Bounds on Transport from Univalence and Pole-Skipping

Sašo Grozdanov [1,2]

[1] *Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2] *Faculty of Mathematics and Physics, University of Ljubljana, Jadranska ulica 19, SI-1000 Ljubljana, Slovenia*

(Received 5 September 2020; revised 4 December 2020; accepted 8 January 2021; published 2 February 2021)

Bounds on transport represent a way of understanding allowable regimes of quantum and classical dynamics. Numerous such bounds have been proposed, either for classes of theories or (by using general arguments) universally for *all* theories. Few are exact and inviolable. I present a new set of methods and sufficient conditions for deriving exact, rigorous, and sharp bounds on all coefficients of hydrodynamic dispersion relations, including diffusivity and the speed of sound. These general techniques combine analytic properties of hydrodynamics and the theory of univalent (complex holomorphic and injective) functions. Particular attention is devoted to bounds relating transport to quantum chaos, which can be established through pole-skipping in theories with holographic duals. Examples of such bounds are shown along with holographic theories that can demonstrate the validity of the conditions involved. I also discuss potential applications of univalence methods to bounds without relation to chaos, such as for example the conformal bound on the speed of sound.

DOI: 10.1103/PhysRevLett.126.051601

Introduction.—The existence of bounds on properties of transport, such as diffusion, has persistently enthralled physicists concerned with time-dependent collective dynamics. Numerous bounds that improved our understanding of quantum and classical dynamics have been proposed. Among them is Sachdev's relaxation time bound [1], the Mott–Ioffe–Regel limit of metallic conductivity [2,3], lower bounds on diffusion and viscosity [4–13], upper bounds on diffusion [9,14,15], and a bound on the speed of sound [16,17]. These bounds are usually heuristic and rely on basic physical principles such as the uncertainty principle and causality. Exact inequalities, even for restricted classes of theories, are rare. An example is Prosen's bound on diffusion [18]. Holographic methods to bound conductivities in disordered theories were developed in [19,20]. Holographic advances in quantum chaos then led to the exact Maldacena–Shenker–Stanford bound on quantum Lyapunov exponents that follows from arguments of analyticity and complex analysis [21]. Another bound on the growth of weak (polynomial) quantum chaos was derived in [22].

Microscopic bounds, such as bounds on quantum chaos, should imply sharp bounds on collective transport. The purpose of this work is to introduce a new set of mathematical techniques from a well-developed theory of univalent functions, which allows for a rigorous derivation of exact inequalities of that type on diffusivity, the speed of sound, and all higher-order coefficients of hydrodynamic dispersion relations. The methods establish sufficient analyticity and microscopic conditions that lead to several long-discussed types of bounds. Due to their generality, univalence methods can also be applied to derive bounds without any reference to chaos.

Univalent functions.—A univalent (or *schlicht*) function $f(z)$ is a complex holomorphic injective function. The condition of injectivity demands that $f(z_1) \neq f(z_2)$ for all $z_1 \neq z_2$. Henceforth, all considered $f(z)$ will be univalent in some simply connected region $U \subset \mathbb{C}$. By the *Riemann mapping theorem*, it is then possible to map U to an open unit disk $\mathbb{D} = \{\zeta | |\zeta| < 1\}$ in the complex ζ plane by a holomorphic invertible conformal map φ: $\zeta = \varphi(z)$ and $z = \varphi^{-1}(\zeta)$. As is conventional, we will use the normalization $f(\zeta = 0) = 0$, and $f'(\zeta = 0) = 1$ for functions in the ζ plane. All such functions admit a power series representation of the following form:

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n. \tag{1}$$

The series is guaranteed to converge for all $|\zeta| < 1$.

Locally, $f(z)$ is univalent if $f'(z) \neq 0$. However, proving local univalence at every $z \in U$ does not guarantee global univalence. Instead, one of numerous sufficient conditions for univalence must be employed [23,24]. Once univalence is established and we have mapped $U \to \mathbb{D}$, then we can resort to theorems bounding univalent functions on $\zeta \in \mathbb{D}$, such as the *growth theorem*:

$$\frac{|\zeta|}{(1+|\zeta|)^2} \le |f(\zeta)| \le \frac{|\zeta|}{(1-|\zeta|)^2}, \tag{2}$$

and the celebrated *de Branges theorem* (originally called the *Bieberbach conjecture*) [25] constraining each coefficient of the power series (1):

$$|b_n| \le n, \quad \text{for all } n \ge 2. \tag{3}$$

The inequalities in Eq. (3) and the growth theorem (2) are saturated by the Koebe function (and its rotations in ζ),

$$f_K(\zeta) = \frac{\zeta}{(1-\zeta)^2} = \sum_{n=1} n\zeta^n, \tag{4}$$

which conformally maps $\mathbb{D} \to \mathbb{C}\backslash(-\infty, -1/4]$.

We will use the condition whereby if $\mathrm{Re} f'(z) > 0$ in any convex $U \subset \mathbb{C}$, then $f(z)$ is univalent in U [26,27]. If, moreover, after φ: $U \to \mathbb{D}$, $\mathrm{Re} f'(\zeta) > 0$, then $f(\zeta)$ satisfies stronger versions of the theorems in Eqs. (2) and (3) [28]:

$$-|\zeta| + 2\ln(1+|\zeta|) \le |f(\zeta)| \le -|\zeta| - 2\ln(1-|\zeta|), \tag{5}$$

$$|b_n| \le \frac{2}{n}, \quad \text{for all } n \ge 2. \tag{6}$$

Hydrodynamics.—Hydrodynamics is an effective theory of collective late-time and long-range excitations in fluids governed by conserved quantities such as energy, momentum, and charges [29–40]. Linearized hydrodynamics predicts the structure of dispersion relations $\omega(\mathbf{q}^2)$, where ω is the frequency and $\mathbf{q}^2$ is the momentum (squared) of a collective mode: diffusion or sound. In theories preserving spatial rotations, classical [41] $\omega(\mathbf{q}^2)$ are infinite Puiseux series in $\mathbf{q}^2$ [44,45]:

$$\omega_{\mathrm{diff}}(z \equiv \mathbf{q}^2) = -i\sum_{n=1}^{\infty} c_n z^n, \tag{7}$$

$$\omega^{\pm}_{\mathrm{sound}}(z \equiv \sqrt{\mathbf{q}^2}) = -i\sum_{n=1}^{\infty} a_n e^{\pm(i\pi n)/2} z^n, \tag{8}$$

where all a_n, $c_n \in \mathbb{R}$. We treat the argument z as complex ($z \in \mathbb{C}$) in both Eqs. (7) and (8). We have $c_1 = D$ (diffusivity) and $a_1 = v_s$ (the speed of sound). Each series converges for $|z| < R \equiv |z_*|$ with $z = z_*$ being the first critical point of the associated complex curve [44–46]. Each fully analytically continued function $\omega(z)$ is holomorphic in the region $z \in H \subset \mathbb{C}$, where H contains $|z| < R$.

Different concepts of wave propagation speeds beyond v_s exist, such as the phase velocity $v_{\mathrm{ph}}(q) \equiv \omega/q$, the front velocity, and the group velocity $v_g(q) \equiv \partial\omega/\partial q$, where $q \equiv \sqrt{\mathbf{q}^2}$. Causality, for example, imposes certain conditions on these speeds (see Ref. [51]). In an analogous spirit, we will sometimes use properties of v_g to define the univalence region of hydrodynamics U.

General bounds.—A hydrodynamic dispersion relation $\omega(z)$ is by Puiseux's theorem invertible at $z = 0$, and thus locally univalent at $z = 0$ [44,45]. Beyond including $z = 0$ in all univalent regions $U \subseteq H$, we assume that U also contains a point $z = z_0$ where $\omega_0 \equiv \omega(z_0)$ is known. U need not be maximal. A convenient way to choose U is through the sufficient condition $\mathrm{Re} f'(z) > 0$, where $f_{\mathrm{diff}}(z) = i\omega_{\mathrm{diff}}(z)$ and $f_{\mathrm{sound}}(z) = \omega_{\mathrm{sound}}(z)$. This implies univalence for $U = \{z||z| < \min[|z_g|, R]\}$, where

$$\text{diffusion:}\ z_g = q_g^2 \equiv \min q^2|\mathrm{Re}v_g\mathrm{Im}q = \mathrm{Im}v_g\mathrm{Re}q, \tag{9}$$

$$\text{sound:}\ z_g = q_g \equiv \min q|\mathrm{Re}v_g = 0, \tag{10}$$

expressed through the properties of the group velocity. If v_g vanishes at $|z_g|$ smaller than those in Eqs. (9) and (10), then univalence is lost locally due to $f'(z_g) = 0$. We have

$$q_g \equiv \min q|v_g = 0. \tag{11}$$

Using a conformal map φ: $U \to \mathbb{D}$ with $\varphi(z) = \zeta$ that preserves the origin [i.e., $\varphi(0) = 0$], we then define

$$f_{\mathrm{diff}}(\zeta) \equiv \frac{i\omega_{\mathrm{diff}}(\varphi^{-1}(\zeta))}{D\partial_\zeta\varphi^{-1}(0)} = \zeta + \sum_{n=2}^{\infty} b_n^{\mathrm{diff}}\zeta^n, \tag{12}$$

$$f_{\mathrm{sound}}(\zeta) \equiv \frac{\omega^{+}_{\mathrm{sound}}(\varphi^{-1}(\zeta))}{v_s\partial_\zeta\varphi^{-1}(0)} = \zeta + \sum_{n=2}^{\infty} b_n^{\mathrm{sound}}\zeta^n. \tag{13}$$

Both Eqs. (12) and (13) have the form of Eq. (1). The growth theorem (2) applied at $\zeta_0 \equiv \varphi(z_0)$ now yields lower and upper bounds on diffusivity and the speed of sound:

$$\frac{|\omega_0|(1-|\zeta_0|)^2}{|\zeta_0||\partial_\zeta\varphi^{-1}(0)|} \le (D \vee v_s) \le \frac{|\omega_0|(1+|\zeta_0|)^2}{|\zeta_0||\partial_\zeta\varphi^{-1}(0)|}, \tag{14}$$

where $(D \vee v_s)$ means either D or v_s, depending on whether we used Eq. (12) or Eq. (13). If, in addition to univalence, $\mathrm{Re} f'(\zeta) > 0$ for $|\zeta| < 1$, then Eq. (5) gives

$$\frac{|\omega_0|}{|\partial_\zeta\varphi^{-1}(0)|\ln[e^{-|\zeta_0|}/(1-|\zeta_0|)^2]} \le (D \vee v_s) \le \frac{|\omega_0|}{|\partial_\zeta\varphi^{-1}(0)|\ln[e^{-|\zeta_0|}(1+|\zeta_0|)^2]}. \tag{15}$$

To bound higher-order coefficients, we use the de Branges theorem (3) on each term of the series (12) or (13). This establishes a chain of inequalities on c_n or a_n in terms of all c_m or a_m with $m < n$. For a diffusive dispersion relation (7), we first use $|b_2| \le 2$ to bound c_2:

$$\left|c_2 + \frac{D}{2}\frac{\partial_\zeta^2 \varphi^{-1}(0)}{[\partial_\zeta \varphi^{-1}(0)]^2}\right| \leq \frac{2D}{|\partial_\zeta \varphi^{-1}(0)|}, \tag{16}$$

further eliminating D through Eq. (14). Next, $|b_3| \leq 3$ is used to bound c_3 and so on for all $c_{n\geq 4}$. If $\mathrm{Re} f'(\zeta) > 0$, then the bound (16) has another factor of $1/2$ on the right-hand side due to $|b_2| \leq 1$ in Eq. (6). An analogous procedure can be used for bounding a_n by v_s and φ. All bounds are determined purely in terms of a single known $\omega_0(z_0)$ and the chosen original region of univalence U through the conformal map φ: $U \to \mathbb{D}$.

Quantum chaos and pole-skipping.—Of particular interest are bounds that stem from the underlying microscopic quantum chaos. While the general relation between transport and chaos is unknown, precise connection has been established through the phenomenon of *pole-skipping* in quantum field theories with a large number of local degrees of freedom (large-N theories) that possess a classical gravitational holographic dual [52–55].

Pole-skipping is an indeterminacy of two-point functions associated with dispersion relations (7)–(8). In the longitudinal channel of energy-momentum fluctuations (e.g., sound or energy diffusion), pole-skipping implies

$$\omega_0(\mathbf{q}_0^2) = i\lambda_L, \qquad \mathbf{q}_0^2 = -\lambda_L^2/v_B^2. \tag{17}$$

Hence, for such modes, we have $q_0 = i\lambda_L/v_B$. Here, λ_L is the maximal Lyapunov exponent $\lambda_L = 2\pi T$, T is the temperature, and v_B is the butterfly velocity characterizing the exponential growth of the out-of-time-ordered correlator used to probe chaos $e^{\lambda_L(t-|\mathbf{x}|/v_B)}$ [21,56]. In neutral theories, a related expression exists also for transverse fluctuations (e.g., momentum diffusion) [45,57]:

$$\omega_0(\mathbf{q}_0^2) = -i\lambda_L, \qquad \mathbf{q}_0^2 = \lambda_L^2/v_B^2. \tag{18}$$

In charged theories, pole-skipping in Eq. (18) at $\omega_0 = -i\lambda_L$ generically exhibits a more complicated q_0 [58]. Since the pole-skipping points can be easily computed from dual gravity, and they relate chaos to transport, we will use them as $\omega_0(z_0)$ in most bounds below.

Diffusion I: maximal univalence.—In our first, simple, and very special example, assume that a diffusive dispersion relation $\omega(z) = \omega_{\mathrm{diff}}(z)$ [cf. Eq. (7)] is maximally univalent ($U = H$) and holomorphic on the entire $z \in \mathbb{C}$ except at a branch point z_* and at $z = \infty$. We define $\omega_* \equiv \omega(z_*)$. Under $\mathrm{Im}z \to -\mathrm{Im}z$, $\mathrm{Re}\omega$ is odd and $\mathrm{Im}\omega$ is even. To have a single z_*, we need $\mathrm{Re}\omega_* = 0$; hence, $z_* \in \mathbb{R}$. For concreteness, we take $z_* > 0$ and choose the branch cut so that $U = \mathbb{C}\backslash[z_*, \infty)$. $R = z_*$ is the radius of convergence of the hydrodynamic series (7). We first use a rescaling Möbius transformation to map $z_* \to -1/4$, keeping $z = \infty$ at ∞. The branch cut is now chosen to lie along $(-\infty, -1/4]$. Next, we use an inverse of the Koebe function (4) to map $\mathbb{C}\backslash(-\infty, -1/4] \to \mathbb{D}$. The full conformal map φ: $U \to \mathbb{D}$ is thus

$$\zeta = \varphi(z) = \frac{z - 2z_* + 2\sqrt{z_*^2 - zz_*}}{z}, \tag{19}$$

$$z = \varphi^{-1}(\zeta) = -4z_* f_K(\zeta) = -\frac{4z_*\zeta}{(1-\zeta)^2}, \tag{20}$$

with $\partial_\zeta^n \varphi^{-1}(0) = -4n^2(n-1)!R$. Using the pole-skipping relations in Eq. (17) or Eq. (18), the diffusivity bounds (14) become

$$z_0 = -\frac{\lambda_L^2}{v_B^2} < 0\colon\ \frac{v_B^2}{\lambda_L} \leq D \leq \frac{v_B^2}{\lambda_L} + \frac{\lambda_L}{R}, \tag{21}$$

$$0 < z_0 = \frac{\lambda_L^2}{v_B^2} < R\colon\ \frac{v_B^2}{\lambda_L} - \frac{\lambda_L}{R} \leq D \leq \frac{v_B^2}{\lambda_L}. \tag{22}$$

Since $[z_*, \infty) \notin U$, we do not consider $z_0 \geq R$. Equations (21) and (22) correspond to the longitudinal (energy diffusion) and, assuming Eq. (18), the transverse (momentum diffusion) channels, respectively. The inequalities are fixed by pole-skipping and the radius of convergence. The lower bound in Eq. (21) and the upper bound in Eq. (22) have the form of the relation between D and v_B^2/λ_L first noticed by Blake [6]. Moreover, our results imply that if a univalent diffusive $\omega(z)$ is entire (holomorphic everywhere except at infinity, so that $R \to \infty$), then $D = v_B^2/\lambda_L$ identically. In terms of quasihydrodynamics [59], small R is related to the relaxation time set by the leading gapped mode. Using Eq. (16) for general R, we can now find bounds on c_2 (a third-order hydrodynamic coefficient [60,61]):

$$0 \leq c_2 \leq \frac{D}{R}. \tag{23}$$

The upper bound from either Eq. (21) or Eq. (22) eliminates D from Eq. (23). Simple algebraic manipulations give further bounds on c_3, c_4, and so on. If we can take $R \to \infty$, then $c_2 = 0$. Moreover, all $c_{n>2} = 0$ in this limit. Hence, for entire univalent $\omega_{\mathrm{diff}}(z)$, the dispersion relation truncates at the first order for all $\mathbf{q}^2$, with D fixed by pole-skipping:

$$\omega_{\mathrm{diff}}(\mathbf{q}^2) = -iD\mathbf{q}^2 = -i\frac{v_B^2}{\lambda_L}\mathbf{q}^2. \tag{24}$$

A theory that exhibits diffusive properties discussed here is a holographic model with broken translational invariance and energy diffusion [62]. At a special self-dual point in the parameter space of the background fields, symmetry enhancement allows us to analytically find the exact diffusive $\omega(z) = -i\pi T[1 - \sqrt{1 - (z/\pi^2 T^2)}]$ [63]. Pole-skipping and hydrodynamic convergence in this theory were studied in [45,54], finding $z_0 = -8\pi^2 T^2$, $v_B^2 = 1/2$, and $z_* = R = \pi^2 T^2$. The bounds implied by Eqs. (21) and (23), along with the bounds on c_3, are then

$$\frac{v_B^2}{\lambda_L} = \frac{1}{4\pi T} \le D \le 9\frac{v_B^2}{\lambda_L} = \frac{9}{4\pi T}, \tag{25}$$

$$0 \le c_2 \le \frac{D}{\pi^2 T^2} \le \frac{9}{4\pi^3 T^3}, \tag{26}$$

$$-\frac{27}{32\pi^5 T^5} \le -\frac{3D}{8\pi^4 T^4} \le c_3 \le \frac{D}{\pi^4 T^4} \le \frac{9}{4\pi^5 T^5}. \tag{27}$$

The actual values of $D = 1/2\pi T$, $c_2 = 1/8\pi^3 T^3$, and $c_3 = 1/16\pi^5 T^5$ all satisfy the inequalities.

Diffusion II: Möbius transformations.—A general diffusive dispersion relation has multiple branch points and branch cuts. Generalizing the scenario in which U is determined by the group velocity conditions in Eqs. (9)–(11), let U of $\omega_{\rm diff}(z)$ be a disk with a center at $z = z_c$ and two boundary points at $z = z_c \pm z_b$ (on its closure), containing $z = 0$ and $z = z_0$, and with $z_c \in \mathbb{C}$ and $z_b \in \mathbb{R}_+$. U can be mapped to $\mathbb{D}$ by the Möbius transformation $\zeta = \varphi(z)$, which we choose to be

$$\varphi(z) = \frac{z_b z}{-z_c z + z_b^2 + z_c^2}, \qquad \varphi^{-1}(\zeta) = \frac{(z_b^2 + z_c^2)\zeta}{z_b + z_c\zeta}, \tag{28}$$

satisfying $\varphi(0) = 0$ and mapping $z_c \pm i z_b \to \pm i$. We have $\partial_\zeta^n \varphi^{-1}(0) = n!(-z_c)^{n-1}(z_b^2 + z_c^2)/z_b^n$. All of the above bounds can now be easily constructed given specific z_0, z_b, and z_c. For example, Eq. (14) becomes

$$\frac{v_B^2}{\lambda_L}\left|1 - \frac{z_c z_0}{z_b^2 + z_c^2}\right|\mathcal{C}_- \le D \le \frac{v_B^2}{\lambda_L}\left|1 - \frac{z_c z_0}{z_b^2 + z_c^2}\right|\mathcal{C}_+, \tag{29}$$

where $z_0 = \pm\lambda_L^2/v_B^2$, depending on whether we use Eq. (17) or Eq. (18). $\mathcal{C}_\pm$ are defined as

$$\mathcal{C}_\pm \equiv (1 \pm |\zeta_0|)^2, \qquad |\zeta_0| = \frac{\lambda_L^2}{v_B^2}\frac{z_b}{|-z_c z_0 + z_b^2 + z_c^2|}. \tag{30}$$

Of particular interest are cases with $z_c = 0$ so that φ rescales a disk of radius $z_b = \min[|z_g|, R]$ to $\mathbb{D}$. The only nonzero $\partial_\zeta^n \varphi^{-1}(0)$ is then $\partial_\zeta \varphi^{-1}(0) = z_b$, and $b_n = z_b^{n-1} c_n / D$ for $n \ge 2$. The bounds on the coefficients of the series (7) follow:

$$\frac{v_B^2}{\lambda_L}\left(1 - \frac{1}{z_b}\frac{\lambda_L^2}{v_B^2}\right)^2 \le D \le \frac{v_B^2}{\lambda_L}\left(1 + \frac{1}{z_b}\frac{\lambda_L^2}{v_B^2}\right)^2, \tag{31}$$

$$-\frac{nD}{z_b^{n-1}} \le c_{n\ge 2} \le \frac{nD}{z_b^{n-1}}. \tag{32}$$

If the pole-skipping $z_0 \in U$, then by taking $z_b \to z_0$, we can at the very least establish that $0 \le D \le 4v_B^2/\lambda_L$. Also, as required, in the $z_b \to \infty$ limit, we again obtain the exact dispersion relation (24). If univalence of $f(\zeta)$ is ensured by ${\rm Re} f'(\zeta) > 0$, then the bounds in Eqs. (31) and (32) are improved:

$$\frac{\lambda_L/z_b}{\ln e^{-\lambda_L^2/z_b v_B^2}\Big/\left(1 - \frac{\lambda_L^2}{z_b v_B^2}\right)^2} \le D \le \frac{\lambda_L/z_b}{\ln e^{-\lambda_L^2/z_b v_B^2}\left(1 + \frac{\lambda_L^2}{z_b v_B^2}\right)^2}, \tag{33}$$

$$-\frac{2D}{n z_b^{n-1}} \le c_{n\ge 2} \le \frac{2D}{n z_b^{n-1}}. \tag{34}$$

If $z_b \to \infty$, $\omega_{\rm diff}(\mathbf{q}^2)$ still reduces to the form in Eq. (24).

To demonstrate the existence of such theories, we consider momentum diffusion in two strongly coupled large-N theories at finite temperature: $3d$ worldvolume theory of M2-branes and $4d$ $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. Diffusive $\omega_{\rm diff}(z)$ is determined by dual transverse metric fluctuations in $4d$ [64] and $5d$ [65] Einstein-Hilbert theories with a negative cosmological constant and anti-de Sitter–Schwarzschild black brane backgrounds. We check numerically that in both theories, ${\rm Re} f'(z) > 0$ on their respective disks of hydrodynamic convergence, thereby establishing univalence for $|z| < z_b = R$. For the $\mathcal{N} = 4$ SYM diffusion, we depict this in Fig. 1. The $3d$ M2-brane case qualitatively matches the plot in Fig. 1, with $R \approx 69.423 T^2$, $\lambda_L = 2\pi T$, and $v_B = \sqrt{3}/2$ [66]. In $4d$ $\mathcal{N} = 4$ SYM theory, $R \approx 87.800 T^2$, $\lambda_L = 2\pi T$, and $v_B = \sqrt{2/3}$ [44,45]. Given these values, we can numerically verify the validity of the bounds in Eqs. (33)–(34). For example, Eq. (33) evaluates to

$$\frac{0.046}{T} \le D = \frac{1}{4\pi T} \approx \frac{0.080}{T} \le \frac{0.201}{T}. \tag{35}$$

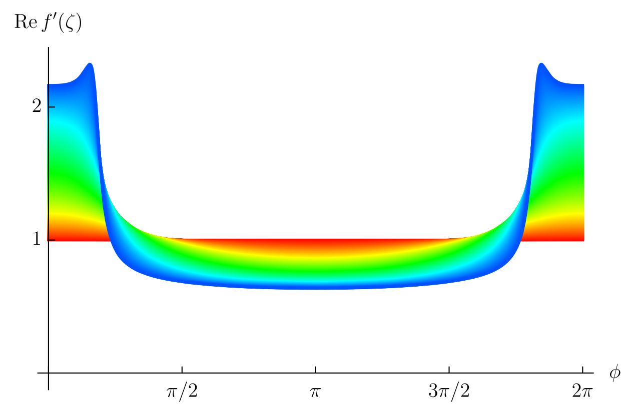

FIG. 1. The univalence condition ${\rm Re} f'(\zeta)$, with $\zeta = |\zeta| e^{i\phi}$, plotted as a function of ϕ for momentum diffusion in $\mathcal{N} = 4$ SYM theory. The color gradient indicates different $|\zeta|$, from $|\zeta| = 0$ (red) to $|\zeta| = 0.92$ (blue). We find that ${\rm Re} f'(\zeta) > 0$ for all $|\zeta| < 1$, with $|\zeta| = 1$ mapped by φ from $|z| = R$.

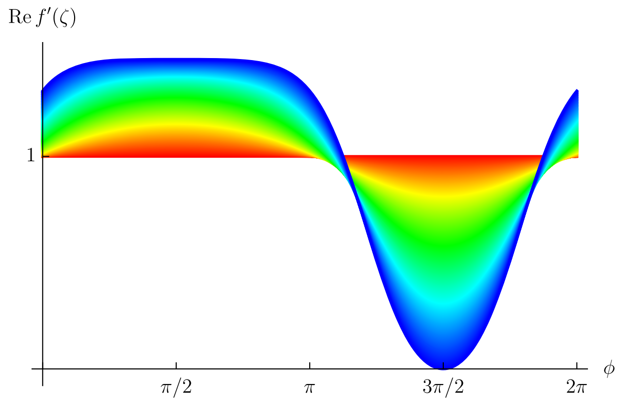

FIG. 2. Re$f'(\zeta)$, with $\zeta = |\zeta|e^{i\phi}$, plotted for sound in $\mathcal{N} = 4$ SYM theory. The color gradient runs from $|\zeta| = 0$ (red) to $|\zeta| = 1$ (blue), with $|\zeta| = 1$ mapped by the $z_c = 0$ Möbius transformation φ from $|z| = z_b = |z_g|$, where $v_g(z_g) = 0$.

Moreover, the bounds become extremely tight as n grows. Assuming that the series coefficients c_n become of the order of the bounds as $n \to \infty$ is consistent with the ratio test for convergence then giving $\lim_{n\to\infty} |c_n/c_{n+1}| = z_b$, which is the radius of convergence of Eq. (7).

Sound.—By extending our holographic analysis to sound in the $\mathcal{N} = 4$ SYM theory, we find that Re$f'(z) \not> 0$ on the hydrodynamic convergence disk $|z| < R$, where $R = 2\sqrt{2}\pi T \approx 8.886T$ [44,45]. Instead, Re$f'(z) > 0$ for $|z| < |z_g| < R$, with $z_g = q_g$ determined by the local condition in Eq. (11). We depict the univalence condition in Fig. 2. Numerically, we find that $z_g \approx -3.791iT$. Since z_g lies within the hydrodynamic radius of convergence, its value can be crudely approximated by conformal first-order hydrodynamics: $z_g \approx -3iv_s/4D = -5.441iT$ with $v_s = 1/\sqrt{3}$ and $D = 1/4\pi T$.

A crucial difference between this case and the diffusion above is that the pole-skipping $z_0 = i\lambda_L/v_B$ [cf. Eq. (17)] is no longer in the $|z| < |z_g|$ disk of univalence U (i.e., $|z_g| < |z_0| = \lambda_L/v_B \approx 7.695T$). However, it can be checked numerically that another univalent disk $z \in U$ can be chosen with $z_c \approx 2.548iT$ and $z_b \approx 6.338T$ [cf. Eq. (28)]. The bounds on $\omega_{\rm sound}(z)$ then follow from Eqs. (14) and (3) [not Eqs. (15) and (6), as Re$f'(\zeta) \not> 0$ for all $|\zeta| < 1$ after φ: $U \to \mathbb{D}$], with $|\omega_0| = \lambda_L$, as well as ζ_0 and the derivatives of $\varphi^{-1}(0)$ computable from Eq. (28).

The maximally univalent sound analog of Eq. (24) is recovered when $z_c = 0$ and $z_b \to \infty$. Then, we find an exact truncated dispersion relation $\omega_{\rm sound}(q) = \pm v_B q$.

Bounds without pole-skipping.—In the absence of pole-skipping considerations, we can derive bounds on transport purely in terms of the wave propagation speeds. For $U = \{z||z| < \min[|z_g|, R]\}$, with z_g given by the group velocity conditions in Eqs. (9)–(10) or in Eq. (11), it follows that if the limit $|\zeta_0| \to 1$ exists, then Eq. (14) implies bounds expressed in terms of the phase velocities and momentum $\bar{q}$: $0 \le D \le 4|v_{\rm ph}(\bar{q}^2)/\bar{q}|$ and $0 \le v_s \le 4|v_{\rm ph}(\bar{q})|$, where $|\bar{q}| = \min[|q_g|, |q_*|]$. If we can use the inequalities from Eq. (15), then 4 in the upper bounds is improved to $1/(2\ln 2 - 1)$. Higher-order coefficients are bounded either by Eq. (3) or Eq. (6). If $\bar{q}$ is the pole-skipping momentum q_0, we again recover the $z_b \to z_0$ limit of Eqs. (31)–(34).

For the final example, assume that there exists a class of theories that has the univalence properties of sound whereby $|\partial_\zeta \varphi^{-1}(0)| = 4|\omega_0(z_0)|\sqrt{d-1}$, with d as the number of spacetime dimensions. Moreover, assume that $\zeta_0 = \varphi(z_0)$ is infinitesimally close to the boundary of $\mathbb{D}$ and that the limit $|\zeta_0| \to 1$ again exists. Intriguingly, for theories satisfying these conditions, the growth theorem (14) would then imply the following conformal upper bound on the speed of sound: $0 \le v_s \le \sqrt{1/(d-1)}$ [16,17], while also ensuring that the lower bound on v_s is at $v_s = 0$. To better understand these simple univalence conditions and their powerful implications, it will be essential to find physical examples of theories (ideally, theories such as quantum chromodynamics [67]) that satisfy them or violate them [68–71]. Of particular interest should be any potential relation between the region U, the map $\varphi^{-1}(0)$ or $\omega_0(z_0)$, and the equation of state of the corresponding quantum field theory.

Discussion.—To use the above construction of bounds, one must first establish univalence in U. Generically, as stated in Eqs. (9)–(11), hydrodynamic dispersion relations will be univalent up to at least the physically motivated group velocity conditions in complexified momentum space, which is sufficient to use the inequalities derived in this work. In holographic theories, this can be checked explicitly by numerical calculations. Finding more efficient methods for identifying (maximal or nonmaximal) regions of univalence, possibly by directly using the associated bulk differential equations, remains an open problem. Another open problem is to explore univalence properties and emergent bounds in weakly coupled field theories and kinetic theory, as well as in quasihydrodynamic theories with long-lived gapped modes [59]. It would also be interesting to understand whether univalence methods can be applied to nonlinear and far-from-equilibrium hydrodynamic flows, as well as shed new light on the universality of the hydrodynamic attractors [72–75].

While pole-skipping was chosen in most examples due to our interest in relating bounds on transport to quantum chaos, as well as for convenience, any known value of $\omega_0(z_0)$ in U could also have been chosen. Two such examples were provided in the last section. Further simple examples can arise from the pole-skipping points without a clear connection to chaos. In fact, such choices may lead to more restrictive bounds. This naturally opens a general problem to find the tightest possible bounds within the scope of univalence techniques. As the univalence methods help pave the way toward more precise analytic explorations of transport, these and other questions will be addressed in the future.

I am grateful to Mike Blake, Richard Davison, Niko Jokela, Pavel Kovtun, Hong Liu, Andrei Starinets, Petar Tadić, and Aleksi Vuorinen for useful and stimulating discussions on related topics. This work was supported by U.S. DOE Grant No. DE-SC0011090 and the research program P1-0402 of Slovenian Research Agency (ARRS).

[1] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, England, 2011).
[2] A. Ioffe and A. Regel, Prog. Semicond. **4**, 237 (1960).
[3] N. Mott, Philos. Mag. **26**, 1015 (1972).
[4] P. K. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).
[5] S. A. Hartnoll, Nat. Phys. **11**, 54 (2015).
[6] M. Blake, Phys. Rev. Lett. **117**, 091601 (2016).
[7] J. Zaanen, SciPost Phys. **6**, 061 (2019).
[8] K. Trachenko and V. Brazhkin, Sci. Adv. **6**, eaba3747 (2020).
[9] M. Baggioli and W.-J. Li, SciPost Phys. **9**, 007 (2020).
[10] P. Kovtun, G. D. Moore, and P. Romatschke, Phys. Rev. D **84**, 025006 (2011).
[11] C. Chafin and T. Schafer, Phys. Rev. A **87**, 023629 (2013).
[12] P. Kovtun, J. Phys. A **48**, 265002 (2015).
[13] M. Martinez and T. Schafer, Phys. Rev. A **96**, 063607 (2017).
[14] A. Lucas and J. Steinberg, J. High Energy Phys. 10 (2016) 143.
[15] T. Hartman, S. A. Hartnoll, and R. Mahajan, Phys. Rev. Lett. **119**, 141601 (2017).
[16] A. Cherman, T. D. Cohen, and A. Nellore, Phys. Rev. D **80**, 066003 (2009).
[17] P. M. Hohler and M. A. Stephanov, Phys. Rev. D **80**, 066002 (2009).
[18] T. Prosen, Phys. Rev. E **89**, 012142 (2014).
[19] S. Grozdanov, A. Lucas, S. Sachdev, and K. Schalm, Phys. Rev. Lett. **115**, 221601 (2015).
[20] S. Grozdanov, A. Lucas, and K. Schalm, Phys. Rev. D **93**, 061901(R) (2016).
[21] J. Maldacena, S. H. Shenker, and D. Stanford, J. High Energy Phys. 08 (2016) 106.
[22] I. Kukuljan, S. Grozdanov, and T. Prosen, Phys. Rev. B **96**, 060301(R) (2017).
[23] P. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften (Springer, New York, 2010).
[24] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Graduate Texts in Mathematics (Springer, New York, 2011).
[25] L. Branges, Acta Math. **154**, 137 (1985).
[26] K. Noshiro, J. Faculty Sci. Hokkaido Imperial University. Ser. 1 Math. **2**, 129 (1934).
[27] S. E. Warschawski, Trans. Am. Math. Soc. **38**, 310 (1935).
[28] T. H. Macgregor, Trans. Am. Math. Soc. **104**, 532 (1962).
[29] L. Landau and E. Lifshits, *Fluid Mechanics* (Pergamon Press, New York, 1987).
[30] P. Kovtun, J. Phys. A **45**, 473001 (2012).
[31] S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, Phys. Rev. D **85**, 085029 (2012).
[32] S. Grozdanov and J. Polonyi, Phys. Rev. D **91**, 105031 (2015).
[33] M. Crossley, P. Glorioso, and H. Liu, J. High Energy Phys. 09 (2017) 095.
[34] P. Glorioso, M. Crossley, and H. Liu, J. High Energy Phys. 09 (2017) 096.
[35] F. M. Haehl, R. Loganayagam, and M. Rangamani, J. High Energy Phys. 01 (2016) 184.
[36] F. M. Haehl, R. Loganayagam, and M. Rangamani, J. High Energy Phys. 04 (2016) 039.
[37] K. Jensen, N. Pinzani-Fokeeva, and A. Yarom, J. High Energy Phys. 09 (2018) 127.
[38] H. Liu and P. Glorioso, Proc. Sci. TASI2017 (**2018**) 008 [arXiv:1805.09331].
[39] S. Grozdanov, D. M. Hofman, and N. Iqbal, Phys. Rev. D **95**, 096003 (2017).
[40] X. Chen-Lin, L. V. Delacretaz, and S. A. Hartnoll, Phys. Rev. Lett. **122**, 091602 (2019).
[41] Classical hydrodynamics is a theory without stochastic noise or loop corrections leading to the breakdown of analyticity [40,42,43].
[42] P. Kovtun and L. G. Yaffe, Phys. Rev. D **68**, 025007 (2003).
[43] L. V. Delacretaz, SciPost Phys. **9**, 034 (2020).
[44] S. Grozdanov, P. K. Kovtun, A. O. Starinets, and P. Tadić, Phys. Rev. Lett. **122**, 251601 (2019).
[45] S. Grozdanov, P. K. Kovtun, A. O. Starinets, and P. Tadić, J. High Energy Phys. 11 (2019) 097.
[46] See also Refs. [47,48] and [49,50].
[47] B. Withers, J. High Energy Phys. 06 (2018) 059.
[48] M. P. Heller, A. Serantes, M. Spaliński, V. Svensson, and B. Withers, arXiv:2007.05524.
[49] N. Abbasi and S. Tahery, J. High Energy Phys. 10 (2020) 076.
[50] A. Jansen and C. Pantelidou, J. High Energy Phys. 10 (2020) 121.
[51] E. Krotscheck and W. Kundt, Commun. Math. Phys. **60**, 171 (1978).
[52] S. Grozdanov, K. Schalm, and V. Scopelliti, Phys. Rev. Lett. **120**, 231601 (2018).
[53] M. Blake, H. Lee, and H. Liu, J. High Energy Phys. 10 (2018) 127.
[54] M. Blake, R. A. Davison, S. Grozdanov, and H. Liu, J. High Energy Phys. 10 (2018) 035.
[55] S. Grozdanov, J. High Energy Phys. 01 (2019) 048.
[56] S. H. Shenker and D. Stanford, J. High Energy Phys. 03 (2014) 067.
[57] M. Blake, R. A. Davison, and D. Vegh, J. High Energy Phys. 01 (2020) 077.
[58] In general, the dispersion relations which give rise to Eqs. (17) and (18) also pass through an infinite sequence of pole-skipping points $\omega(\mathbf{q}_n^2) = -2\pi T i n$ for all $n \in \mathbb{Z}_+ \cup \{0\}$ [45,57].
[59] S. Grozdanov, A. Lucas, and N. Poovuttikul, Phys. Rev. D **99**, 086012 (2019).
[60] S. Grozdanov and N. Kaplis, Phys. Rev. D **93**, 066012 (2016).
[61] S. M. Diles, L. A. Mamani, A. S. Miranda, and V. T. Zanchin, J. High Energy Phys. 05 (2020) 019.
[62] T. Andrade and B. Withers, J. High Energy Phys. 05 (2014) 101.
[63] R. A. Davison and B. Gouteraux, J. High Energy Phys. 01 (2015) 039.
[64] C. P. Herzog, J. High Energy Phys. 12 (2002) 026.

[65] G. Policastro, D. T. Son, and A. O. Starinets, J. High Energy Phys. 09 (2002) 043.
[66] Convergence of hydrodynamics in the holographic M2-brane theory is analyzed by using the methods from Refs. [44,45]. The transverse channel pole-skipping in Eq. (18) follows from the methods of [45,54,57]. We can prove analytically that $\omega_{\rm diff}(z)$ passes through an infinite sequence of pole-skipping points: $\omega_{\rm diff}(z_n) = -2\pi T i n$ at $z_n = \mathbf{q}_n^2 = 16\pi^2 T^2 \sqrt{n}/3$ for all $n \in \mathbb{Z}_+ \cup \{0\}$.
[67] E. Annala, T. Gorda, A. Kurkela, J. Nättilä, and A. Vuorinen, Nat. Phys. **16**, 907 (2020).
[68] M. Kulaxizi and A. Parnachev, Nucl. Phys. **B815**, 125 (2009).
[69] C. Hoyos, N. Jokela, D. Rodríguez Fernández, and A. Vuorinen, Phys. Rev. D **94**, 106008 (2016).
[70] C. Ecker, C. Hoyos, N. Jokela, D. Rodríguez Fernández, and A. Vuorinen, J. High Energy Phys. 11 (2017) 031.
[71] T. Ishii, M. Jarvinen, and G. Nijs, J. High Energy Phys. 07 (2019) 003.
[72] M. P. Heller and M. Spalinski, Phys. Rev. Lett. **115**, 072501 (2015).
[73] M. P. Heller, A. Kurkela, M. Spaliński, and V. Svensson, Phys. Rev. D **97**, 091503(R) (2018).
[74] P. Romatschke, Phys. Rev. Lett. **120**, 012301 (2018).
[75] G. S. Denicol and J. Noronha, Phys. Rev. Lett. **124**, 152301 (2020).