

Operational Resource Theory of Imaginarity

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Wave-particle duality is one of the basic features of quantum mechanics, giving rise to the use of complex numbers in describing states of quantum systems and their dynamics and interaction. Since the inception of quantum theory, it has been debated whether complex numbers are essential or whether an alternative consistent formulation is possible using real numbers only. Here, we attack this long-standing problem theoretically and experimentally, using the powerful tools of quantum resource theories. We show that, under reasonable assumptions, quantum states are easier to create and manipulate if they only have real elements. This gives an operational meaning to the resource theory of imaginarity. We identify and answer several important questions, which include the state-conversion problem for all qubit states and all pure states of any dimension and the approximate imaginarity distillation for all quantum states. As an application, we show that imaginarity plays a crucial role in state discrimination, that is, there exist real quantum states that can be perfectly distinguished via local operations and classical communication but that cannot be distinguished with any nonzero probability if one of the parties has no access to imaginarity. We confirm this phenomenon experimentally with linear optics, discriminating different two-photon quantum states by local projective measurements. Our results prove that complex numbers are an indispensable part of quantum mechanics.

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Introduction.—Complex numbers, which originated in mathematics, are widely used in mechanics, electrodynamics, and optics, allowing for an elegant formulation of these theories. The rise of quantum mechanics as a unified picture of waves and particles further strengthened the prominent role of complex numbers in physics. According to the postulates of quantum mechanics, a state of a quantum system is described by a wave function $\Psi(x) = |\Psi(x)|e^{-i\phi(x)}$ with probability amplitude $|\Psi(x)|^2$ and phase $\phi(x)$. The wave-based point of view provides an important set of tools for the formulation and construction of quantum physics. Therefore, it is natural to ask whether the complex arithmetic in quantum mechanics arising from the imaginary part of $e^{i\phi}$ is necessary to describe the fundamental properties and dynamics of a quantum system. In other words, can quantum physics be restated in a formalism using real numbers only?

One approach to address this question is to use the standard rules of quantum mechanics but to require all states and measurement operators to have real elements

only [1–9]. The aim of this approach, then, is to find physical effects and applications that are possible in standard quantum mechanics but impossible in a version restricted to real numbers [10,11]. It has been noted that this real-vector-space quantum theory is fundamentally different from the standard one in various aspects, e.g., it is bilocally tomographic [3], a rebit (real qubit) can be maximally entangled with many rebits [4,6,7], and it allows optimal transport of information from preparation to measurement [8].

Another reason to distinguish between complex and real quantum states is the effort required to establish them in experimental setups. An important example is polarization-encoded photonic systems, where we can realize an arbitrary rotation around the y axis by a single half-wave plate, while for a rotation around the z axis two additional quarter-wave plates are needed. The fact that a certain type of transformation is easy to perform is the basic feature of any quantum resource theory [12–14]. This justifies the study of the “resource theory of imaginarity” [15] using the

framework of general quantum resource theories that has been successfully applied to investigate basic properties and applications of quantum entanglement [16], quantum coherence [17], and quantum thermodynamics [18,19].

The aim of this Letter is twofold. First, we provide the resource theory of imaginarity with an operational meaning, discussing the experimental role of complex and real operations, i.e., quantum operations that do not create imaginarity. Second, we identify and answer several important questions within this theory. As an application, we show that imaginarity plays a crucial role for local quantum state discrimination when complex numbers are allowed in the measurement. We show that there exist real bipartite states that can be perfectly distinguished via local operations and classical communication (LOCC) but that cannot be distinguished with any nonzero probability via LOCC restricted to real local measurements. In the context of quantum tomography, a similar effect has been observed previously in [10]. By experimentally measuring the optimal distinguishing probability for different families of mixed states, our results clearly demonstrate that complex numbers play a distinguished role in quantum theory, allowing for phenomena that would not be possible with real quantum mechanics alone.

Resource theory of imaginarity.—The first step to formulating any resource theory is to identify the free states of the theory, i.e., quantum states that, within the theory under study, can be created at no cost. Similar to the resource theory of coherence [17,20], we specify a particular basis $\{|j\rangle\}$, and a pure quantum state can be written as $|\psi\rangle = \sum_j c_j |j\rangle$ with complex coefficients c_j that satisfy $\sum_j |c_j|^2 = 1$. The natural choice for free states in the theory of imaginarity are “real states,” i.e., quantum states with all coefficients c_j being real (up to a non-observable overall phase) [15]. Mixed real states can be identified as convex combinations of real pure states $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$. The set of all real states will be denoted by \mathcal{R} . It can also be characterized as the set of states with a real density matrix [15].

The formulation of a resource theory is completed by defining an appropriate set of free operations corresponding to physical transformations of quantum systems that are easy to implement. In general, quantum operations can be specified by a set of Kraus operators $\{K_j\}$ satisfying the completeness relation $\sum_j K_j^\dagger K_j = \mathbb{1}$. In the case of probabilistic transformations, the Kraus operators satisfy the more general condition $\sum_j K_j^\dagger K_j \leq \mathbb{1}$. As the free operations of imaginarity theory, we identify quantum operations that admit a Kraus decomposition having only real elements in the free basis [15]: $\langle m|K_j|n\rangle \in \mathbb{R}$ for all j, m, n . Such transformations are called “real operations” [15]. This definition guarantees that real operations cannot create imaginarity, even if interpreted as a general quantum measurement. In this case, the postmeasurement state will

be real for any real initial state, regardless of the measurement outcome.

A desirable feature of a quantum resource theory is the existence of a golden unit: a quantum state that can be converted into any other state via free operations. In the resource theory of imaginarity, the golden unit is the maximally imaginary state $|\hat{\dagger}\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$. Interestingly, via real operations it is possible to convert $|\hat{\dagger}\rangle$ into any state of arbitrary dimension [15]. Another maximally imaginary state is given by $|\hat{\ominus}\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$. A detailed discussion of the main features of quantum resource theories, including resource quantifiers and state conversion properties under free operations, is provided in the accompanying paper [21]. If not stated otherwise, we consider quantum systems of arbitrary but finite dimension in the following.

Quantum state conversion.—We will now present a complete solution for the conversion problem via real operations for all qubit states, characterizing when a qubit state ρ can be converted into another qubit state σ via real operations. To this end, recall that any single-qubit state can be represented by a real three-dimensional Bloch vector. Now, the transition $\rho \rightarrow \sigma$ is possible via real operations if and only if

$$s_y^2 \leq r_y^2, \quad \frac{1 - s_z^2 - s_x^2}{s_y^2} \geq \frac{1 - r_z^2 - r_x^2}{r_y^2}, \quad (1)$$

where \mathbf{r} and \mathbf{s} are the Bloch vectors of the initial and the target state, respectively. The proof of this statement is given in [21] and relies on methods developed earlier within the resource theory of quantum coherence [22–24].

Notably, there exist states σ that cannot be obtained from a given state ρ via real operations. In this case, it might still be possible to achieve the conversion probabilistically. Defining the conjugated state $|\psi^*\rangle = \sum_j c_j^* |j\rangle$, we now present the optimal conversion probability via real operations for any two pure states.

Theorem 1: The maximum probability for a pure state transformation $|\psi\rangle \rightarrow |\phi\rangle$ via real operations is given by

$$P(|\psi\rangle \rightarrow |\phi\rangle) = \min \left\{ \frac{1 - |\langle\psi^*|\psi\rangle|}{1 - |\langle\phi^*|\phi\rangle|}, 1 \right\}. \quad (2)$$

The proof of the theorem makes use of properties of general resource quantifiers, we refer to [21] for more details. We note that the theorem holds true for systems of arbitrary finite dimension. Moreover, if the target state $|\phi\rangle$ is real, the conversion probability is 1 for any initial state.

Approximate imaginarity distillation.—So far we have discussed exact transformations between quantum states via real operations, both deterministically and stochastically. We will now go one step further and consider approximate transformations in the cases when an exact transformation is impossible. Typically, one aims to convert

a state ρ into the most valuable quantum state, which in the resource theory of imaginarity is the maximally imaginary state $|\hat{\dagger}\rangle$. This leads us to the ‘‘fidelity of imaginarity,’’ quantifying the maximal fidelity between a state ρ and the maximally imaginary state, achievable via real operations Λ :

$$F_I(\rho) = \max_{\Lambda} \langle \hat{\dagger} | \Lambda[\rho] | \hat{\dagger} \rangle. \quad (3)$$

In the following, we will provide a closed expression for F_I , establishing at the same time a close connection to the robustness of imaginarity, defined as [15]

$$\mathcal{J}_R(\rho) = \min_{\tau} \left\{ s \geq 0 : \frac{\rho + s\tau}{1+s} \in \mathcal{R} \right\}, \quad (4)$$

where the minimum is taken over all quantum states τ and all $s \geq 0$. We note that similar quantifiers have been studied earlier in entanglement theory [25–27] and the resource theory of coherence [28,29].

Theorem 2: For any quantum state ρ , the fidelity of imaginarity is given as

$$F_I(\rho) = \frac{1 + \mathcal{J}_R(\rho)}{2} = \frac{1}{2} + \frac{1}{4} \|\rho - \rho^T\|_1, \quad (5)$$

where T denotes transposition and $\|M\|_1 = \text{Tr} \sqrt{M^\dagger M}$ is the trace norm.

This theorem provides at the same time a closed formula for the fidelity of imaginarity and the robustness of imaginarity for systems of arbitrary finite dimension. The proof of Theorem 2 further implies that the maximum in Eq. (3) can be achieved with a real operation Λ having a two-dimensional image space. Interestingly, these results can be extended to a general class of quantum resource theories (see Corollary 14 in [30]). The proof of Theorem 2 is presented in [21], where we also give more details on the robustness measure and its role in general quantum resource theories.

Applications.—We will now discuss applications of imaginarity as a resource for discrimination of quantum states and quantum channels. Channel discrimination can be seen as a game, where one has access to a ‘‘black box’’ with the promise that it implements a quantum channel Λ_j with probability p_j . The goal of the game is to guess Λ_j by applying the black box to a quantum state ρ , followed by a suitably chosen positive operator valued measure (POVM) $\{M_j\}$ that satisfies $\sum_j M_j = \mathbb{1}$, $M_j \geq 0$. The measurement outcome j then serves as a basis for guessing that the black box has implemented the channel Λ_j in the corresponding realization of the experiment. The probability of correctly guessing an ensemble of channels $\{p_j, \Lambda_j\}$ in this procedure is given by $p_{\text{succ}}(\rho, \{p_j, \Lambda_j\}, \{M_j\}) = \sum_j p_j \text{Tr}[M_j \Lambda_j(\rho)]$.

Recently, it has been shown that any quantum resource provides an operational advantage in some channel

discrimination task [31,32]. Specifically, for the resource theory of imaginarity, it holds that

$$\max_{\{p_j, \Lambda_j\}, \{M_j\}} \frac{p_{\text{succ}}(\rho, \{p_j, \Lambda_j\}, \{M_j\})}{\max_{\sigma \in \mathcal{R}} p_{\text{succ}}(\sigma, \{p_j, \Lambda_j\}, \{M_j\})} = 1 + \mathcal{J}_R(\rho). \quad (6)$$

Eq. (6) implies that for any quantum state ρ that has nonreal elements there exists a set of channels such that the optimal guessing probability is strictly larger than for any $\sigma \in \mathcal{R}$.

Another closely related task is ‘‘quantum state discrimination’’ [33], where one aims to distinguish between quantum states ρ_j , each given with probability p_j . To this end, one performs quantum measurements described by a POVM $\{M_j\}$. The average probability for correctly guessing the state is $p_{\text{succ}}(\{p_j, \rho_j\}, \{M_j\}) = \sum_j p_j \text{Tr}[M_j \rho_j]$. In general, one aims to find a strategy $\{M_j\}$ that maximizes the success probability for a given ensemble of states and probabilities $\{p_j, \rho_j\}$. For a broad class of quantum resource theories, the performance of this task can be quantified by extending the robustness quantifier from states to measurements [32,34,35].

Going one step further, we will now show that complex numbers play an indispensable role in ‘‘local state discrimination’’ [10,36]. Assume that the states to be discriminated are shared by two distant parties, Alice and Bob. It was shown in [37] that any pair of pure orthogonal states can be perfectly distinguished via LOCC. To perfectly distinguish the states $\{\rho_j^{AB}\}$ via LOCC, there must exist a POVM with elements $\{M_j\}$ of the form $M_j = \sum_k A_{j,k} \otimes B_{j,k}$ and the property $\text{Tr}(M_j \rho_k^{AB}) = \delta_{jk}$ for all j and k . If the states $\{\rho_j^{AB}\}$ are real, we are particularly interested in perfect discrimination with local real operations and classical communication (LRCC) where all $A_{j,k}, B_{j,k}$ must be real and symmetric. Indeed, if two states are pure, orthogonal, and real, such perfect LRCC discrimination is possible (see [21] for more details).

For some real mixed states, instead, the situation is radically different. Consider the states

$$\begin{aligned} \rho_1^{AB} &= \frac{1}{2} (|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|), \\ \rho_2^{AB} &= \frac{1}{2} (|\phi^+\rangle\langle\phi^+| + |\psi^-\rangle\langle\psi^-|) \end{aligned} \quad (7)$$

with the Bell states $|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. These states can be perfectly distinguished via LOCC. To see this, we express the states as follows:

$$\begin{aligned} \rho_1^{AB} &= \frac{1}{4} (1 + \sigma_y \otimes \sigma_y), \\ \rho_2^{AB} &= \frac{1}{4} (1 - \sigma_y \otimes \sigma_y). \end{aligned} \quad (8)$$

Consider now the POVM defined by

$$\begin{aligned} M_1 &= |\hat{+}\rangle\langle\hat{+}| \otimes |\hat{+}\rangle\langle\hat{+}| + |\hat{-}\rangle\langle\hat{-}| \otimes |\hat{-}\rangle\langle\hat{-}|, \\ M_2 &= |\hat{+}\rangle\langle\hat{+}| \otimes |\hat{-}\rangle\langle\hat{-}| + |\hat{-}\rangle\langle\hat{-}| \otimes |\hat{+}\rangle\langle\hat{+}|. \end{aligned} \quad (9)$$

This POVM can be implemented as an LOCC protocol if Alice and Bob perform local measurements in the $\{|\hat{+}\rangle, |\hat{-}\rangle\}$ basis and share their measurement outcomes via a classical channel. We verify the equality $\text{Tr}(M_j \rho_k^{AB}) = \delta_{jk}$, implying that this POVM perfectly discriminates the states in Eq. (7). On the other hand, the states in Eq. (7) cannot be distinguished via LRCC with any nonzero probability. To see this, note that $\text{Tr}[S\sigma_y] = 0$ for any real symmetric 2×2 matrix S . It follows that, for any POVM element $M_j = \sum_k A_{j,k} \otimes B_{j,k}$ with real symmetric matrices $A_{j,k}$, it holds

$$\text{Tr}(M_j \rho_1^{AB}) = \text{Tr}(M_j \rho_2^{AB}) = \frac{1}{4} \text{Tr}(M_j). \quad (10)$$

This means that the states in Eq. (7) are completely indistinguishable via LRCC even if we consider imperfect state discrimination with finite error.

The states in Eq. (7) show the role of imaginarity for quantum state discrimination in an extreme way. It is clear from the above discussion that this effect is also observed if only one of the parties is limited to real operations and the other party has access to all quantum operations locally. Nevertheless, the states can be perfectly distinguished by LOCC if both Alice and Bob can perform general quantum measurements locally.

These results further highlight the relevance of complex numbers in quantum mechanics. Note that the states in Eq. (7) have real elements in the computational basis. This means that they are also valid states in “real quantum theory” [1–9], which is the restriction of quantum theory to real states and real measurements. In such a theory, two remote parties would not be able to distinguish these states with any nonzero probability, whereas they are perfectly distinguishable in reality.

Experimental relevance of imaginarity.—Here, we perform a comparison of real operations and general quantum operations in optical experiments, focusing on the single-photon interferometer setup with half-wave plates (HWP), quarter-wave plates (QWPs) and polarizing beam splitters as the building blocks. Under these assumptions, a real quantum operation acting on path degree of a d dimensional system can be implemented with $(d^6 - d^3)/2$ unset wave plates, whereas a general quantum operation requires at least $d^6 - 1$ unset wave plates for the implementation. We assume that both operations are implemented via a unitary dilation. For large d , this implementation allows one to reduce the number of HWPs (QWPs) by half if we restrict ourselves to real operations (see the accompanying article [21] for more details). Similar results are found in implementing a real n outcome generalized measurement on a

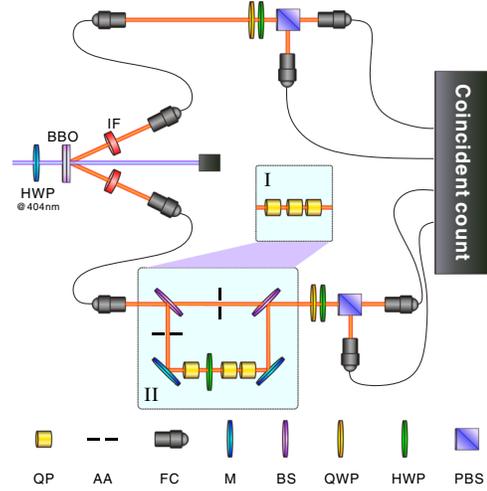


FIG. 1. Experimental setup for local state discrimination. The experiments are carried out using linear optics. We prepare entangled photon sources and perform local projective measurements on each photon, identifying the successful guessing probability by classical communications. The optical elements are HWP, QWP, β -BaB₂O₄ (BBO), interference filter (IF), adjustable aperture (AA), beam splitter (BS), mirror (M), quartz plate (QP), and fiber coupler (FC). We refer to Ref. [41] for more details.

single polarization-encoded qubit [21]. Even in this case, using real measurements instead of general measurements reduces the number of unset wave plates by half in the limit of large n . These results show that under assumptions commonly applied in optical experiments, real states are easier to create and real operations are easier to perform compared to general states and operations. This justifies the choice of real states (operations) as the free states (operations) of the resource theory of imaginarity [21].

Experimental local state discrimination.—As discussed earlier, imaginarity plays an important role for quantum state discrimination. We devised an experimental setup for local state discrimination using two entangled photons. The experimental setup is shown in Fig. 1. After generating polarization-entangled photon pairs, we prepare different mixed states for discrimination experiments (light blue regions I and II in Fig. 1). Local projective measurements are performed on each photon via the combination of a QWP, an HWP, and a polarizing beam splitter. From the photon detecting and coincident system, we can experimentally determine the guessing probability [38–40].

Our experiment consists of two parts. In the first part, we consider the discrimination of the two states in Eq. (7). These states can be perfectly distinguished by performing local projective measurements in the maximally imaginary basis $\{|\hat{+}\rangle, |\hat{-}\rangle\}$ and sharing the measurement outcomes via a classical channel. The experimental results are shown in Figs. 2(a) and 2(b). The experimental success probability reads $P = 0.984 \pm 0.002$. To demonstrate that the states

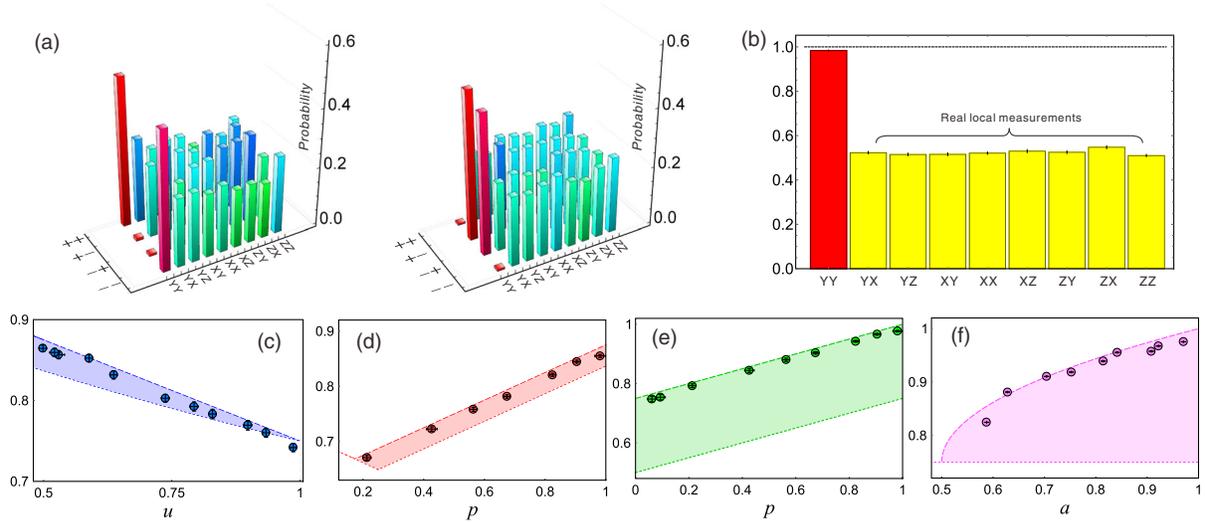


FIG. 2. Experimental results for local state discrimination. All discrimination tasks concern output states after some local free operations, when the initial state is prepared as a Bell state. (a),(b) present the probabilities for the extreme examples in Eq. (7). (b) shows the experimentally measured guessing probabilities under local measurements. The horizontal axes of (a) represent the measurement results $\{++, +-, -+, --\}$, where $+, -$ corresponds to the result ± 1 of local observable $A \in \{X, Y, Z\}$. (c)–(f) present results for experimental discrimination of mixed states via local projective measurements and classical communication. The families of states are prepared as follows: (c) $|\phi^+\rangle\langle\phi^+|$ and $[u|\phi^+\rangle\langle\phi^+| + (1-u)|\psi\rangle\langle\psi| + |\phi^-\rangle\langle\phi^-|]/2$, where $|\psi\rangle$ is a nontrivial entangled pure state; (d) $p|\phi^+\rangle\langle\phi^+| + (1-p)\mathbb{1}/4$ and $(|\phi^+\rangle\langle\phi^+| + |\psi\rangle\langle\psi| + 2|\phi^-\rangle\langle\phi^-|)/4$; (e) $(|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|)/2$ and $p|\phi^+\rangle\langle\phi^+| + (1-p)\mathbb{1}/4$; (f) $(|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|)/2$ and $s|\phi^+\rangle\langle\phi^+| + (1-s)|\phi^-\rangle\langle\phi^-|$, with purity $a = 1-2s + 2s^2$. Dashed and dotted lines represent bounds for guessing probabilities using general projective measurements and real projective measurements, respectively. The shaded areas represent the advantage of using complex measurements over real ones. In (d), general projective measurements do not provide any advantage compared to real projective measurements in the range $p < 0.173$. See [41] for more details.

cannot be distinguished with real local measurements, we show the experimentally measured success probabilities under the σ_x and σ_z measurements, since real measurements can be written as a combination of $\mathbb{1}$, σ_x , and σ_z . In this case, the output is nearly a uniform distribution. From Fig. 2(b), we extract experimentally determined guessing probabilities under different local Pauli measurements. We can see that imaginarity is necessary in the measurement of both subsystems to improve the guessing probability.

In Figs. 2(c)–2(f), we show the experimental results for distinguishing different families of mixed states given in the caption of Fig. 2. All dashed lines represent theoretically derived maximum guessing probabilities via LOCC; dotted lines represent the aforementioned probabilities under LRCC (see the caption of Fig. 2 and Ref. [41] for more details). Note that all states considered here have only real elements in the computational basis.

These results clearly demonstrate the indispensable role of imaginarity in local state discrimination, even if the states to be distinguished have real density matrices.

Discussion.—In this Letter, we investigate the resource theory of imaginarity, studying the role of complex numbers in quantum mechanics in an operational way. We demonstrate the usefulness of our methods in local state discrimination, where two remote parties aim to distinguish states by applying local operations and classical

communication. We show theoretically and experimentally that there exist real quantum states that can be perfectly distinguished in this setup if imaginarity is used in the local measurements. However, when restricted to only real measurements, the states cannot be distinguished with any nonzero probability. This demonstrates that complex numbers are an essential ingredient of quantum mechanics.

The usefulness of complex numbers in quantum mechanics is worth an in-depth study also in the light of the recent advances in quantum technologies. An important example is quantum computers, which can solve certain problems of interest significantly faster than any classical computer [46,47]. As of today, the reason for this quantum advantage is not completely understood, especially when it comes to quantum computers operating on noisy states [48–52]. A quantitative analysis of imaginarity in quantum computers can shed new light on the quantum features required for the quantum speedup.

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