = 1, \ldots, n-1$. Note that the Hamiltonian is invariant under the particle-hole transformation $c_{x,\sigma} \to (-1)^x c^\dagger_{x,\sigma}$, where $(-1)^x = 1$ if $x \in A$ and $(-1)^x = -1$ if $x \in B$.

To state our first theorem, let us define SU(N) singlet states. To this end, we introduce the operators $F^{\sigma,\tau} = \sum_{x\in\Lambda} c^\dagger_{x,\sigma}c_{x,\tau}$. Here, $F^{\sigma,\sigma}$ is the total number operator of fermions with flavor σ, while $F^{\sigma,\tau}$ ($\sigma \neq \tau$) are flavor-raising and -lowering operators. Since all $F^{\sigma,\tau}$ operators commute with the Hamiltonian H, it has the global $\mathrm{U}(N) = \mathrm{U}(1) \times \mathrm{SU}(N)$ symmetry. A state $|\Phi_{\rm singlet}\rangle$ is an SU(N) singlet with the total fermion number N_f if $F^{\sigma,\tau}|\Phi_{\rm singlet}\rangle = 0$ for all $\sigma \neq \tau$ and $F^{\sigma,\sigma}|\Phi_{\rm singlet}\rangle = (N_f/N)|\Phi_{\rm singlet}\rangle$ for all $\sigma = 1, \ldots, N$ [48]. Our first theorem is stated as follows.

Theorem 1.—Consider the attractive SU(N) Hubbard model with the Hamiltonian (1) with $N \geq 3$. When $N_A \neq N_B$, there are exactly two ground states in the whole Fock space. The two ground states are SU(N) singlets and their total fermion numbers are NN_A and NN_B, respectively. When $N_A = N_B$, there are at most two ground states, each of which is an SU(N) singlet and whose total fermion number is $NN_A (= NN_B)$.

We can also show an inequality for a correlation function for the ground state. Let us define an operator $S_{x,y}$ for a pair of sites $x, y \in \Lambda$ (including the case $x = y$) as

$$S_{x,y} = (-1)^x(-1)^y\left(n_x - \frac{N}{2}\right)\left(n_y - \frac{N}{2}\right), \qquad (4)$$

where $(-1)^x = 1$ if $x \in A$ and $(-1)^x = -1$ if $x \in B$. Then, our second theorem is stated as follows.

Theorem 2.—Under the same conditions as in Theorem 1, we have for any ground state $|\Phi_{\rm GS}\rangle$ and for $x, y \in \Lambda$ that

$$\langle\Phi_{\rm GS}|S_{x,y}|\Phi_{\rm GS}\rangle > 0. \qquad (5)$$

The correlation function $\langle\Phi_{\rm GS}|S_{x,y}|\Phi_{\rm GS}\rangle$ is a measure of the charge density wave order. Note that this inequality does not necessarily imply the presence of the long-range order in the thermodynamic limit.

However, when $|N_A - N_B|$ is macroscopically large, we can prove the presence of the long-range order. Assume that $|N_A - N_B| = aN_s$ with a constant a such that $0 \leq a < 1$. Note that $N_s = N_A + N_B$. The order parameter for the charge density wave is

$$S_{\rm CDW} = \sum_{x\in\Lambda}(-1)^x\left(n_x - \frac{N}{2}\right). \qquad (6)$$

Then, our third theorem is stated as follows.

Theorem 3.—Under the same conditions as in Theorem 1, we have for any ground state $|\Phi_{\rm GS}\rangle$ that

$$\langle\Phi_{\rm GS}|(S_{\rm CDW})^2|\Phi_{\rm GS}\rangle > \left(\frac{aNN_s}{2}\right)^2. \qquad (7)$$

Since the right-hand side of the inequality (7) is proportional to N_s^2 for $0 < a < 1$, this theorem shows that the system has long-range order.

Theorem 3 follows from Theorems 1 and 2.

Proof of Theorem 3.—First one finds

$$(S_{\rm CDW})^2 = \sum_{x,y\in\Lambda} S_{x,y}. \qquad (8)$$

By using the inequality (5),

$$\begin{aligned}
&\langle\Phi_{\rm GS}|\sum_{x,y\in\Lambda} S_{x,y}|\Phi_{\rm GS}\rangle \\
&> \langle\Phi_{\rm GS}|\sum_{x,y\in\Lambda}(-1)^x(-1)^y S_{x,y}|\Phi_{\rm GS}\rangle \\
&= \langle\Phi_{\rm GS}|\left[\sum_{x,y\in\Lambda}\left(n_x - \frac{N}{2}\right)\left(n_y - \frac{N}{2}\right)\right]|\Phi_{\rm GS}\rangle \\
&= \langle\Phi_{\rm GS}|\left[\sum\nolimits_{x\in\Lambda}\left(n_x - \frac{N}{2}\right)\right]^2|\Phi_{\rm GS}\rangle. \qquad (9)
\end{aligned}$$

From Theorem 1, the total fermion number of the ground state is NN_A or NN_B. Substituting $\sum_{x\in\Lambda} n_x = NN_A$ or NN_B into Eq. (9) and using $|N_A - N_B| = aN_s$, we obtain Eq. (7). ■

To prove Theorems 1 and 2, we use a matrix representation of eigenstates introduced by Wei *et al.* [47]. First, we will show the following lemma.

Lemma 4.—Consider the attractive SU(N) Hubbard model with the Hamiltonian (1) with $N \geq 3$. When NN_s is odd, there are exactly two ground states. When NN_s is even, there are at most two ground states.

In the following discussion, we only consider the case where NN_s is odd. For even NN_s, see the Supplemental Material [49].

The Majorana representation.—A complex fermion can be decomposed into two Majorana fermions. We define $\gamma^{(1)}_{x,\sigma} = c_{x,\sigma} + c^\dagger_{x,\sigma}$, $\gamma^{(2)}_{x,\sigma} = -i(c_{x,\sigma} - c^\dagger_{x,\sigma})$ at sublattice A and

$\gamma^{(1)}_{x,\sigma} = -i(c_{x,\sigma} - c^\dagger_{x,\sigma})$, $\gamma^{(2)}_{x,\sigma} = c_{x,\sigma} + c^\dagger_{x,\sigma}$ at sublattice B. They satisfy the relations

$$\gamma^{(j)\dagger}_{x,\sigma} = \gamma^{(j)}_{x,\sigma}, \{\gamma^{(j)}_{x,\sigma}, \gamma^{(k)}_{y,\tau}\} = 2\delta_{j,k}\delta_{x,y}\delta_{\sigma,\tau} \tag{10}$$

for all $x, y \in \Lambda$, $\sigma, \tau = 1, \ldots, N$, j, $k = 1$, 2. Using the Majorana representation, we can rewrite Eqs. (2) and (3) as

$$H_{\rm hop} = \sum_{x\in A, y\in B} \sum_{\sigma=1}^{N} t_{x,y} \left(\frac{i}{2}\gamma^{(1)}_{x,\sigma}\gamma^{(1)}_{y,\sigma} - \frac{i}{2}\gamma^{(2)}_{x,\sigma}\gamma^{(2)}_{y,\sigma} \right), \tag{11}$$

$$H_{\rm int} = \sum_{x\in\Lambda} \sum_{\sigma,\tau=1}^{N} U_x \left(\frac{i}{2}\gamma^{(1)}_{x,\sigma}\gamma^{(1)}_{x,\tau} \right) \left(-\frac{i}{2}\gamma^{(2)}_{x,\sigma}\gamma^{(2)}_{x,\tau} \right). \tag{12}$$

The operators on the whole Fock space form a complex vector space. We write this vector space as $\mathcal{O}$. Note that the dimension of $\mathcal{O}$ is 2^{NN_s}. We introduce the Hilbert-Schmidt inner product for $O_1, O_2 \in \mathcal{O}$ as

$$\langle O_1, O_2 \rangle = \frac{1}{2^{NN_s}} {\rm Tr}[O_1^\dagger O_2]. \tag{13}$$

Then, the operators defined by

$$\Gamma^{(1)}_\alpha = i^{\lfloor l(\alpha)/2 \rfloor} \gamma^{(1)}_{x_1,\sigma_1} \cdots \gamma^{(1)}_{x_{l(\alpha)},\sigma_{l(\alpha)}}, \tag{14}$$

$$\Gamma^{(2)}_\alpha = (-i)^{\lfloor l(\alpha)/2 \rfloor} \gamma^{(2)}_{x_1,\sigma_1} \cdots \gamma^{(2)}_{x_{l(\alpha)},\sigma_{l(\alpha)}} \tag{15}$$

form an orthonormal basis of $\mathcal{O}$. Here, $\alpha = ((x_1, \sigma_1), \ldots, (x_{l(\alpha)}, \sigma_{l(\alpha)}))$ denotes a subset of $\Lambda \times \{1, 2, \ldots, N\}$ ordered according to an arbitrary order introduced in $\Lambda \times \{1, 2, \ldots, N\}$. We wrote the length of α as $l(\alpha)$, and $\lfloor l(\alpha)/2 \rfloor$ is the largest integer less than or equal to $l(\alpha)/2$. We write the set of α as $\mathcal{C}$, and the set of even- (odd-) length α as $\mathcal{C}_{\rm even(odd)}$. Here, $|\mathcal{C}| = 2^{NN_s}$ and $|\mathcal{C}_{\rm even}| = |\mathcal{C}_{\rm odd}| = 2^{NN_s-1}$ [50]. We also define the parity operators,

$$\Delta^{(1)} = i^{\lfloor NN_s/2 \rfloor} \prod_{x\in\Lambda} \prod_{\sigma=1}^{N} \gamma^{(1)}_{x,\sigma}, \tag{16}$$

$$\Delta^{(2)} = (-i)^{\lfloor NN_s/2 \rfloor} \prod_{x\in\Lambda} \prod_{\sigma=1}^{N} \gamma^{(2)}_{x,\sigma}, \tag{17}$$

which commute with the Hamiltonian. Here, we assumed that the product is ordered in the same order as $\Gamma^{(1)}_\alpha$ and $\Gamma^{(2)}_\alpha$. Note that $\Delta^{(1)}$ commutes (anticommutes) with $\Delta^{(2)}$ when NN_s is even (odd), and $(\Delta^{(1)})^2 = (\Delta^{(2)})^2 = 1$.

Next, we define the *eigenoperators* of the Hamiltonian.

Definition 5.—An operator $O \in \mathcal{O}$ is said to be an eigenoperator of H with eigenvalue E when $HO = OH = EO$. We denote by $\mathcal{O}^E$ the subspace of $\mathcal{O}$ spanned by the eigenoperators of H with eigenvalue E.

Let us consider the relation between the eigenoperator formalism and the ordinary eigenvector formalism. Let $\{|E, j\rangle | j = 1, \ldots, n_E\}$ be the complete set of eigenvectors of H with eigenvalue E. Then, the subspace of $\mathcal{O}$ spanned by $\{|E, j\rangle\langle E, k| | j, k = 1, \ldots, n_E\}$ corresponds to $\mathcal{O}^E$. Therefore, if the degeneracy of the ground-state eigenvectors is n_E, the degeneracy of the ground-state eigenoperators is n_E^2.

The eigenoperator can be decomposed into four sectors because the Hamiltonian preserves the parity (even or odd) of the number of $\gamma^{(1)}$ and $\gamma^{(2)}$, respectively,

$$\mathcal{O} = \mathcal{O}_{\rm even,even} \oplus \mathcal{O}_{\rm even,odd} \oplus \mathcal{O}_{\rm odd,even} \oplus \mathcal{O}_{\rm odd,odd}, \tag{18}$$

where $\mathcal{O}_{\rm even(odd),even(odd)}$ is the subspace of $\mathcal{O}$ spanned by $\{\Gamma^{(1)}_\alpha \Gamma^{(2)}_\beta | \alpha \in \mathcal{C}_{\rm even(odd)}, \beta \in \mathcal{C}_{\rm even(odd)}\}$.

When NN_s is odd, each parity operator $\Delta^{(1)}$ and $\Delta^{(2)}$ contains an odd number of Majorana operators. Thus, they define maps between different sectors. For example, if O is in the even-even sector, $\Delta^{(1)}O$, $\Delta^{(2)}O$, and $\Delta^{(1)}\Delta^{(2)}O$ are in the odd-even sector, even-odd sector, and odd-odd sector, respectively. Furthermore, these are maps between eigenoperators with the same energy because they commute with the Hamiltonian. Therefore, if the ground-state eigenoperator is unique in the even-even sector, the total degeneracy of the ground-state eigenoperators is four. This means that the ground-state eigenvectors are twofold degenerate.

In the following discussion, we focus on the even-even sector. In this sector, an operator is expressed as

$$O(W) = \sum_{\alpha,\beta\in\mathcal{C}_{\rm even}} W_{\alpha,\beta} \Gamma^{(1)}_\alpha \Gamma^{(2)}_\beta, \tag{19}$$

where W is a $|\mathcal{C}_{\rm even}| \times |\mathcal{C}_{\rm even}|$ matrix. This matrix representation plays an essential role in the proof.

Let an operator $O(W) \in \mathcal{O}_{\rm even,even}$ be an eigenoperator of H with eigenvalue E. Then W satisfies the following two equations [49]:

$$KW + WK + \sum_{x\in\Lambda} \sum_{\sigma,\tau=1}^{N} U_x L_{xx,\sigma\tau} W L_{xx,\sigma\tau} = EW, \tag{20}$$

$$K^\top W + WK^\top + \sum_{x\in\Lambda} \sum_{\sigma,\tau=1}^{N} U_x L^\top_{xx,\sigma\tau} W L^\top_{xx,\sigma\tau} = EW, \tag{21}$$

where $\top$ denotes the transpose. $L_{xy,\sigma\tau}$ and K are $|\mathcal{C}_{\rm even}| \times |\mathcal{C}_{\rm even}|$ Hermitian matrices defined by

$$(L_{xy,\sigma\tau})_{\alpha,\beta} = \langle \Gamma^{(1)}_\alpha, \frac{i}{2}\gamma^{(1)}_{x,\sigma}\gamma^{(1)}_{y,\tau}\Gamma^{(1)}_\beta \rangle, \tag{22}$$

$$(K)_{\alpha,\beta} = \sum_{x\in A,y\in B}\sum_{\sigma=1}^{N} t_{x,y}(L_{xy,\sigma\sigma})_{\alpha,\beta}. \qquad (23)$$

Since $L_{xy,\sigma\tau}$ and K are Hermitian, we find $W^\dagger$ also satisfies Eqs. (20) and (21), and hence, $O(W^\dagger)$ is also an eigenoperator with eigenvalue E. Thus, W can be symmetrized or antisymmetrized to be Hermitian.

Let us define the normalization condition for operators $O \in \mathcal{O}$ as $\langle O, O\rangle = 1$, where the inner product is defined by Eq. (13). Since $\langle O(W), O(W)\rangle = \mathrm{Tr}[W^\dagger W]$, the normalization condition for Hermitian W is $\mathrm{Tr}[W^2] = 1$. Under the normalization condition, the expectation value of H with respect to O is defined by $\langle O, HO\rangle$. We define $E(W) = \langle O(W), HO(W)\rangle$ for a normalized Hermitian matrix W. Then, $E(W)$ is calculated as

$$E(W) = 2\mathrm{Tr}[KW^2] + \sum_{x\in\Lambda}\sum_{\sigma,\tau=1}^{N} U_x \mathrm{Tr}[WL_{xx,\sigma\tau}WL_{xx,\sigma\tau}]. \qquad (24)$$

For a Hermitian matrix W, by diagonalizing it by a unitary matrix U like $W = UDU^\dagger, D = \mathrm{diag}(\lambda_1, \ldots, \lambda_{|\mathcal{C}_{\mathrm{even}}|})$, we can define a new matrix $|W| = U|D|U^\dagger$, where $|D| = \mathrm{diag}(|\lambda_1|, \ldots, |\lambda_{|\mathcal{C}_{\mathrm{even}}|}|)$. Then we have $E(|W|) \le E(W)$. If W is normalized, $|W|$ is also normalized. Therefore, by the variational principle, if $O(W)$ is a ground-state eigenoperator, then $O(|W|)$ is also a ground-state eigenoperator.

Implication of connectivity.—Here we prove the following lemmas, which is essential in the proof of Lemma 4 and Theorem 2.

Lemma 6.—Consider the attractive SU(N) Hubbard model with $N \ge 3$. If a positive semidefinite matrix W satisfies Eqs. (20) and (21), then W is either positive definite or zero.

See the Supplemental Material for a proof [49]. The condition $N \ge 3$ comes from this lemma. From Lemma 6, we can show the following lemma.

Lemma 7.—Consider the attractive SU(N) Hubbard model with $N \ge 3$. If a ground-state eigenoperator is $O(W)$, then W is either positive or negative definite.

Proof of Lemma 7.—Let $O(W) \in \mathcal{O}_{\mathrm{even,even}}$ be a ground-state eigenoperator in the even-even sector. Then, $O(|W|)$ is also a ground-state eigenoperator. Thus, $|W| - W$ is a positive semidefinite matrix which satisfies Eqs. (20) and (21) with $E = E_{\mathrm{GS}}$. Here, E_{GS} is the ground-state energy. From Lemma 6, $|W| - W$ is either positive definite or zero. If $|W| - W$ is positive definite, all eigenvalues of W are strictly negative, which means that W is negative definite. If $|W| - W$ is zero, then $W = |W|$. By using Lemma 6, $|W|$ is positive definite because $|W|$ is a nonvanishing positive semidefinite matrix which satisfies Eqs. (20) and (21). Thus, W is also positive definite. Therefore, W is positive or negative definite. ∎

We can prove Lemma 4 from Lemma 7.

Proof of Lemma 4 for odd NN_s.—Suppose that the ground-state eigenoperators in the even-even sector are degenerate. Then, we pick two orthogonal ground-state eigenoperators $O(W_1)$ and $O(W_2)$. Then, $\mathrm{Tr}[W_1^\dagger W_2] = \langle O(W_1), O(W_2)\rangle = 0$. But Lemma 7 implies that $\mathrm{Tr}[W_1^\dagger W_2] \neq 0$ [51]. Since this is a contradiction, the ground-state eigenoperator is unique in the even-even sector. Therefore, there are exactly two ground states in total. ∎

We will complete the proof of Theorem 1 for odd NN_s by identifying the SU(N) quantum number and the total fermion number of the ground states.

Proof of Theorem 1 for odd NN_s.—First, we determine the SU(N) quantum numbers of the ground states. Note that the ground-state degeneracy in an SU(N) invariant model is at least N unless the ground states are SU(N) singlets. This, together with Lemma 4, implies that the two ground states are SU(N) singlets.

To determine the fermion number, we consider a toy model on the same lattice with long-range interactions. The Hamiltonian of the model is

$$H_{\mathrm{toy}} = \sum_{x\in A,y\in B}\left(n_x - \frac{N}{2}\right)\left(n_y - \frac{N}{2}\right). \qquad (25)$$

The ground states of the model are twofold degenerate. Let us write the two ground states as $|\Phi_\pm\rangle$. Then,

$$n_x|\Phi_\pm\rangle = \begin{cases} \frac{(N\pm N)}{2}|\Phi_\pm\rangle & \text{if } x \in A, \\ \frac{(N\mp N)}{2}|\Phi_\pm\rangle & \text{if } x \in B. \end{cases} \qquad (26)$$

As shown in Ref. [22] (see the Supplemental Material [49]), $|\Phi_+\rangle$ and $|\Phi_-\rangle$ are also the ground states of the attractive SU(N) Hubbard model in the large-U_x limit. The fermion numbers of $|\Phi_+\rangle$ and $|\Phi_-\rangle$ are NN_A and NN_B, respectively. The eigenoperators are written as $O_\pm = |\Phi_\pm\rangle\langle\Phi_\pm| = 2^{-NN_s}\prod_{x\in\Lambda}\prod_{\sigma=1}^{N}(1 \pm i\gamma_{x,\sigma}^{(1)}\gamma_{x,\sigma}^{(2)})$. Then, $2^{NN_s-1}(O_+ + O_-)$ is in the $\mathcal{O}_{\mathrm{even,even}}$ sector and written as $O(I)$, where I is the identity matrix of size $|\mathcal{C}_{\mathrm{even}}|$. Let $O(W_{\mathrm{GS}}) \in \mathcal{O}_{\mathrm{even,even}}$ be the ground-state eigenoperator of the original Hamiltonian H. By Lemma 7, $\langle O(I), O(W_{\mathrm{GS}})\rangle = \mathrm{Tr}[W_{\mathrm{GS}}] \neq 0$, because W_{GS} is positive or negative definite. Suppose we expand $O(W_{\mathrm{GS}})$ in an orthonormal basis of $\mathcal{O}$ including $|\Phi_+\rangle\langle\Phi_+|$ and $|\Phi_-\rangle\langle\Phi_-|$. Since $\langle O(I), O(W_{\mathrm{GS}})\rangle \neq 0$, the coefficient of either $|\Phi_+\rangle\langle\Phi_+|$ or $|\Phi_-\rangle\langle\Phi_-|$ is nonzero.

Let $P_{A(B)}$ be the projection operator onto states with NN_A (NN_B) fermions. Then, either $P_A O(W_{\mathrm{GS}}) P_A$ or $P_B O(W_{\mathrm{GS}}) P_B$ is nonzero. Since $P_{A(B)}$ commutes with the Hamiltonian, the projected operators are also ground-state eigenoperators. Therefore, there is a ground state whose fermion number is NN_A or NN_B. Because of the particle-hole symmetry, if there is a ground state with the fermion number NN_A (NN_B), there must be another ground state with the fermion number NN_B (NN_A). Note that $NN_A \neq NN_B$ when NN_s is odd. This, together with Lemma 4, implies that there are exactly two ground states, and the fermion numbers of the two ground states are NN_A and NN_B, respectively. ∎

To prove Theorem 2, we use the following lemma [52].

Lemma 8.—Let M, M' be $D \times D$ Hermitian matrices. If M is positive or negative definite and M' is nonvanishing, then

$$\mathrm{Tr}[MM'MM'] > 0. \tag{27}$$

Proof of Theorem 2 for odd NN_s.—We consider the ground-state expectation value of the operator $S_{x,y}$ defined as Eq. (4). First, using the Majorana representation, $S_{x,y}$ is expressed as

$$S_{x,y} = \sum_{\sigma,\tau=1}^{N} \left(\frac{i}{2}\gamma^{(1)}_{x,\sigma}\gamma^{(1)}_{y,\tau}\right)\left(-\frac{i}{2}\gamma^{(2)}_{x,\sigma}\gamma^{(2)}_{y,\tau}\right). \tag{28}$$

When NN_s is odd, the ground-state eigenoperators are fourfold degenerate in total and in the $\mathcal{O}_{\mathrm{even,even}}$, $\mathcal{O}_{\mathrm{even,odd}}$, $\mathcal{O}_{\mathrm{odd,even}}$, $\mathcal{O}_{\mathrm{odd,odd}}$ sectors, respectively. Let us first consider the $\mathcal{O}_{\mathrm{even,even}}$ sector. Assume that the ground-state eigenoperator in this sector is expressed as $O(W_{\mathrm{GS}})$, where W_{GS} is a Hermitian matrix. Then, the expectation value for $S_{x,y}$ is calculated as

$$\langle O(W_{\mathrm{GS}}), S_{x,y} O(W_{\mathrm{GS}})\rangle = \sum_{\sigma,\tau=1}^{N} \mathrm{Tr}[W_{\mathrm{GS}} L_{xy,\sigma\tau} W_{\mathrm{GS}} L_{xy,\sigma\tau}], \tag{29}$$

where $L_{xy,\sigma\tau}$ is a Hermitian matrix defined as Eq. (22). From Lemma 7, W_{GS} is positive or negative definite. Using Lemma 8, we obtain $\mathrm{Tr}[W_{\mathrm{GS}} L_{xy,\sigma\tau} W_{\mathrm{GS}} L_{xy,\sigma\tau}] > 0$. Thus, one finds $\langle O(W_{\mathrm{GS}}), S_{x,y} O(W_{\mathrm{GS}})\rangle > 0$.

We next note that the ground-state eigenoperators in the $\mathcal{O}_{\mathrm{even,odd}}$, $\mathcal{O}_{\mathrm{odd,even}}$, $\mathcal{O}_{\mathrm{odd,odd}}$ sectors are $\Delta^{(1)} O(W_{\mathrm{GS}})$, $\Delta^{(2)} O(W_{\mathrm{GS}})$, $\Delta^{(1)}\Delta^{(2)} O(W_{\mathrm{GS}})$, respectively. Since both of $\Delta^{(1)}$ and $\Delta^{(2)}$ commute with $S_{x,y}$, the expectation value of $S_{x,y}$ does not depend on the choice of the ground state. If two operators $O_1, O_2 \in \mathcal{O}$ are in different sectors, $\langle O_1, S_{x,y} O_2\rangle$ is zero because each term of $S_{x,y}$ has the even number of $\gamma^{(1)}$ and $\gamma^{(2)}$ fermions. Therefore, we obtain Eq. (5) for any ground state when NN_s is odd. ∎

Summary.—We presented the degeneracy, the fermion number, and the SU(N) quantum number of the ground state of the attractive SU(N) Hubbard model with particle-hole symmetry. We also showed that the ground state has the charge density wave long-range order when $|N_A - N_B|$ is macroscopically large. One can easily extend our results to include attractive (repulsive) interactions between two sites in the same (different) sublattice. Although we focused on a model with SU(N) symmetry, we expect that our approach will find further applications to N-component fermionic models with flavor-dependent hopping and interaction [53]. It would also be interesting to consider the application of the method to other multi-component fermionic systems such as SO(5) symmetric models [54].

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*hironobu-yoshida57@g.ecc.u-tokyo.ac.jp

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[48] These conditions can also be expressed with the SU(N) version of spin operators. From the operators $F^{\sigma,\tau}$, one can construct new operators as $F = \sum_{\sigma=1}^{N} F^{\sigma,\sigma}$ and $T^a = \sum_{\sigma,\tau=1}^{N} \mathcal{T}^a_{\sigma,\tau} F^{\sigma,\tau}$ for $a = 1, \ldots, N^2 - 1$, where $\mathcal{T}^a_{\sigma,\tau}$ are the generators of SU(N) Lie algebra. Here, F is the total fermion number operator, and T^a ($a = 1, \ldots, N^2 - 1$) are the SU(N) version of spin operators. The conditions for an SU(N) singlet with fermion number N_f are written as $T^a|\Phi_{\text{singlet}}\rangle = 0$ and $F|\Phi_{\text{singlet}}\rangle = N_f|\Phi_{\text{singlet}}\rangle$.
[49] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.100201 for detailed derivations and proofs.
[50] $|S|$ denotes the number of elements in S.
[51] See p. 364 of Ref. [7] for a proof.
[52] Here, we provide a proof of Lemma 8. Take the orthonormal basis $\{\boldsymbol{u}_j\}_{j=1,\ldots,D}$, which consists of the eigenvectors of M, i.e., $M\boldsymbol{u}_j = \lambda_j \boldsymbol{u}_j$. If M is positive (negative) definite, $\lambda_j > 0$ ($\lambda_j < 0$) for all $j = 1, \ldots, D$. Then, $\mathrm{Tr}[MM'MM'] = \sum_{j,k=1}^{D} \boldsymbol{u}_j^\dagger MM'M\boldsymbol{u}_k \boldsymbol{u}_k^\dagger M'\boldsymbol{u}_j = \sum_{j,k=1}^{D} \lambda_j\lambda_k |\boldsymbol{u}_k^\dagger M'\boldsymbol{u}_j|^2 > 0$, because $|\boldsymbol{u}_k^\dagger M'\boldsymbol{u}_j|^2 > 0$ for some j and k..
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