Exact Critical Casimir Amplitude of Anisotropic Systems from Conformal Field Theory and Self-Similarity of Finite-Size Scaling Functions in $d \geq 2$ Dimensions

Volker Dohm and Stefan Wessel

Institute for Theoretical Physics, RWTH Aachen University, 52056 Aachen, Germany

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The exact critical Casimir amplitude is derived for anisotropic systems within the $d = 2$ Ising universality class by combining conformal field theory with anisotropic φ^4 theory. Explicit results are presented for the general anisotropic scalar φ^4 model and for the fully anisotropic triangular-lattice Ising model in finite rectangular and infinite strip geometries with periodic boundary conditions. These results demonstrate the validity of multiparameter universality for confined anisotropic systems and the nonuniversality of the critical Casimir amplitude. We find an unexpected complex form of self-similarity of the anisotropy effects near the instability where weak anisotropy breaks down. This can be traced back to the property of modular invariance of isotropic conformal field theory for $d = 2$. More generally, for $d > 2$ we predict the existence of self-similar structures of the finite-size scaling functions of $O(n)$-symmetric systems with planar anisotropies and periodic boundary conditions both in the critical region for $n \geq 1$ as well as in the Goldstone-dominated low-temperature region for $n \geq 2$.

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Fluctuation-induced thermodynamic forces are ubiquitous in confined condensed matter systems [1]. They exist in both isotropic systems such as fluids, superfluids, and binary liquid mixtures [2,3] as well as in anisotropic systems such as liquid crystals [1,4], superconductors [5], and compressible solids [6]. Near a critical point, so-called critical Casimir forces [2,3] arise from long-range critical fluctuations, which generate a universal finite-size critical behavior that can be classified in universality classes with universal critical exponents [7]. Within a universality class there exist subclasses [8,9] of isotropic and weakly anisotropic d-dimensional systems—the latter have d independent nonuniversal correlation-length amplitudes in d principal directions. While the Casimir force amplitude at criticality is widely believed to be a universal quantity [2,3,7,10–14], this is not valid for weakly anisotropic $O(n)$-symmetric systems with an n-component order parameter in $2 < d < 4$ dimensions [8,9,15–17]. Furthermore, low-temperature Casimir forces due to Goldstone modes [18] exhibit nonuniversal anisotropy effects [8]. Recently the hypothesis of multiparameter universality for weakly anisotropic systems has been put forward [8] but no proof has been given for confined systems and no detailed analysis has been performed near the instability where weak anisotropy breaks down. In particular, the universality properties of the critical Casimir amplitude of finite anisotropic systems in $d = 2$ have remained unexplored in the literature.

Two-dimensional systems are of fundamental theoretical interest since conformal field theory (CFT) is capable of deriving rigorous results for critical Casimir amplitudes of *isotropic* systems on a strip [12–14,19,20] and for the partition function at the critical temperature T_c on a parallelogram [21–25]. In this Letter our focus is on the critical Casimir force in weakly anisotropic ($d = 2$, $n = 1$) Ising-like systems for which CFT has not made any prediction so far. We show how to combine an exact result of CFT for the isotropic Ising model on a torus [22,25] with an exact shear transformation of anisotropic φ^4 theory [16] which, on the basis of multiparameter universality [8,26], leads to exact predictions for all weakly anisotropic systems with periodic boundary conditions (PBC) in the ($d = 2$, $n = 1$) universality class. We discover unexpected self-similar structures in the critical Casimir amplitude near the instability where weak anisotropy breaks down. They can be traced back to the modular invariance of isotropic CFT. We also demonstrate the validity of multiparameter universality for confined systems. More generally, we find self-similar structures in the $O(n)$-symmetric φ^4 theory with PBC for $1 \leq n \leq \infty$ in $d > 2$ dimensions in the presence of planar anisotropies not only near T_c but also in the Goldstone-dominated low-temperature region of anisotropic systems with $2 \leq n \leq \infty$.

We consider systems with short-range interactions in a rectangular $L_\parallel^{d-1} \times L$ geometry with PBC near an ordinary critical point. The total free energy $\mathcal{F}_{\rm tot}$ (divided by $k_B T$) can be decomposed into singular and nonsingular parts. We are interested in the singular part $\mathcal{F}_c$ of $\mathcal{F}_{\rm tot}$ at T_c. It is well known that the critical free-energy density $f_c = \mathcal{F}_c/(L_\parallel^{d-1} L)$ has the large-L behavior $f_c(L_\parallel, L) = L^{-d} F_c(\rho)$ at fixed aspect ratio $\rho = L/L_\parallel$ [7,10] with a finite amplitude $F_c(\rho)$, which implies that

$$\mathcal{F}_c = \rho^{1-d} F_c(\rho) \tag{1}$$

is a finite quantity in the large-L limit. The critical Casimir force in the vertical direction is obtained as

$$F_{\mathrm{Cas},c} = -\partial(L f_c)/\partial L = L^{-d} X_c(\rho), \tag{2}$$

where the derivative is taken at fixed $L_\parallel$. This yields the critical Casimir amplitude [8,27]

$$X_c(\rho) = (d-1)F_c(\rho) - \rho\partial F_c(\rho)/\partial\rho = -\rho^d \partial\mathcal{F}_c/\partial\rho. \tag{3}$$

If two-scale-factor universality [7,10,11] is valid the amplitudes $\mathcal{F}_c$, F_c, and X_c, for given geometry and BC, are universal. In this Letter we show that these amplitudes exhibit a nonuniversal dependence on microscopic couplings with a complex self-similar structure if the systems are anisotropic. From CFT we derive exact results for $d=2$ for both the scalar φ^4 model and the Ising model which belong to the same universality class.

We outline our strategy in the schematic Fig. 1 for the case $\rho = 1$. The anisotropic φ^4 model is characterized by two important nonuniversal parameters (see also Fig. 2): the angle Ω describing the orientation of the two principal axes and the ratio $q = \xi_{0\pm}^{(1)}/\xi_{0\pm}^{(2)}$ of the two principal correlation lengths [26] $\xi_\pm^{(\beta)} = \xi_{0\pm}^{(\beta)}|t|^{-1}, \beta = 1, 2, t = (T - T_c)/T_c$. For the anisotropic Ising model the corresponding parameters are denoted by Ω^{Is} and q^{Is}. Step 1 uses a shear transformation of the anisotropic φ^4 model on a square to an isotropic φ^4 model on a parallelogram that leaves the critical free energy $\mathcal{F}_c$ invariant [16]. Step 2 is based on two-scale-factor universality [7] implying that the critical free energy $\mathcal{F}_c^{\mathrm{iso}}$ of the isotropic φ^4 model is the same as $\mathcal{F}_c^{\mathrm{CFT}}$ of the isotropic Ising model on the same parallelogram described by CFT. Step 3 employs the hypothesis of multiparameter universality [8] predicting that $\mathcal{F}_c^{\mathrm{Is}}$ of the anisotropic Ising model with $\rho = 1$ is obtained from $\mathcal{F}_c(q, \Omega)$ of the anisotropic φ^4 model by the substitution $q \to q^{\mathrm{Is}}$, $\Omega \to \Omega^{\mathrm{Is}}$. Overall, these steps are equivalent to an effective shear transformation (dashed arrow in Fig. 1) between the isotropic Ising model on a parallelogram and the anisotropic Ising model on a square.

Step 1: We first consider the anisotropic scalar φ^4 lattice model in a $L_\parallel \times L$ rectangle on a square lattice with lattice spacing $\tilde{a}$ and PBC. The Hamiltonian and the total free energy divided by $k_B T$ on $N = L_\parallel L/\tilde{a}^2$ lattice points $\mathbf{x}_i \equiv (x_{i1}, x_{i2})$ are defined by [9,16]

$$H = \tilde{a}^2\left[\sum_i\left(\frac{r_0}{2}\varphi_i^2 + u_0\varphi_i^4\right) + \sum_{i,j}\frac{K_{i,j}}{2}(\varphi_i - \varphi_j)^2\right]$$

and by $\mathcal{F}_{\mathrm{tot}} = -\ln\{\prod_{i=1}^N \int_{-\infty}^{\infty} d\varphi_i \exp(-H)\}$. The large-distance anisotropy is described by the symmetric anisotropy matrix $\mathbf{A} = (\mathbf{A}_{\alpha\beta}) = \binom{a\ c}{c\ b}$,

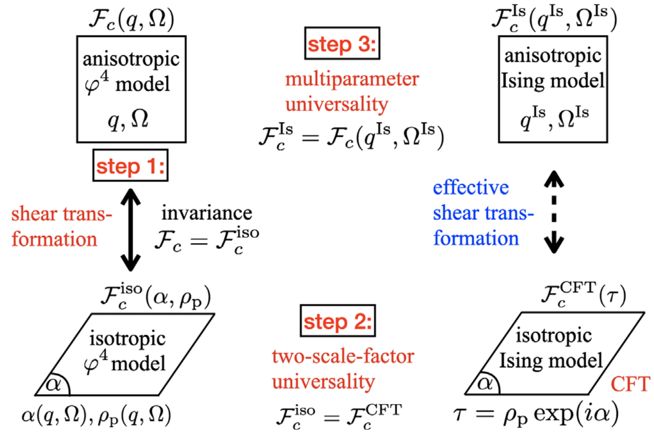

FIG. 1. Steps of argumentation for the exact relationships between the critical free energies $\mathcal{F}_c$ of finite anisotropic and isotropic φ^4 and Ising models in $d = 2$ (see also Fig. 2).

$$\mathbf{A}_{\alpha\beta} = \lim_{N\to\infty} N^{-1}\sum_{i,j}(x_{i\alpha} - x_{j\alpha})(x_{i\beta} - x_{j\beta})K_{i,j}. \tag{4}$$

Weak anisotropy requires positive eigenvalues $\lambda_1 > 0, \lambda_2 > 0$ of $\mathbf{A}$, i.e., $\det\mathbf{A} > 0$ [28]. It is known [8,9,15–17,26,29] that anisotropy effects near T_c are described by the reduced anisotropy matrix $\bar{\mathbf{A}} = \mathbf{A}/(\det\mathbf{A})^{1/d}$ which for $d = 2$ has the form [26]

$$\bar{\mathbf{A}}(q,\Omega) = \begin{pmatrix} qc_\Omega^2 + q^{-1}s_\Omega^2 & (q - q^{-1})c_\Omega s_\Omega \\ (q - q^{-1})c_\Omega s_\Omega & qs_\Omega^2 + q^{-1}c_\Omega^2 \end{pmatrix}, \tag{5}$$

with $q = (\lambda_1/\lambda_2)^{1/2} = \xi_{0\pm}^{(1)}/\xi_{0\pm}^{(2)}$ and the abbreviations $c_\Omega \equiv \cos\Omega$, $s_\Omega \equiv \sin\Omega$ where Ω determines the principal axes described by the eigenvectors $\mathbf{e}^{(1)} = (c_\Omega, s_\Omega)^\intercal$, $\mathbf{e}^{(2)} = (-s_\Omega, c_\Omega)^\intercal$ of $\mathbf{A}$. The exact dependence of Ω and q on the couplings $K_{i,j}$ through a, b, c has been derived in [26]. A shear transformation can be performed such that the transformed φ^4 model on a parallelogram (Fig. 2) has changed second moments $\mathbf{A}'_{\alpha\beta} = \delta_{\alpha\beta}$ representing an isotropic system [9,15,26,30]. The transformations $\boldsymbol{L}_p = \boldsymbol{\lambda}^{-1/2}\mathbf{U}\boldsymbol{L}$ and $\boldsymbol{L}_{p\parallel} = \boldsymbol{\lambda}^{-1/2}\mathbf{U}\boldsymbol{L}_\parallel$ of the vertical and horizontal sides $\boldsymbol{L} = L\mathbf{e}_v$ and $\boldsymbol{L}_\parallel = L_\parallel\mathbf{e}_h$ yield the

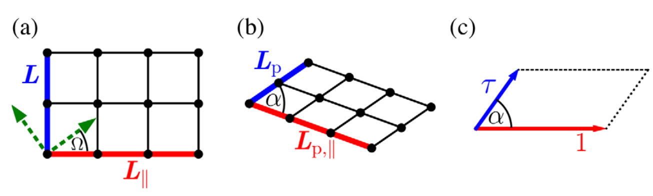

FIG. 2. (a) Square lattice with vertical (horizontal) side $\boldsymbol{L}$ ($\boldsymbol{L}_\parallel$) and aspect ratio $\rho = |\boldsymbol{L}|/|\boldsymbol{L}_\parallel|$. Arrows represent the unit vectors $\mathbf{e}^{(1)}, \mathbf{e}^{(2)}$ of the principal axes for $\Omega = \pi/5$. (b) Transformed lattice for $q = 1/2$ with sides $\boldsymbol{L}_p, \boldsymbol{L}_{p\parallel}$, and the angle α and aspect ratio $\rho_p = |\boldsymbol{L}_p|/|\boldsymbol{L}_{p\parallel}|$. (c) Parallelogram parametrized by the complex modular parameter τ (11).

corresponding transformed sides $\boldsymbol{L}_p$ and $\boldsymbol{L}_{p\parallel}$ of the parallelogram where the rotation and rescaling matrices

$$\mathbf{U} = \begin{pmatrix} \cos\Omega & \sin\Omega \\ -\sin\Omega & \cos\Omega \end{pmatrix}, \qquad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (6)$$

are employed. The parallelogram is characterized by the angle $0 < \alpha < \pi$ between $\boldsymbol{L}_p$ and $\boldsymbol{L}_{p\parallel}$ and the transformed aspect ratio $\rho_p = |\boldsymbol{L}_p|/|\boldsymbol{L}_{p\parallel}|$. We find

$$\cot\alpha(q,\Omega) = -\bar{\mathbf{A}}(q,\Omega)_{12} = (q^{-1} - q)\cos\Omega\sin\Omega, \quad (7)$$

$$[\rho_p(\rho,q,\Omega)]^2 = \rho^2\frac{\bar{\mathbf{A}}(q,\Omega)_{11}}{\bar{\mathbf{A}}(q,\Omega)_{22}} = \rho^2\frac{\tan^2\Omega + q^2}{1 + q^2\tan^2\Omega}, \quad (8)$$

for arbitrary ρ, q, Ω which is valid for arbitrary BC. The singular part $\mathcal{F}_c^{\rm iso}(\alpha,\rho_p)$ of the total free energy at T_c of the isotropic parallelogram is a function of α and ρ_p. The shear transformation leaves both the Hamiltonian and the singular part $\mathcal{F}_c$ of the total free energy $\mathcal{F}_{\rm tot}$ at T_c invariant [9,16], thus $\mathcal{F}_c$ is determined by

$$\mathcal{F}_c(\rho,q,\Omega) = \mathcal{F}_c^{\rm iso}[\alpha(q,\Omega),\rho_p(\rho,q,\Omega)] = \rho^{-1}F_c(\rho,q,\Omega). \quad (9)$$

In the strip limit the shear transformation yields [31]

$$\lim_{\rho\to 0} X_c(\rho,q,\Omega) = (q\cos^2\Omega + q^{-1}\sin^2\Omega)^{-1}X_{c,\rm strip}^{\rm iso}, \quad (10)$$

where $X_{c,\rm strip}^{\rm iso}$ is the amplitude on an isotropic strip. Equations (9) and (10) demonstrate that $\mathcal{F}_c$, F_c, and X_c depend on microscopic details via $q(a,b,c)$ and $\Omega(a,b,c)$, thus violating two-scale-factor universality. So far it is unknown how to calculate the dependence of $\mathcal{F}_c^{\rm iso}$ on α and ρ_p.

Step 2: At this point we invoke two-scale-factor universality for *isotropic* systems [8,10] which means that isotropic φ^4 and Ising models have the same singular parts $\mathcal{F}_c^{\rm iso}$ and $\mathcal{F}_c^{\rm Is,iso}$. For the Ising model exact information is available from CFT [22,25]. Via an isotropic continuum description in terms of a free fermion field an exact contribution $Z^{\rm CFT}(\tau)$ to the partition function of the $d = 2$ isotropic Ising model on a torus at T_c has been derived. We choose the same parameters α and ρ_p as for the isotropic φ^4 model. The Ising parallelogram is described by a complex torus modular parameter [25]

$$\tau(\alpha,\rho_p) = {\rm Re}\,\tau + i\,{\rm Im}\,\tau = \rho_p\exp(i\alpha), \quad (11)$$

where α is the angle shown in Fig. 2(c) and $\rho_p = |\tau|$ is the aspect ratio of the Ising parallelogram. The partition function is expressed in terms of Jacobi theta functions $\theta_i(0|\tau) \equiv \theta_i(\tau)$ (in the notation of [25], see [31]) as [22]

$$Z^{\rm CFT}(\tau) = [|\theta_2(\tau)| + |\theta_3(\tau)| + |\theta_4(\tau)|]/[2|\eta(\tau)|], \quad (12)$$

with $\eta(\tau) = [\frac{1}{2}\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)]^{1/3}$, from which we obtain $\mathcal{F}_c^{\rm CFT}(\tau) = -\ln Z^{\rm CFT}(\tau)$. The singular part of the total free energy of the isotropic Ising model at T_c is

$$\mathcal{F}_c^{\rm Is,iso}[\tau(\alpha,\rho_p)] = \mathcal{F}_c^{\rm CFT}[\tau(\alpha,\rho_p)] = \mathcal{F}_c^{\rm iso}(\alpha,\rho_p), \quad (13)$$

where, owing to two-scale-factor universality, the last equation applies to the transformed φ^4 model on the parallelogram. We define $\tau(\rho,q,\Omega) = \tau[\alpha(q,\Omega),\rho_p(\rho,q,\Omega)]$ with $\alpha(q,\Omega)$ and $\rho_p(\rho,q,\Omega)$ given by (7) and (8). Then we obtain from (13), (9), (1), and (3) our exact result for the Casimir amplitude X_c of the anisotropic φ^4 model as

$$X_c(\rho,q,\Omega) = -\rho^2\,\partial\mathcal{F}_c(\rho,q,\Omega)/\partial\rho \quad (14)$$

with $\mathcal{F}_c(\rho,q,\Omega) = \mathcal{F}_c^{\rm CFT}[\tau(\rho,q,\Omega)]$ where the nonuniversal expressions for $q(a,b,c)$ and $\Omega(a,b,c)$ [26] have to be inserted. In the strip limit we obtain [31]

$$X_c(0,q,\Omega) = -\pi/[12(q\cos^2\Omega + q^{-1}\sin^2\Omega)], \quad (15)$$

in accord with the CFT result $-\pi/12$ [12,13] for $q = 1$.

In Fig. 3 we present contour plots of X_c in the complete Ω-q plane for $\rho = 1$ and $\rho = 0.5$. Contrary to the simple (Ω, q) dependence (15) in strip geometry and to the claim that the effects of weak anisotropy are fairly harmless [32], we find unexpectedly complex structures exhibiting the feature of self-similarity in the regions $q \ll 1$ and $q \gg 1$ near the border lines $q = 0$ and $q = \infty$, where $\det\mathbf{A} = 0$ or $\lambda_\alpha = 0$, i.e., where weak anisotropy breaks down [28]. This self-similarity can be traced back to the property of modular invariance [25] $Z^{\rm CFT}(\tau) = Z^{\rm CFT}(\tau+1)$, $Z^{\rm CFT}(\tau) = Z^{\rm CFT}(-1/\tau)$ for the partition function of the isotropic Ising model at T_c in a parallelogram geometry with PBC, i.e., on a torus, which implies $\mathcal{F}_c^{\rm Is,iso}(\tau) = \mathcal{F}_c^{\rm Is,iso}(-1/\tau) = \mathcal{F}_c^{\rm Is,iso}(\tau+1)$. This is illustrated by the periodic structure of $\mathcal{F}_c^{\rm Is,iso}$ in the complex τ plane [Fig. 4(a)] which generates a self-similar structure in the (ρ_p,α) plane [Fig. 4(b)]. The modular transformation

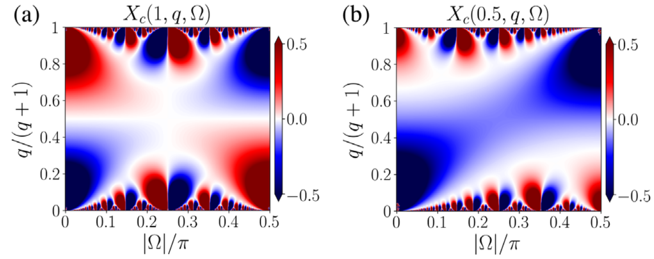

FIG. 3. Universal contour plot of the critical Casimir amplitude X_c of the anisotropic $d = 2$ φ^4 model from (14) for $\rho = 1$ in (a) and for $\rho = 0.5$ in (b). For the anisotropic $d = 2$ Ising model, (16) yields the same plots if $q \to q^{\rm Is}$, $\Omega \to \Omega^{\rm Is}$.

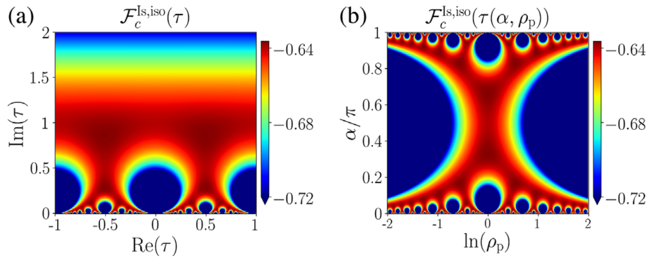

FIG. 4. Critical amplitude $\mathcal{F}_c^{\mathrm{Is,iso}}$ (13) of the isotropic Ising model in parallelogram geometry (a) in the complex τ plane and (b) in the ρ_p-α plane. The same plot (b) holds for $\mathcal{F}_c^{\mathrm{iso}}$ (13) of the transformed isotropic φ^4 model.

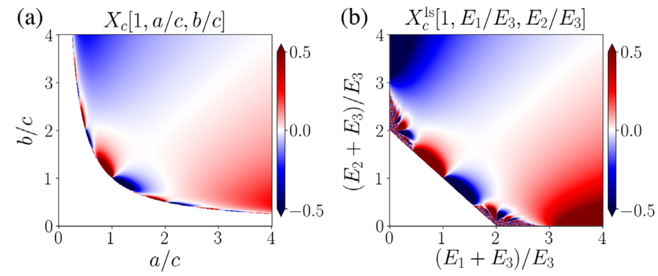

FIG. 5. Nonuniversal critical Casimir amplitudes X_c and X_c^{Is} for $\rho = 1$ of the $d = 2$ φ^4 and Ising models for (a) $\det \mathbf{A} = ab - c^2 > 0$, (b) $E_1 + E_2 > 0$, $E_1 + E_3 > 0$, $E_2 + E_3 > 0$ [40].

$\tau \to -1/\tau$ corresponds to $\rho_p \to 1/\rho_p$, $\alpha \to \pi - \alpha$ which yields equivalent parallelograms. The Dehn twist [25] $\tau \to \tau + 1$ yields parallelograms with different ρ_p and α, but the invariance of $\mathcal{F}_c^{\mathrm{Is,iso}}$ can be understood geometrically since a given torus can be cut in different ways such that different parallelograms with PBC are generated which all have the same critical free energy on the same torus. By two-scale-factor universality, the same result applies to $\mathcal{F}_c^{\mathrm{iso}}$ of the isotropic φ^4 model on the same torus. The dependence on (ρ_p, α) for the isotropic system in Figs. 2(b) and 4(b) is transferred by the shear transformation (7), (8), and by (14) to a corresponding dependence of $X_c(\rho, q, \Omega)$ on (q, Ω) as is shown in Fig. 3. We note that so far no assumption has been made other than the validity of two-scale-factor universality for isotropic systems.

Step 3: We proceed to the anisotropic triangular Ising model on an $L_\parallel \times L$ rectangle with the Hamiltonian [26,33] $H^{\mathrm{Is}} = -\sum_{j,k}[E_1\sigma_{j,k}\sigma_{j,k+1} + E_2\sigma_{j,k}\sigma_{j+1,k} + E_3\sigma_{j,k}\sigma_{j+1,k+1}]$ with spin variables $\sigma_{j,k} = \pm 1$ on a square lattice with PBC. Both the angle $\Omega^{\mathrm{Is}}(E_1, E_2, E_3)$ of the principal axes and the ratio of the principal correlation lengths $q^{\mathrm{Is}}(E_1, E_2, E_3) = \xi_{0\pm}^{(1)\mathrm{Is}}/\xi_{0\pm}^{(2)\mathrm{Is}}$ are known functions of E_1, E_2, E_3 [26]. Multiparameter universality was proven for bulk systems in [26], thus the exact critical bulk correlation function is governed by the Ising anisotropy matrix $\bar{\mathbf{A}}^{\mathrm{Is}} = \bar{\mathbf{A}}(q^{\mathrm{Is}}, \Omega^{\mathrm{Is}})$ with the same matrix $\bar{\mathbf{A}}$ as in (5) for the φ^4 model, but with q and Ω replaced by q^{Is} and Ω^{Is}. Since bulk and finite-size properties are governed by the same anisotropy matrix [8] we predict that the exact critical Casimir amplitude of the anisotropic Ising model is given by

$$X_c^{\mathrm{Is}}(\rho, q^{\mathrm{Is}}, \Omega^{\mathrm{Is}}) = -\rho^2 \partial \mathcal{F}_c(\rho, q^{\mathrm{Is}}, \Omega^{\mathrm{Is}})/\partial\rho, \qquad (16)$$

where $\mathcal{F}_c(\rho, q^{\mathrm{Is}}, \Omega^{\mathrm{Is}})$ is the same function as in (14) but now the results for $q^{\mathrm{Is}}(E_1, E_2, E_3)$ and $\Omega^{\mathrm{Is}}(E_1, E_2, E_3)$ of the Ising model [26] have to be inserted. Our predictions go far beyond all previous special results [12,13,34–39] for confined isotropic and anisotropic Ising models. Here we have succeeded in treating the general anisotropic case of an arbitrary direction of the principal axes described by a nonzero angle Ω^{Is} in a finite geometry with an arbitrary aspect ratio ρ. This is of physical relevance for general anisotropic systems with more complicated interactions whose principal axes generically have skew directions relative to the symmetry axes of the underlying lattice. This advance is made possible by our new approach of combining exact relations of anisotropic φ^4 lattice theory with exact results of CFT. Specifically our predictions agree with Ising-model results for isotropic strips [12,13,34], rectangles [35], and parallelograms [36] as well as for anisotropic strips [37] and rectangles [38,39] which constitutes a direct confirmation of multiparameter universality for confined systems. Thus we predict that the results in Fig. 3 for the φ^4 model are valid also for the Ising model after substituting $q \to q^{\mathrm{Is}}$, $\Omega \to \Omega^{\mathrm{Is}}$, i.e., the (q, Ω) representation has a universal character that is applicable to all weakly anisotropic systems in the $(d = 2,\ n = 1)$ universality class. It becomes nonuniversal if the dependence of (q, Ω) and $(q^{\mathrm{Is}}, \Omega^{\mathrm{Is}})$ on a, b, c and E_1, E_2, E_3 is inserted. We denote these Casimir amplitudes by $X_c[\rho, a/c, b/c]$ and $X_c^{\mathrm{Is}}[\rho, E_1/E_3, E_2/E_3]$. They are shown in Fig. 5 for $\rho = 1$. The nonuniversal differences between X_c and X_c^{Is} confirm the prediction [8,9,15,16] that the Casimir amplitude X_c for weakly anisotropic systems is not a universal quantity. Even if it is known for one anisotropic system it cannot be predicted for other anisotropic systems of the same universality class since Ω generically depends in an unknown nonuniversal way on the anisotropic interactions [26].

In the following we demonstrate that self-similar structures exist more generally for $O(n)$-symmetric systems with PBC for $d = 3$, $n \geq 1$ with a 3×3 anisotropy matrix $\mathbf{A_3}$. Consider a $L_\parallel^2 \times L$ geometry with $\rho = L/L_\parallel$. In [8] the scaling function $X(\hat{x}, \rho, \bar{\mathbf{A}}_\mathbf{3})$ of the Casimir force of the φ^4 model has been derived for $n \geq 1$, where $\hat{x} \propto (T - T_c)L^{1/\nu}$ is the scaling variable. This includes the low-temperature amplitude $X_0(\rho, \bar{\mathbf{A}}_\mathbf{3}) \equiv X(-\infty, \rho, \bar{\mathbf{A}}_\mathbf{3})$ due to the Goldstone modes for $n \geq 2$. In particular, the exact result $\hat{X} = \lim_{n\to\infty} X/n$ has been derived in the large-n limit. At fixed $\hat{x}$ the anisotropy effect is completely contained in the function

$$K_3(y, \hat{\mathbf{C}}) = \sum_{\mathbf{m}} \exp(-y\,\mathbf{m} \cdot \hat{\mathbf{C}}\mathbf{m}), \qquad (17)$$

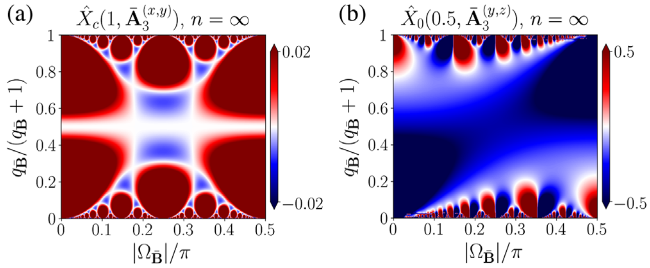

FIG. 6. Universal contour plots of critical and low-temperature Casimir amplitudes of the $d = 3$ φ^4 theory for $n = \infty$ [8,31] with planar anisotropies (18): $\hat{X}_c$ for $\rho = 1$ with $\bar{\mathbf{A}}_3^{(\mathbf{x},\mathbf{y})}$ in (a) and $\hat{X}_0$ for $\rho = 0.5$ with $\bar{\mathbf{A}}_3^{(\mathbf{y},\mathbf{z})}$ in (b).

where y is independent of $\bar{\mathbf{A}}_3$. The sum $\sum_{\mathbf{m}}$ runs over $\mathbf{m} = (m_1, m_2, m_3), m_\alpha = 0, \pm 1, \ldots, \pm\infty$ and the 3×3 matrix $\hat{\mathbf{C}}$ has the elements $\hat{C}_{\alpha\beta} = \rho_\alpha \rho_\beta (\bar{\mathbf{A}}_3)_{\alpha\beta}$ with $\rho_1 = \rho_2 = \rho, \rho_3 = 1$. We consider two types of planar anisotropies as described by the anisotropy matrices

$$\bar{\mathbf{A}}_3^{(\mathbf{x},\mathbf{y})} = \begin{pmatrix} \bar{\mathbf{B}}_\mathbf{h} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \bar{\mathbf{A}}_3^{(\mathbf{y},\mathbf{z})} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{\mathbf{B}}_\mathbf{v} \end{pmatrix}. \tag{18}$$

In (18) the 2×2 submatrices $\bar{\mathbf{B}}_\mathbf{h}$ and $\bar{\mathbf{B}}_\mathbf{v}$ describe anisotropies in the "horizontal" x-y and "vertical" y-z planes, respectively. The difference between these cases is that the Casimir force defined in (2) is perpendicular to the x-y anisotropy in case of $\bar{\mathbf{A}}_3^{(\mathbf{x},\mathbf{y})}$ whereas it is parallel to the y-z anisotropy in case of $\bar{\mathbf{A}}_3^{(\mathbf{y},\mathbf{z})}$. Both $\bar{\mathbf{B}}_\mathbf{h}$ and $\bar{\mathbf{B}}_\mathbf{v}$ have the same form as in (5), with (q, Ω) replaced by $(q_{\bar{\mathbf{B}}}, \Omega_{\bar{\mathbf{B}}})$. This suggests that self-similar structures exist for $d = 3$ like those found for $d = 2$. This is indeed verified by evaluating the exact results for $\hat{X}_c$ and $\hat{X}_0$ of [8] for $d = 3, n = \infty$ with the planar anisotropies (18) as shown in Fig. 6. We find similar structures from [8] for any finite n and $\hat{x}$. The self-similar structures of Fig. 6 disappear in the film limit $\rho \to 0$ [8] at finite L but are maintained in the cylinder limit $\rho \to \infty$ [6] at finite $L_\parallel$.

We find that, to some extent, this self-similarity can be traced back to the modular invariance of $K_3(y, \hat{\mathbf{C}})$ (17). Since a symmetric matrix $\bar{\mathbf{B}}$ with det $\bar{\mathbf{B}} = 1$ contains only two independent matrix elements, it can be expressed as

$$\bar{\mathbf{B}} = \frac{1}{\mathrm{Im}(\tau_{\bar{\mathbf{B}}})} \begin{pmatrix} |\tau_{\bar{\mathbf{B}}}|^2 & -\mathrm{Re}(\tau_{\bar{\mathbf{B}}}) \\ -\mathrm{Re}(\tau_{\bar{\mathbf{B}}}) & 1 \end{pmatrix}, \tag{19}$$

where $\tau_{\bar{\mathbf{B}}} = \mathrm{Re}(\tau_{\bar{\mathbf{B}}}) + i\mathrm{Im}(\tau_{\bar{\mathbf{B}}})$ is a complex number with $\mathrm{Im}(\tau_{\bar{\mathbf{B}}}) > 0$. Based on the one-to-one relation (19) between $\bar{\mathbf{B}}$ and $\tau_{\bar{\mathbf{B}}}$, we can relate a modular transformation $\tau_{\bar{\mathbf{B}}} \to \tau_{\bar{\mathbf{B}}'}$ to a corresponding matrix $\bar{\mathbf{B}}'$, with, e.g., $\tau_{\bar{\mathbf{B}}'} = \tau_{\bar{\mathbf{B}}} + 1$ for the Dehn twist. The function K_3 remains invariant under such transformations for $\bar{\mathbf{A}}_3 = \bar{\mathbf{A}}_3^{(\mathbf{y},\mathbf{z})}$, $\rho = 1$ and for $\bar{\mathbf{A}}_3 = \bar{\mathbf{A}}_3^{(\mathbf{x},\mathbf{y})}$ and arbitrary ρ [31]. This is parallel to the modular invariance of $Z^{\mathrm{CFT}}(\tau)$. More generally, we expect self-similar structures also for $d = 3$ systems with nonplanar anisotropies and PBC.

Conclusion and outlook.—We have studied the dependence of finite-size effects on the principal correlation lengths and principal axes for the case of PBC. For $d = 2$, we have achieved a breakthrough by identifying unexpected self-similar structures via the combination of isotropic CFT with anisotropic φ^4 theory. For $d = 3$, our analysis paves the way toward an exploration of finite-size effects near the borderlines where weak anisotropy breaks down not only near T_c but also in the Goldstone-dominated region. On the basis of $d = 3$ finite-size theories [8,9,41,42] and owing to multiparameter universality we predict that in all $O(n)$-symmetric systems with weak anisotropies and PBC the self-similar structures described in this Letter appear also in various physical quantities such as the specific heat and susceptibility. Self-similar structures do not appear for simple anisotropies with $\Omega = 0, \pi/4$ studied previously [30,38,39,43,44] although two-scale-factor universality is violated in these cases. Our results provide strong motivation for investigating the case of other boundary conditions and to study finite-size effects in anisotropic systems such as superconductors, magnetic materials, solids with structural phase transitions, and near magnetic-field-induced phase transitions [45] where the interplay between spatial and spin anisotropy is relevant. In particular, it would be important to explore the crossover from weak anisotropy to strong anisotropy of cooperative phenomena such as those near Lifshitz points.

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