Absence of Fast Scrambling in Thermodynamically Stable Long-Range Interacting Systems

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In this study, we investigate out-of-time-order correlators (OTOCs) in systems with power-law decaying interactions such as $R^{-\alpha}$, where R is the distance. In such systems, the fast scrambling of quantum information or the exponential growth of information propagation can potentially occur according to the decay rate α. In this regard, a crucial open challenge is to identify the optimal condition for α such that fast scrambling cannot occur. In this study, we disprove fast scrambling in generic long-range interacting systems with $\alpha > D$ (D: spatial dimension), where the total energy is extensive in terms of system size and the thermodynamic limit is well defined. We rigorously demonstrate that the OTOC shows a polynomial growth over time as long as $\alpha > D$ and the necessary scrambling time over a distance R is larger than $t \gtrsim R^{[(2\alpha-2D)/(2\alpha-D+1)]}$.

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Introduction.—Information scrambling, which characterizes the inaccessibility of local information after time evolution, is a central research topic in interdisciplinary problems ranging from thermalization in quantum many-body systems [1–4] to the black hole information problem [5–7]. In the recent developments on the connection between quantum chaos and information theory, out-of-time-order correlators (OTOCs) were found to be a useful quantitative tool for characterizing information scrambling [8–11].

For quantum lattice models, the OTOC has the form [11]

$$C(R,t) := \frac{1}{\mathrm{tr}(\hat{1})}\mathrm{tr}([W_i(t), V_{i'}]^\dagger [W_i(t), V_{i'}]), \qquad (1)$$

where $W_i(t) = e^{iHt}W_i e^{-iHt}$, H denotes the system Hamiltonian, and the operators W_i and $V_{i'}$ are defined on the sites i and i', respectively; they are separated from each other by a distance R. When the Hamiltonian H includes only short-range interactions, the OTOC grows as $C(R,t) \propto e^{\lambda_L(t-R/v_B)}$, where λ_L and v_B are referred to as quantum analogs of the Lyapunov exponent [10] and the butterfly speed [12], respectively. On the butterfly speed v_B, the Lieb-Robinson bound [13–15] yields the simplest upper bound for generic quantum many-body systems. The exploration of the universal behaviors of the OTOC has been one of the most fascinating and essential topics in modern physics [16–26]. Moreover, along with theoretical developments, the experimental observations of the OTOC have been proposed and realized in various setups [27–32].

When the Hamiltonian consists of only short-range interactions, the OTOC exhibits a ballistic spreading of the wavefront with a butterfly speed v_B [12,33–39]. However, when the Hamiltonian includes long-range (or power-law decaying) interactions proportional to $R^{-\alpha}$ with the distance R between two particles, the wavefront can spread superlinearly with time [40–52]. From the analogy of the short-range interacting systems, the following exponential growth of the OTOC may be inferred:

$$C(R,t) \propto e^{\lambda_L t}/R^{\alpha}. \qquad (2)$$

It results in the so-called *fast scrambling* which implies that local quantum information is spread over the entire regime of the system with a timescale of $t_s \approx \log(n)/\lambda_L$, where n is the system size. Indeed, the well-known Lieb-Robinson bound [53,54] for long-range interacting systems gives the upper bound in the form of Eq. (2). Recent studies have focused on the universal laws of fast scrambling, specifically in the context of black hole physics [7,55,56]. Starting with the exact solution of the Sachdev-Ye-Kitaev model [9,57], intensive studies have been conducted to determine the types of quantum many-body systems that permit or prohibit the fast scrambling [58–68].

Fast scrambling implies that a system can relax arbitrarily fast under a local perturbation, whereas it is difficult to imagine that such extremely fast information propagation usually occurs in nature. Systems with very large α are categorized as short-range systems, and hence, the OTOC cannot be described accurately for the entire regime of α by

Eq. (2). Indeed, a more accurate description of the OTOC for long-range interacting systems may lead to the following polynomial growth [69–80] instead of an exponential growth (2):

$$C(R,t) \leq (\lambda_L t/R^{\zeta})^{\tilde{\alpha}}, \tag{3}$$

where $\zeta \leq 1$ and $\zeta\tilde{\alpha} \leq \alpha$. This inequality yields scrambling time that is algebraic with respect to the system size, i.e., $\approx n^{\zeta/D}/\lambda_L$. For sufficiently large α, several numerical [69–72,81] and theoretical [73–80] studies indicate polynomial growth.

From the above background, the following fundamental question naturally arises: what is the optimal condition for α to prohibit the fast scrambling of the OTOC given in Eq. (2)? Because numerical calculations have already indicated that polynomial growth of the OTOC might break down for $\alpha \leq D$ [70,72], we expect that the condition $\alpha > D$ is at least necessary. Moreover, this condition defines natural long-range interacting systems that are thermodynamically stable such that the total energy is extensive with regard to the system size and the thermodynamic limit is well defined [82,83].

In previous studies, theoretical analyses have been mostly limited to the regime of $\alpha > 2D$ [73–80]. For $\alpha > 2D$, Foss-Feig *et al.* proved that ζ in Eq. (3) is lower-bounded by $[(\alpha - 2D)/(\alpha - D + 1)]$ [73], which was improved to $[(\alpha - 2D)/(\alpha - D)]$ in Refs. [76,77]. Furthermore, for $\alpha > 2D + 1$, even the existence of the finite butterfly speed (i.e., $\zeta = 1$) has been proven in generic long-range interacting systems [78–80]. The sequence of these achievements has demonstrated that fast scrambling (2) is prohibited in long-range interacting systems when α is above a threshold, i.e., $\alpha = 2D$.

In contrast, fast scrambling conditions in regimes of $D < \alpha \leq 2D$ are highly elusive. In this regime, a sub-exponential speed of the quantum-state-transfer is in principle possible by a clever protocol employing quantum many-body long-range interactions [84]. In addition, when exponent α approaches D, the effective system dimensions become infinitely large, and hence different physics can appear. For example, various studies on one-dimensional systems have shown that the long-range interactions can qualitatively change the fundamental physical properties for $\alpha \leq 2$ both in static [85–89] and dynamical phases [90,91]. Therefore, physics induced by long-range interactions in this regime is quite nontrivial and can yield unexpected consequences. Nevertheless, various observations have indicated the prohibition of fast scrambling in this regime. As a partial solution, Tran *et al.* have disproved fast scrambling for a condition $\alpha > 3/2$ in one dimension [79].

In the present Letter, we prove that under the condition $\alpha > D$ fast scrambling is prohibited in arbitrary long-range interacting systems. Thus, by combining the counterexamples for $\alpha \leq D$ [70,72], we identify $\alpha > D$ as the optimal condition for the polynomial growth (3) of the OTOC (see also Ref. [92]). As a general upper bound, we derive the polynomial growth of the OTOC with exponent ζ expressed as $\zeta = [(2\alpha - 2D)/(2\alpha - D + 1)]$. Our analyses consist of the following two parts: (i) A simple connection technique for the unitary time operators for small times, which is utilized in Ref. [93] and (ii) the Lieb-Robinson bound for short-time evolution. Using these techniques, we can not only prove our main result, but also develop a considerably simple proof for the state-of-the-art Lieb-Robinson bound for $2D < \alpha \leq 2D + 1$ in Refs. [76,77]. Our result verifies the empirical hypothesis that thermodynamically natural class of long-range interactions cannot induce fast scrambling.

Setup and main result.—Let us consider a quantum spin system with n spins, where each spin is located on one vertex of the D-dimensional graph (or D-dimensional lattice) with Λ of the total spin set, i.e., $|\Lambda| = n$. For simplicity, we consider (1/2)-spin systems; however, the extension to a general finite spin dimension d is straightforward. For a partial set $X \subseteq \Lambda$, we denote the cardinality, i.e., the number of vertices contained in X, by $|X|$ (e.g., $X = \{i_1, i_2, \ldots, i_{|X|}\}$). Further, we denote the complementary subset of X as $X^{\mathrm{c}} \coloneqq \Lambda \backslash X$. For two arbitrary spins i and i', we define distance $d_{i,i'}$ as the shortest path length on the lattice that connects i and i'. We define $i[r]$ as the ball region with radius r from site i (Fig. 1).

$$i[r] \coloneqq \{i' \in \Lambda | d_{i,i'} \leq r\}, \tag{4}$$

where $i[0] = i$ and r is an arbitrary positive integer.

We consider a general system having at most k-body long-range interactions with finite k. For example, we give the Hamiltonian with $k = 2$, which is described as

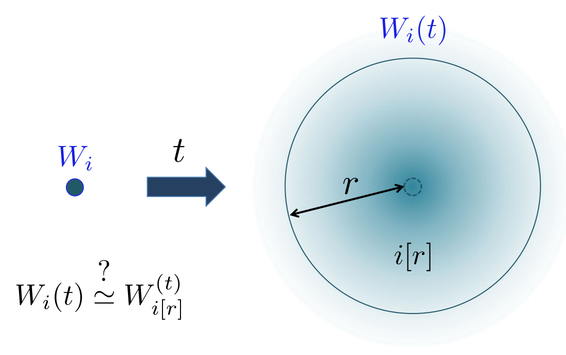

FIG. 1. The OTOC (1) roughly determines the spreading of local operator W_i by time evolution. We aim to approximate $W_i(t)$ in a local region $i[r]$, which has a maximum distance of r from site i [Eq. (4)]: If operator $W_i(t)$ is well approximated by using $W^{(t)}_{i[r]}$ as long as $t \lesssim \mathcal{O}(r^{\zeta})$ ($\zeta < 1$), the OTOC exhibits polynomial growth, as in Eq. (3), because of Eq. (7).

$$H = \sum_{i<i'} h_{i,i'} + \sum_{i=1}^{n} h_i, \qquad \|h_{i,i'}\| \le \frac{J_0}{(d_{i,i'}+1)^\alpha}, \quad (5)$$

for $\forall\, i,\ i' \in \Lambda$, where $\{h_{i,i'}\}_{i<i'}$ are interaction operators acting on the spins $\{i, i'\}$, and $\|\cdots\|$ is the operator norm. One of the simple examples is the long-range transverse Ising model, which has a form of Eq. (5) by choosing $h_{i,i'} = J\sigma_i^x \sigma_{i'}^x / d_{i,i'}^\alpha$ and $h_i = B\sigma_i^z$. Such long-range interactions have been realized in various experimental setups such as atomic, molecular, and optical systems [94–107]. In this Letter, we are in particular interested in the regime of $D < \alpha \le 2D$, which is also experimentally important as it includes several realistic long-range interactions, such as dipole-dipole interactions ($D = 2,\ \alpha = 3$) and van der Waals interactions ($D = 3,\ \alpha = 6$).

In our analyses, we focus on time evolution by the Hamiltonian H. A key strategy for estimating the OTOC is using the local approximation of the time-evolved operator $W_i(t) \coloneqq e^{iHt} W_i e^{-iHt}$ (Fig. 1). We approximate the operator $W_i(t)$ using another operator $W_{i[r]}^{(t)}$ which is supported on the local subset $i[r]$. The error of this approximation is estimated by

$$\|W_i(t) - W_{i[r]}^{(t)}\|_p, \quad (6)$$

where $\|\cdots\|_p$ is the Schatten-p norm, which is defined as $\|O\|_p \coloneqq [\mathrm{tr}(O^\dagger O)^{p/2}]^{1/p}$. For $p = \infty$, the Schatten norm $\|\cdots\|_\infty$ corresponds to the standard operator norm, while the case of $p = 2$ corresponds to the Frobenius norm, which is of interest. For an arbitrary operator $V_{i'}$ with $d_{i,i'} = R$ and $\|V_{i'}\| = 1$, one can easily show

$$C(R,t) \le 4\|W_i(t) - W_{i[R-1]}^{(t)}\|_F^2, \quad (7)$$

where we define the normalized Frobenius norm $\|\cdots\|_F \coloneqq \|\cdots\|_2 / [\mathrm{tr}(\hat{1})]^{1/2}$ and use $[W_{i[R-1]}^{(t)}, V_{i'}] = 0$ for $d_{i,i'} = R$.

Our main result provides the efficiency guarantee for the local approximation of a time-evolved operator $W_i(t)$ in the region $i[r]$ (see Supplemental Material [108], Sec. S.II for more details).

Theorem 1.—Let us consider Hamiltonians with few-body interactions and power-law decay exponent $\alpha > D$. Then, for an arbitrary operator W_i ($\|W_i\| = 1$) and the corresponding time evolution of $W_i(t)$, there exists an operator $W_{i[r]}^{(t)}$ that approximates $W_i(t)$ on a region $i[r]$ as

$$\|W_i(t) - W_{i[r]}^{(t)}\|_F \le C r^{-\alpha+D} t^{\alpha-[(D-1)/2]}, \quad (8)$$

where C is an $\mathcal{O}(1)$ constant.

From the inequalities in Eqs. (7) and (8), we obtain the upper bound of the OTOC as

$$C(R,t) \lesssim \left(\frac{C' t}{R^{\frac{2\alpha-2D}{2\alpha-D+1}}}\right)^{\alpha-[(D-1)/2]},$$

where C' is a constant of $\mathcal{O}(1)$. This gives the polynomial growth in Eq. (3) with $\zeta = [(2\alpha - 2D)/(2\alpha - D + 1)]$ and $\tilde{\alpha} = \alpha - (D-1)/2$.

In the above theorem, we consider an on-site operator W_i; however, the theorem can be generalized to an operator W_X supported on an arbitrary subset $X \subset \Lambda$. Let us consider the case where the subset X satisfies $X \subseteq i[r_0]$ for particular choices of i and r_0. Then, for $W_X(t)$, we obtain an inequality that is similar to Eq. (8) as

$$\|W_X(t) - W_{i[r_0+r]}^{(t)}\|_F \le \frac{C t^{\alpha-[(D-1)/2]} (r + r_0)^{[(D-1)/2]}}{r^{\alpha-[(D+1)/2]}}.$$

For $D = 1$, the above inequality reduces to

$$\|W_X(t) - W_{i[r_0+r]}^{(t)}\|_F \le \frac{C t^\alpha}{r^{\alpha-1}}.$$

Concept of the proof.—A central technique in our proof is the connection of unitary time evolutions addressed in Ref. [93] (Fig. 2). Following Ref. [93], we decompose the time to m_t pieces, and we define $t_m \coloneqq m\Delta t$ and $t_{m_t} \coloneqq t$ where $\Delta t = t/m_t$. We assume Δt as a small constant. For fixed r and $i \in \Lambda$, we define lengths Δr, r_m, and subset X_m as

$$\Delta r \coloneqq r/m_t, \qquad X_m \coloneqq i[m\Delta r]. \quad (9)$$

Using these notations, we approximate $W_i(t_m)$ with another operator supported on subset X_m.

For the approximation, we adopt the following recursive procedure. For $m = 1$, we define operator $W_{X_1}^{(1)}$ as an approximation of $W_i(\Delta t)$ onto the subset X_1:

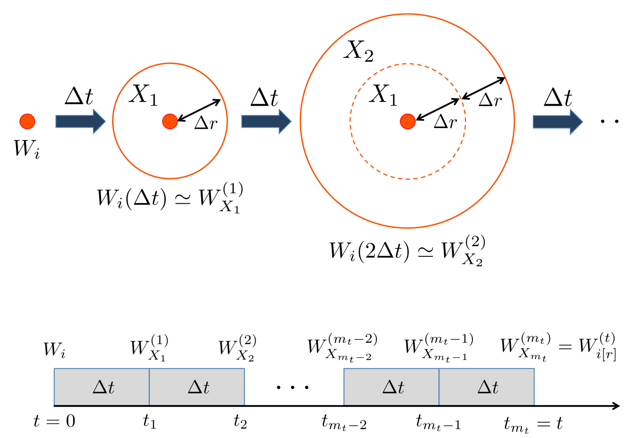

FIG. 2. We decompose time t and length r to m_t pieces, namely, $\Delta t \coloneqq t/m_t$ and $\Delta r \coloneqq r/m_t$. We start from time evolution $W_i(\Delta t)$ and approximate it by $W_{X_1}^{(1)}$, which is supported on an extended region X_1 as in Eq. (9). Then, we iteratively approximate $W_{X_m}^{(m)}(\Delta t)$ by $W_{X_{m+1}}^{(m+1)}$, which finally yields the approximation (12). The main advantage of this method is that we need to estimate the local approximation of the time-evolved operators only for a short time.

$$W_{X_1}^{(1)} \coloneqq W_i(\Delta t, X_1),$$

where we define notation $W_i(t, X_1)$ as

$$W_i(t, X_1) \coloneqq \frac{1}{\mathrm{tr}_{X_1^c}(\hat{1})} \mathrm{tr}_{X_1^c}[W_i(t)] \otimes \hat{1}_{X_1^c}. \tag{10}$$

Note that $W_i(\Delta t, X_1)$ is now supported on subset X_1. For $m = 2$, we adopt the second-step approximation $W_{X_2}^{(2)} \coloneqq W_{X_1}^{(1)}(\Delta t, X_2)$, which is similar to Eq. (10). We then obtain the approximation error as

$$\begin{aligned} &\|W_i(2\Delta t) - W_{X_2}^{(2)}\|_p \\ &\quad \le \|W_i(2\Delta t) - W_{X_1}^{(1)}(\Delta t) + W_{X_1}^{(1)}(\Delta t) - W_{X_2}^{(2)}\|_p \\ &\quad \le \|W_i(\Delta t) - W_{X_1}^{(1)}\|_p + \|W_{X_1}^{(1)}(\Delta t) - W_{X_2}^{(2)}\|_p, \end{aligned} \tag{11}$$

with $W_{X_1}^{(1)}(\Delta t) \coloneqq e^{iH\Delta t} W_{X_1}^{(1)} e^{-iH\Delta t}$, where we use the triangle inequality and unitary invariance for the Schatten-p norm.

By repeating this procedure, we define operator $W_{X_m}^{(m)}$ recursively as $W_{X_m}^{(m)} = W_{X_{m-1}}^{(m-1)}(\Delta t, X_m)$. Then, similar to Eq. (11), we obtain the following inequality:

$$\|W_i(m_t\Delta t) - W_{X_{m_t}}^{(m_t)}\|_p \le \sum_{m=0}^{m_t-1} \|W_{X_m}^{(m)}(\Delta t) - W_{X_{m+1}}^{(m+1)}\|_p, \tag{12}$$

where we define $W_{X_0}^{(0)} \coloneqq W_i$. The problem now reduces to estimating the approximation error of $\|W_{X_m}^{(m)}(\Delta t) - W_{X_{m+1}}^{(m+1)}\|_p$ only for short-time evolution, which is a critical point to derive our main results.

As the simplest exercise, let us consider the case with $p = \infty$, which provides the standard operator norm. The resulting wavefront shape for information propagation is the same as that obtained in Refs. [76,77]; however, our derivation is considerably simpler and can be applied to a more general class of Hamiltonians. For the short-time evolution, we can utilize the well-known simple Lieb-Robinson bound as in Refs. [53,54]. Using their results, we can readily derive the following approximation error (see Supplemental Material [108], Sec. S.III A for the derivation):

$$\|W_{X_m}^{(m)}(\Delta t) - W_{X_{m+1}}^{(m+1)}\|_\infty \le c|\partial X_m| e^{c'\Delta t}(\Delta r)^{-\alpha+D+1}, \tag{13}$$

where c and c' are the constants of $\mathcal{O}(1)$, which depend on only the details of the system. Note that ∂X_m is the surface region of subset X_m. For a sufficiently large Δt, the bound (13) eventually yields an exponential growth; however, Δt is now selected to be as small as $\mathcal{O}(1)$, and hence, $e^{c'\Delta t}$ is given by a constant.

Thus, by introducing geometric parameter γ that yields $|\partial X_m| \le |\partial i[r]| \le \gamma r^{D-1}$, we obtain

$$\|W_{X_m}^{(m)}(\Delta t) - W_{X_{m+1}}^{(m+1)}\|_\infty \le \tilde{c} r^{2D-\alpha} t^{\alpha-D-1},$$

where $\tilde{c} \coloneqq c\gamma e^{c'\Delta t}(\Delta t)^{-\alpha+D+1}$, and we use $\Delta r = \Delta t(r/t)$. Therefore, we reduce the upper bound in Eq. (12) to

$$\|W_i(t) - W_{i[r]}^{(m_t)}\|_\infty \le \tilde{c}' r^{2D-\alpha} t^{\alpha-D}, \tag{14}$$

where $\tilde{c}' \coloneqq \tilde{c}/\Delta t$, and we use $m_t = t/\Delta t$. The time step, Δt, is selected as an $\mathcal{O}(1)$ constant, and hence, $\tilde{c}'$ is also an $\mathcal{O}(1)$ constant. Using the upper bound, information propagation is restricted to a region with diameter $R \approx |t|^{[(\alpha-D)/(\alpha-2D)]}$, which is the same as the state-of-the-art estimation obtained in Refs. [76,77], namely, the improved version of Refs. [73–75]. Note that the result above is more general; we *do not* have to assume the few-body interactions of the Hamiltonian in deriving Eq. (13) because the upper bound in Eq. (13) is applied to the Hamiltonians without the assumption of few-body interactions (see Ref. [53], Assumption 2.1).

Finally, we explain why the condition of $\alpha > 2D$ appears instead of $\alpha > D$ to obtain a meaningful upper bound. This condition originated from coefficient $|\partial X_m|$ in Eq. (13). When we consider the time evolution of an operator supported on subset $X \subset \Lambda$ (e.g., O_X), the Lieb-Robinson bound unavoidably includes the subset dependence [13–15]. This subset dependence is the primary obstacle that resists the rigorous proof of the polynomial growth of the information propagation for $\alpha < 2D$. In the case where the Frobenius norm ($p = 2$) is considered, this subset dependence is significantly improved, as shown in Eq. (15). This provides a breakthrough in deriving the strictest condition, namely, $\alpha > D$, for the polynomial growth of the OTOC.

Proof of Theorem 1 (Case with $p = 2$ and $\alpha > D$).—For proving our main theorem, we start from the inequality in Eq. (12). Thus, our task is to derive a local approximation for short-time evolution. Here, let O_X be an arbitrary operator on subset X with $\|O_X\| = 1$. We aim to approximate $O_X(t)$ by $O_X(t, X[r])$, where $X[r]$ is an extended subset defined as $X[r] \coloneqq \bigcup_{i\in X} i[r]$. The key technical ingredient is the following inequality for short-time evolution in terms of the Frobenius norm (see the Supplemental Material [108], Theorem 3)

$$\|O_X(t) - O_X(t, X[r])\|_F \le c_0|t|\sqrt{|\partial X[r]| \cdot r^{-2\alpha+D+1}}, \tag{15}$$

with c_0 as an $\mathcal{O}(1)$ constant, where $\partial X[r]$ is the surface region of $X[r]$, and time t is assumed to be smaller than a certain threshold. Most parts of the proof are dedicated to deriving Eq. (15), as shown in the Supplemental Material [108], Secs. S.IV and S.V.

With the inequality in Eq. (15), we can easily prove the main Theorem 1 in the same manner as that used for deriving Eq. (14) for $p = \infty$. Here, Δt is sufficiently small such that the inequality (15) holds. Applying inequality (15) to Eq. (12), we obtain

$$\|W_{X_m}^{(m)}(\Delta t) - W_{X_{m+1}}^{(m+1)}\|_F \le \tilde{c}_0 r^{-\alpha+D} t^{\alpha-[(D+1)/2]},$$

with $\tilde{c}_0$ being an $\mathcal{O}(1)$ constant, where we use $W_{X_{m+1}}^{(m+1)} = W_{X_m}^{(m)}(\Delta t, X_m[\Delta r])$ and $|\partial(X_m[\Delta r])| \le |\partial(i[2r])| \le \gamma(2r)^{D-1}$. The above inequality reduces inequality in Eq. (15) to the main inequality given in Eq. (8) using $m_t := t/\Delta t$ as

$$\|W_i(t) - W_{i[r]}^{(m_t)}\|_F \le (\tilde{c}_0/\Delta t) r^{-\alpha+D} t^{\alpha-[(D-1)/2]}.$$

This completes the proof of Theorem 1.□

Conclusion.—In this work, we investigated the polynomial growth of the OTOC represented in Eq. (3) for all long-range interacting systems with $\alpha > D$, where the existence of a well-defined thermodynamic limit is ensured. We comprehensively disproved fast scrambling in this natural class of long-range interactions. Our results indicate the lower bound of the scrambling time as $n^{\zeta/D}$ with $\zeta = [(2\alpha - 2D)/(2\alpha - D + 1)]$.

This study has two future directions. First, our condition of $\alpha > D$ for the polynomial growth of the OTOC is expected to be qualitatively tight; however, the quantitative estimation of ζ still has scope for improvement. In particular, it is an intriguing problem to identify the critical value of α_c above which the ballistic propagation of information scrambling (i.e., $\zeta = 1$) is ensured. For the operator norm [i.e., $p = \infty$ in Eq. (6)], the critical α_c is proven to be equal to $2D + 1$ [78–80]. For the Frobenius norm, it has been conjectured that the critical α_c is equal to $3D/2 + 1$, where the case of $D = 1$ has been indeed proved [79]. We hope that our current analysis will be further refined to identify the optimal value of ζ in the future.

Second, we considered the most common form of the OTOC in Eq. (1), which adopts the average for a uniformly mixed state. In experimental application, if we would be able to prepare the uniform mixed state as the initial state, Theorem 1 appropriately predicts the growth of the OTOC. On the other hand, if the initial state is prepared as a finite temperature result, we need to consider the following generalization for a finite-temperature state:

$$C_\beta(x,t) := \frac{1}{\mathrm{tr}(e^{-\beta H})} \mathrm{tr}(e^{-\beta H}[W_i(t), V_{i'}]^\dagger [W_i(t), V_{i'}]).$$

The inequality in Eq. (12) is applied to this case, and we expect that the same polynomial growth can be obtained above a temperature threshold by using the cluster expansion technique [111,112].

Finally, throughout the Letter, we consider the Hamiltonian dynamics e^{-iHt}. It is an intriguing to extend our result to Markovian quantum dynamics [113,114]. If the uniform mixed state is a steady state, our formalism in Eq. (12) is applied and we expect to derive a similar upper bound for the OTOC.

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