# Information Scrambling over Bipartitions: Equilibration, Entropy Production, and Typicality

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In recent years, the out-of-time-order correlator (OTOC) has emerged as a diagnostic tool for information scrambling in quantum many-body systems. Here, we present exact analytical results for the OTOC for a typical pair of random local operators supported over two regions of a bipartition. Quite remarkably, we show that this "bipartite OTOC" is equal to the operator entanglement of the evolution, and we determine its interplay with entangling power. Furthermore, we compute long-time averages of the OTOC and reveal their connection with eigenstate entanglement. For Hamiltonian systems, we uncover a hierarchy of constraints over the structure of the spectrum and elucidate how this affects the equilibration value of the OTOC. Finally, we provide operational significance to this bipartite OTOC by unraveling intimate connections with average entropy production and scrambling of information at the level of quantum channels.

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*Introduction.*—A characteristic feature of certain quantum many-body systems is their ability to quickly spread "localized" information over subsystems, thereby making it inaccessible to local observables. Although unitary evolution retains all information, this local inaccessibility manifests itself as equilibration in closed systems and has been termed "information scrambling" [1–5].

For Hamiltonian quantum dynamics, scrambling can be probed by examining the overlap of a time-evolved local operator $V(t) \coloneqq U_t^\dagger V U_t$ with a second static operator $W$. This overlap is commonly quantified via the strength of the commutator

$$C_{V,W}(t) \coloneqq \frac{1}{2}\mathrm{Tr}\big([V(t), W]^\dagger [V(t), W]\rho_\beta\big), \tag{1}$$

where $\rho_\beta$ denotes the thermal state at inverse temperature $\beta$ (in fact, $C_{V,W}(t) = \frac{1}{2}\|[V(t), W]\|^2$ for the norm associated with the inner product $\langle X, Y\rangle_\beta = \mathrm{Tr}(X^\dagger Y \rho_\beta)$, $\beta < \infty$). From the perspective of information spreading, $C_{V,W}(t)$ is a natural quantity to consider, since it constitutes a state-dependent variant of the Lieb-Robinson scheme; the latter enforces a fundamental restriction on the speed of correlations spreading in nonrelativistic quantum systems [6–9]. In Eq. (1), it is convenient to consider pairs of operators $V$, $W$ which at $t = 0$ act nontrivially on different subsystems and, thus, commute; we follow this convention here.

The commutator $C_{V,W}(t)$ is intimately linked to the out-of-time-order correlator (OTOC) [10,11] which is a four-point function with an unconventional time ordering

$$F_{V,W}(t) \coloneqq \mathrm{Tr}[V^\dagger(t) W^\dagger V(t) W \rho_\beta]. \tag{2}$$

The connection between the two arises when $V$, $W$ are unitary; Eq. (1) then immediately reduces to $C_{V,W}(t) = 1 - \mathrm{Re}[F_{V,W}(t)]$. In this Letter, we focus on the infinite-temperature, $\beta = 0$ case.

Through the years, several key signatures of quantum chaos [12–15] have been introduced. The initial exponential growth of the OTOC was proposed as a diagnostic of quantum chaos [16–23]. However, a careful analysis has revealed that information scrambling does not always necessitate chaos [24–29].

*Per se*, the OTOC's ability to probe dynamical features clearly depends on the choice of operators $V$, $W$. However, it is desirable to be able to capture these features as independently as possible from the specific choice of operators. This insensitivity can be achieved by averaging over a set of operators, a strategy also considered in Refs. [22,30–35]. It is crucial to remark that, for the averaged OTOC to faithfully capture information spreading, the averaging process must *preserve the initial locality of the system*, i.e., which subsystems $V$, $W$ initially act upon —an observation that was quintessential in revealing the correct behavior of the OTOC and its connection with Loschmidt echo [35].

Given a bipartition of a finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, we will henceforth focus on averaging $C_{V_A,W_B}(t)$ over the (independent) unitary operators $V_A$ and $W_B$, whose support is over subsystems $A$ and $B$, respectively. The resulting quantity

 

$$G(t) := 1 - \frac{1}{d}\mathrm{Re}\int dVdW\mathrm{Tr}[V_A^\dagger(t)W_B^\dagger V_A(t)W_B] \quad (3)$$

depends only on the dynamics and the Hilbert space cut, where we denote $V_A = V \otimes I_B$ and $W_B = I_A \otimes W$ and the averaging is performed according to the Haar measure [36]. We will refer to $G(t)$ for brevity as the *bipartite OTOC*, and analyzing its properties will be the focus of the present Letter.

It was recently shown in Ref. [35], where $G(t)$ was first introduced, under the assumptions of (i) weak coupling between $A$ and $B$ and (ii) Markovianity, that $G(t)$ exhibits a close connection with the Loschmidt echo [37,38]; the latter has been widely employed to characterize chaos [39,40]. Here, we first show, without any of the previous assumptions, that $G(t)$ is, in fact, amenable to exact analytical treatment, and we uncover its direct relation with entropy production, information spreading, and entanglement. We also rigorously prove that the average case is also the typical one, hence justifying the averaging process. Our main results are stated in the theorems that follow. All proofs of the claims appearing in the text can be found in Supplemental Material [41].

*The bipartite OTOC.*—We begin by bringing $G(t)$ in a more explicit form which will be the starting point for a sequence of results. This can be achieved by working on the doubled space $\mathcal{H}\otimes\mathcal{H}'$, where $\mathcal{H}' = \mathcal{H}_{A'}\otimes\mathcal{H}_{B'}$ is a replica of the original Hilbert space.

**Theorem 1:** Let $S_{AA'}$ be the operator over $\mathcal{H}\otimes\mathcal{H}'$ that swaps $A$ with its replica $A'$ and $d = \dim(\mathcal{H})$. Then

$$G(t) = 1 - \frac{1}{d^2}\mathrm{Tr}(S_{AA'}U_t^{\otimes 2}S_{AA'}U_t^{\dagger\otimes 2}). \quad (4)$$

The analogous expression for $BB'$ also holds.

The above formula immediately exposes a connection between the bipartite OTOC and the *operator entanglement* of the evolution $E_{\mathrm{op}}(U_t)$, as defined in Ref. [47] (see also [41] for the relevant definitions). The two quantities, remarkably, coincide exactly. This observation also allows one to express the *entangling power* [48] $e_P(U_t)$ as a function of the bipartite OTOC for the symmetric case $d_A = d_B$. The former quantifies the average entanglement produced by the evolution and has been established as an indicator of global chaos in few-body systems [49–52].

**Theorem 2:** Let $G_U$ denote the bipartite OTOC for the evolution $U$. Then, (i) $E_{\mathrm{op}}(U_t) = G_{U_t}$, and (ii) for a symmetric bipartition $d_A = d_B$,

$$e_P(U_t) = \frac{d}{(\sqrt{d}+1)^2}(G_{U_t} + G_{U_tS_{AB}} - G_{S_{AB}}). \quad (5)$$

For the finite-temperature case, Eq. (4) admits a straightforward generalization which we report in Ref. [41]. However, a direct connection with operator entanglement and entangling power may not be so simple.

*How informative is the average $G(t)$?*—Usually, one is interested in behavior of the OTOC for a typical choice of random unitary operators. Because of measure concentration [53], we prove that the two essentially coincide; i.e., the probability that a random instance deviates significantly from the mean is exponentially suppressed as the dimension of either of the subsystems $A$ and $B$ grows large.

**Proposition 3:** Let $P(\epsilon)$ be the probability that a random instance of $C_{V_A,W_B}(t)$ deviates from its Haar average $G(t)$ more than $\epsilon$. Then,

$$P(\epsilon) \leq 2\exp\left(-\frac{\epsilon^2 d_{\max}}{64}\right), \quad (6)$$

where $d_{\max} = \max\{d_A, d_B\}$.

In the definition of the bipartite OTOC and to obtain the replica formula Eq. (4), we have so far considered averaging over the uniform (Haar) ensemble which continuously extends over the whole unitary group. Although natural from a mathematical viewpoint, this choice can turn out to be rather complicated on physical and numerical grounds [54]. Nonetheless, we show in Ref. [41] that Haar averaging can be replaced by any unitary ensemble that forms a 1-design [55–58] without altering $G(t)$. Such ensembles mimic the Haar randomness only up to the first moment, which is the depth of randomness that the OTOC can probe [22]. The latter assumption is thus much weaker than Haar randomness. For instance, consider the case of a spin-1/2 many-body system split into two parts, $A$ and $B$. Instead of averaging over Haar random unitaries $V_A$ and $W_B$, that typically do not factor, the 1-design (equivalent) picture prescribes to instead consider only fully factorized unitaries with support over $A$ and $B$, e.g., products of local Pauli matrices.

*Time averaging the bipartite OTOC.*—In finite-dimensional quantum systems, nontrivial quantum expectation values or quantities such as $C_{V,W}(t)$ do not converge to a limit for $t\to\infty$. Instead, after a long time they typically oscillate around an equilibrium value [59–64] which can be extracted by time averaging $\overline{X(t)} := \lim_{T\to\infty}(1/T)\int_0^T dtX(t)$. We now turn to examine this long-time behavior $\overline{G(t)}$ of the bipartite OTOC as a function of the Hamiltonian and the Hilbert space cut.

Let us begin with the case of a chaotic dynamics, which entails level repulsion statistics [15] and an "incommensurable" relation among the energy levels. As such, chaotic Hamiltonians satisfy (either exactly or to very good approximation) the no-resonance condition (NRC): The energy levels and energy gaps feature nondegeneracy. This has important implications for the long-time behavior of their bipartite OTOC, as we will see soon.

Let us spectrally decompose $H = \sum_k E_k|\phi_k\rangle\langle\phi_k|$ and use $\rho_k^{(\chi)} := \mathrm{Tr}_{\bar{\chi}}(|\phi_k\rangle\langle\phi_k|)$ to denote the reduced density operator over $\chi = A, B$ corresponding to the $k$th Hamiltonian eigenstate ($\bar{\chi}$ corresponds to the complement).

Below, $\langle X,Y\rangle := \mathrm{Tr}(X^\dagger Y)$ denotes the Hilbert-Schmidt inner product [65], which gives rise to the operator 2-norm $\|X\|_2 := \sqrt{\langle X,X\rangle}$.

**Proposition 4:** Consider a Hamiltonian satisfying the NRC. Then

$$\overline{G(t)}^{\mathrm{NRC}} = 1 - \frac{1}{d^2}\sum_{\chi\in\{A,B\}}\left(\|R^{(\chi)}\|_2^2 - \frac{1}{2}\|R_D^{(\chi)}\|_2^2\right), \quad (7)$$

where $R^{(\chi)}$ is the Gram matrix of the reduced Hamiltonian eigenstates $\{\rho_k^{(\chi)}\}_{k=1}^d$, i.e.,

$$R_{kl}^{(\chi)} := \langle \rho_k^{(\chi)}, \rho_l^{(\chi)}\rangle, \quad (8)$$

while $(R_D^{(\chi)})_{kl} := R_{kl}^{(\chi)}\delta_{kl}$.

Let us first point out some basic, yet important properties of the above formula. The matrix $R^{(\chi)}$ is real and symmetric, while $R_D^{(\chi)}$ is positive semidefinite and diagonal. Moreover, the completeness of the Hamiltonian eigenvectors imposes $\sum_k \rho_k^{(\chi)} = d_{\tilde{\chi}} I$; thus, the rescaled $\tilde{R}^{(\chi)} := R^{(\chi)}/d_{\tilde{\chi}}$ are doubly stochastic, i.e., $\sum_i \tilde{R}_{ij}^{(\chi)} = \sum_i \tilde{R}_{ji}^{(\chi)} = 1\ \forall\ j$. As $\tilde{R}^{(\chi)}$ is a (rescaled) Gram matrix, its eigenvalues are non-negative and upper bounded by 1, and at most $d_\chi^2$ of them are nonzero [65]. This last property follows from the fact that $\mathrm{Rank}\,\tilde{R}^{(\chi)} = \dim\mathrm{Span}\{\rho_k^{(\chi)}\}_k \le d_\chi^2$. Observe also that $\|R_D^{(A)}\|_2^2 = \|R_D^{(B)}\|_2^2$ as two states $\rho_k^{(A)}$ and $\rho_k^{(B)}$ always have the same spectrum (up to irrelevant zeroes).

*Bipartite OTOC and entanglement.*—Proposition 4 makes it possible to bridge the long-time behavior of the bipartite OTOC with the entanglement structure of the Hamiltonian eigenstates. Let us begin with the symmetric case where $d_A = d_B$ and all $|\phi_k\rangle$ are maximally entangled with respect to the $A$-$B$ Hilbert space cut. This limit uniquely determines the time average for the NRC case, regardless of the exact Hamiltonian eigenbasis. In general, however, knowledge of the entanglement is not enough to uniquely determine the equilibration value; the inner products $R_{kl}^{(\chi)}$ go beyond probing just the spectrum of the reduced states. A simple substitution in Eq. (7) gives for the maximally entangled case $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}} = (1-1/d)^2$. We will later show the upper bound $G(t) \le 1 - 1/d_{\min}^2$; therefore, the equilibrium value for the bipartite OTOC in this case is nearly maximal, as expected for highly entangled models (e.g., Refs. [66,67]).

How robust is this conclusion for chaotic Hamiltonians with a possibly asymmetric bipartition? Typical eigenstates of chaotic Hamiltonians, as also predicted by the eigenstate thermalization hypothesis [68–70], are believed to obey a volume law for the entanglement entropy. Moreover, their entanglement properties in the bulk resemble those of Haar random pure states [71–73]. We will now show that high entanglement for the Hamiltonian eigenstates necessarily implies that the deviation of the actual equilibration value from $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}}$ is small.

It is convenient for this purpose to quantify the amount of entanglement via the linear entropy [74,75] of the reduced state $E(|\psi_{AB}\rangle) := S_{\mathrm{lin}}(\mathrm{Tr}_\chi|\psi_{AB}\rangle\langle\psi_{AB}|)$, where $S_{\mathrm{lin}}(\rho) := 1 - \mathrm{Tr}(\rho^2)$. The latter will also emerge naturally later when we express the bipartite OTOC in terms of entropy production. Notice that $E \le 1 - 1/d_{\max} := E_{\max}$, which is achievable only for $d_A = d_B$.

**Proposition 5:** If $E_{\max} - E(|\phi_k\rangle) \le \epsilon$ holds for at least a fraction $\alpha$ of the Hamiltonian eigenstates, then $|\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}} - \overline{G(t)}^{\mathrm{NRC}}| \le \alpha J + (1-\alpha)K$, where

$$J := \frac{6\epsilon}{d_{\min}} + \frac{5\epsilon^2}{2} + 2\frac{\lambda^2-1}{d_{\max}^2}, \quad (9a)$$

$$K := \left(1 + \frac{2}{d_{\min}}\right)(1-\alpha) + \frac{2}{d} + 4(\epsilon + \sqrt{\epsilon}), \quad (9b)$$

and $\lambda = d_{\max}/d_{\min}$.

The above bound provides a sufficient condition such that the bipartite OTOC equilibrates around $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}}$. It is expressed in terms of the fraction $\alpha$ of the highly entangled eigenstates, their entanglement, and the asymmetry of the $A$-$B$ bipartition. Notice that the bound simplifies considerably for the case $\alpha = 1$ and $d_{\min} = d_{\max} = \sqrt{d}$, that is, $|\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}} - \overline{G(t)}^{\mathrm{NRC}}| \le \epsilon(6/\sqrt{d} + 5\epsilon/2)$, which should hold to a good approximation for Hamiltonians with high entanglement in the bulk of the energies. Applied to chaotic Hamiltonians, the bound of Proposition 5 indicates that the bipartite OTOC will equilibrate near $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}}$, with deviations up to $O(1/d_{\min}^2)$. Here, chaoticity concretely means that the Hamiltonian spectrum satisfies the NRC and that the entanglement of the typical eigenvectors in the bulk, which determine the equilibration value, resembles that of Haar random vectors [76,77]; i.e., $\mathrm{Tr}(\rho_\chi^2) \approx (d_A + d_B)/(d+1)$ and, thus, $\epsilon = O(1/d_{\min})$ and $\alpha \approx 1$. For a fixed ratio $\lambda$ and as $d$ grows, $\overline{G(t)}^{\mathrm{NRC}}$ hence converges to $\overline{G_{\mathrm{ME}}(t)}^{\mathrm{NRC}}$ for all chaotic systems. Since $G(t) \le 1 - 1/d_{\min}^2$, fluctuations around the time average are necessarily insignificant, justifying the term equilibration.

*Beyond chaotic Hamiltonians.*—We now relax the "strong" level repulsion, i.e., NRC, criterion and uncover how a hierarchy of constraints, each implying a different strength of chaos, is reflected in the equilibration value of the bipartite OTOC.

Integrable models, which possess a structured spectrum, are expected to violate the NRC. Nevertheless, notice that Eq. (7), although derived under the NRC, can still be evaluated for an (arbitrary) choice of orthonormal eigenvectors of the Hamiltonian. We will refer to the resulting value as the *NRC estimate* of the time average, and we will

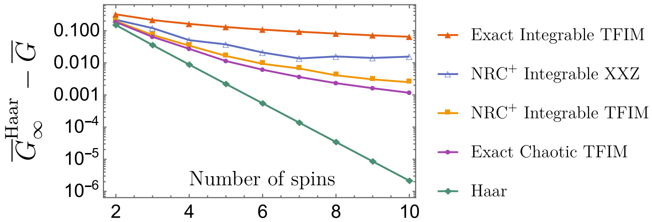


FIG. 1. Logarithmic plot of various $\bar{G}$ estimates, along with the exact time average, for fixed $d_A = 2$ as a function of the total number of spins $n$. $\bar{G}_\infty^{\rm Haar} = 3/4$ corresponds to the Haar estimate for $n \to \infty$. For the chaotic phase of the TFIM ($g = -1.05$, $h = 0.5$), the NRC constitutes a satisfactory, though imperfect, approximation. The chaotic and integrable phases ($h = 0$) can be clearly distinguished through the equilibration behavior of the bipartite OTOC. For the integrable $XXZ$ model (we set $J = 0.4$, $\Delta = 2.5$), the NRC$^+$ estimate coincides (up to numerical error) with the exact time average. Inequality (11) holds valid in all cases.

shortly show that this estimate always constitutes an upper bound of the actual equilibration value (and coincides with it for chaotic Hamiltonians). This is of both conceptual and practical importance, as evaluating the NRC estimate is considerably less intensive than calculating the exact value.

In fact, one can make a broader claim. For that, we first sketch three types of averaging processes over $G$, increasingly shifting away from the strong chaoticity limit. Each of them gives rise to a corresponding estimate for the (exact) equilibration time-average value $\overline{G(t)}$. (i) $\bar{G}^{\rm Haar}$: Averaging over (global) Haar random unitary operators $U \in U(d)$ in place of the time evolution. This averaging process is "beyond chaos," in the sense that it does not conserve energy, in contrast with time averaging over any Hamiltonian evolutions. Its estimate (only a function of the dimension) is given later in Eq. (10). (ii) $\overline{G(t)}^{\rm NRC}$: Time average, assuming the Hamiltonian has nondegenerate energy levels and nondegenerate energy gaps. The corresponding estimate is Eq. (7). (iii) $\overline{G(t)}^{\rm NRC^+}$: As before, but assuming the Hamiltonian may have a degenerate spectrum, but the energy gaps (between the different levels) are nondegenerate. Its estimate depends only on the eigenprojectors of the Hamiltonian and can be found in Ref. [41].

The value of the Haar average can be performed exactly, with the result

$$\bar{G}^{\rm Haar} = \frac{(d_A^2 - 1)(d_B^2 - 1)}{d^2 - 1}. \tag{10}$$

The following ordering holds.

**Theorem 6:** For any given Hamiltonian, the corresponding estimates are related with the exact time average $\overline{G(t)}$ as

$$\bar{G}^{\rm Haar} \geq \overline{G(t)}^{\rm NRC} \geq \overline{G(t)}^{\rm NRC^+} \geq \overline{G(t)}. \tag{11}$$

The above constitutes a proof that coincidences in the spectrum of a Hamiltonian up to the "gaps of gaps" (i.e., degeneracy over the energy levels and their gaps) always *reduce* the equilibration value of the bipartite OTOC.

Let us now numerically compare each of the estimates for two models of spin-1/2 chains with open-boundary conditions: (i) transverse-field Ising model (TFIM) with nearest-neighbor interaction, $H_I = -\sum_i(\sigma_i^z\sigma_{i+1}^z + g\sigma_i^x + h\sigma_i^z)$ and (ii) nearest-neighbor $XXZ$ interaction $H_{XXZ} = -J\sum_i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z)$. Recall that $H_I$ for $h = 0$ is integrable in terms of free fermions, while $H_{XXZ}$ by Bethe ansatz techniques. The two types of solutions yield qualitatively different spectra; free fermion solutions necessarily violate nondegeneracy of the gaps. This is reflected in the accuracy of the estimates (see Fig. 1). Although the NRC estimate provides essentially the exact equilibration values for the chaotic phase of the TFIM, it overestimates them in the integrable phase. On the other hand, NRC$^+$ is essentially exact for the integrable case of the $H_{XXZ}$ due to the lack of coincidences in the gaps. The results obtained here corroborate existing studies in the literature, where the (short- and) long-time behavior of the OTOC was studied for various many-body systems; see Refs. [78–80].

*Bipartite OTOC and subsystem evolution.*—We have so far focused on examining the behavior of the bipartite OTOC from the perspective of closed systems, i.e., over the full bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. One can instead express $G(t)$ as a function of the reduced time dynamics over only either $\mathcal{H}_A$ or $\mathcal{H}_B$ (and the corresponding duplicate), at the expense of giving up unitarity. This can be easily realized by formally performing a partial trace in Eq. (4), which immediately results in the following equivalent expression for the bipartite OTOC.

**Proposition 7:** Let $\Lambda_t^{(A)}(\rho_A) ≔ {\rm Tr}_B[U_t(\rho_A \otimes I_B/d_B)U_t^\dagger]$ be the reduced dynamics over $A$ when the environment $B$ is initialized in a maximally mixed state. Then,

$$G(t) = 1 - \frac{1}{d_A^2}{\rm Tr}[S_{AA'}(\Lambda_t^{(A)})^{\otimes 2}(S_{AA'})]. \tag{12}$$

The analogous expression for $BB'$ also holds.

The quantum map $\Lambda_t^{(\chi)}$ is unital; i.e., the maximally mixed state is a fixed point. As such, the transformation $\rho_\chi \mapsto \Lambda_t^{(\chi)}(\rho_\chi)$ results always in an output state whose spectrum is more disordered than the input one [81]. As a result, when $\rho_\chi$ is pure, the effect of the reduced time dynamics is to scramble and, hence, produce entropy. Let us now turn to examine this connection more closely.

*Bipartite OTOC as entropy production.*—We now show that the bipartite OTOC $G(t)$ is nothing but a measure of the average entropy production over pure states, with the latter quantified by linear entropy $S_{\rm lin}$.

**Theorem 8:**

$$G(t) = \frac{d_\chi + 1}{d_\chi} \int dU S_{\mathrm{lin}}[\Lambda_t^{(\chi)}(|\psi_U\rangle\langle\psi_U|)], \quad (13)$$

where $\chi = A$, $B$ and $|\psi_U\rangle \coloneqq U|\psi_0\rangle$ corresponds to Haar random pure states over $\mathcal{H}_\chi$.

In this manner, the bipartite OTOC can be fully characterized by linear entropy measurements over any of the $A$, $B$ subsystems. To obtain a satisfactory estimate of the mean in the rhs of Eq. (13), one does not, in practice, need to sample over the full Haar ensemble. An adequate estimate can be obtained with a rapidly decreasing number of necessary samples, as the dimension $d_\chi$ grows. More precisely, let $\tilde{P}(\epsilon)$ be the probability of the entropy $S_{\mathrm{lin}}[\Lambda_t^{(\chi)}(|\psi\rangle\langle\psi|)]$ deviating from $[d_\chi/(d_\chi+1)]G(t)$ more than $\epsilon$ for an instance of a random state. We show in Ref. [41] that

$$\tilde{P}(\epsilon) \leq \exp\left(-\frac{d_\chi \epsilon^2}{64}\right). \quad (14)$$

The linear entropy, although, *per se,* a nonlinear functional, can be turned into an ordinary expectation value if two (uncorrelated) copies of the quantum state are simultaneously available, $1 - S_{\mathrm{lin}} = \mathrm{Tr}(S\rho^{\otimes 2})$ for $S = S_{AA'}S_{BB'}$. This fact can be exploited to simplify its experimental accessibility [82–86]. More recently, protocols based on correlating measurements over random bases have also been developed to measure entropies [87–90], as well as OTOCs [91,92]. As a result, Theorem 8 and the typicality result Eq. (14) suggest that the bipartite OTOC is, in turn, tractable via linear entropy measurements. We provide more details in Ref. [41].

From Eq. (13), one can also infer the upper bound $G(t) \leq 1 - 1/d_\chi^2 \coloneqq G_{\max}^{(\chi)}$ announced earlier that follows from the range of the linear entropy function. The bound is thus achievable only when $\Lambda_t^{(\chi)}$ is equal to the completely depolarizing map $\mathcal{T}^{(\chi)}(\cdot) \coloneqq \mathrm{Tr}(\cdot)(I_\chi/d_\chi)$.

Finally, we remark that linear entropy occurs rather naturally in relation with the bipartite OTOC, as demonstrated by Theorem 2 (where it lies implicitly in the definition of operator entanglement and entangling power) and Theorem 8. This fact has its roots in the definition of the OTOC, which is intimately related to the Frobenius norm. Relevant relations for the linear entropy have been also reported in Ref. [31]. Starting from the inequality $S_{\mathrm{lin}}(\rho) \leq S(\rho)$ between the linear and von Neumann entropies ($S(\rho) \coloneqq -\mathrm{Tr}[\rho \log(\rho)]$), one can also obtain the corresponding estimates for the latter.

*Bipartite OTOC and information spreading.*—The bipartite OTOC measures the average ability of the reduced time evolution to erase information, as captured by the entropy production over a random pure state. This naturally raises the question as to whether $G(t)$ can also be understood as a measure of distance between $\Lambda_t^{(\chi)}$ and the depolarizing map $\mathcal{T}^{(\chi)}$, that is, in the space of quantum channels [i.e., completely positive and trace preserving (CPTP) maps [93] ].

A straightforward answer can be obtained by resorting to the duality between quantum states and operations [93]. Let $\rho_{\mathcal{E}} \coloneqq \mathcal{E} \otimes \mathcal{I}(|\phi^+\rangle\langle\phi^+|)$ denote the (Choi) state corresponding to the CPTP map $\mathcal{E}$, where $|\phi^+\rangle \coloneqq d^{-1/2}\sum_{i=1}^{d}|ii\rangle$ is a maximally entangled state.

**Proposition 9:** The bipartite OTOC is a measure of the distance between the reduced time evolution and the depolarizing map:

$$G(t) = G_{\max}^{(\chi)} - \|\rho_{\Lambda_t^{(\chi)}} - \rho_{\mathcal{T}^{(\chi)}}\|_2^2. \quad (15)$$

As an application, the proposition above can be utilized to bound the distance $\|\Lambda_t^{(\chi)} - \mathcal{T}^{(\chi)}\|_\diamond$ given by the diamond norm [94,95]; the latter is a well-established measure of distance between quantum channels, since it admits an operational interpretation in terms of discrimination on the level of quantum processes [96]. Bounding the difference in terms of the quantum processes also constrains the distinguishability in terms of states: $\|\mathcal{E}_1(\rho) - \mathcal{E}_2(\rho)\|_1 \leq \|\mathcal{E}_1 - \mathcal{E}_2\|_\diamond$ for all states and quantum processes. The distinguishability of the two operations satisfies $\|\Lambda_t^{(\chi)} - \mathcal{T}^{(\chi)}\|_\diamond \leq d_\chi^{3/2}\sqrt{G_{\max}^{(\chi)} - G(t)}$ (see [41]); therefore, if $G_{\max}^{(\chi)} - G(t)$ decays faster than $d_\chi^{-3}$, then asymptotically the two channels are essentially indistinguishable.

*Summary.*—We showed that the bipartite OTOC is amenable to exact analytical treatment and, quite remarkably, is equal to the operator entanglement of the dynamics. This identity allows one to establish a rigorous quantitative connection between the OTOC and the notion of entangling power, a well-established quantifier of few-body chaos. This may provide insights into recent work involving "dual unitaries" and many-body chaos [97–100]; the latter maximize operator entanglement [100,101]. We then turned to late-time averages of the bipartite OTOC and provided a hierarchy of estimates for systems that violate the conditions of a "generic spectrum." Finally, we unraveled the operational significance of the OTOC by establishing intimate connections with entropy production and information scrambling at the level of quantum channels. Possible future directions include applying further these theoretical tools to concrete many-body systems and uncovering relations with thermalization, localization, and other many-body phenomena.

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[1] D. N. Page, Average Entropy of a Subsystem, Phys. Rev. Lett. **71**, 1291 (1993).

[2] P. Hayden and J. Preskill, Black holes as mirrors: Quantum information in random subsystems, J. High Energy Phys. 09 (2007) 120.

[3] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, Chaos in quantum channels, J. High Energy Phys. 02 (2016) 004.

[4] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Operator Hydrodynamics, OTOCs, and Entanglement Growth in Systems without Conservation Laws, Phys. Rev. X **8**, 021013 (2018).

[5] S. Moudgalya, T. Devakul, C. W. von Keyserlingk, and S. L. Sondhi, Operator spreading in quantum maps, Phys. Rev. B **99**, 094312 (2019).

[6] E. H. Lieb and D. W. Robinson, The finite group velocity of quantum spin systems, in *Statistical Mechanics* (Springer, New York, 1972), pp. 425–431.

[7] M. B. Hastings, Lieb-Schultz-Mattis in higher dimensions, Phys. Rev. B **69**, 104431 (2004).

[8] D. A. Roberts and B. Swingle, Lieb-Robinson Bound and the Butterfly Effect in Quantum Field Theories, Phys. Rev. Lett. **117**, 091602 (2016).

[9] N. Lashkari, D. Stanford, M. Hastings, T. Osborne, and P. Hayden, Towards the fast scrambling conjecture, J. High Energy Phys. 04 (2013) 022.

[10] A. Larkin and Y. N. Ovchinnikov, Quasiclassical method in the theory of superconductivity, Sov. Phys. JETP **28**, 1200 (1969).

[11] A. Kitaev, A simple model of quantum holography, in *KITP Program: Entanglement in Strongly-Correlated Quantum Matter* (2015), http://online.kitp.ucsb.edu/online/entangled15/kitaev/

[12] S. Fishman, D. R. Grempel, and R. E. Prange, Chaos, Quantum Recurrences, and Anderson Localization, Phys. Rev. Lett. **49**, 509 (1982).

[13] S. Adachi, M. Toda, and K. Ikeda, Quantum-Classical Correspondence in Many-Dimensional Quantum Chaos, Phys. Rev. Lett. **61**, 659 (1988).

[14] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).

[15] F. Haake, *Quantum Signatures of Chaos* (Springer, New York, 2013).

[16] J. Maldacena, S. H. Shenker, and D. Stanford, A bound on chaos, J. High Energy Phys. 08 (2016) 106.

[17] D. A. Roberts and D. Stanford, Diagnosing Chaos Using Four-Point Functions in Two-Dimensional Conformal Field Theory, Phys. Rev. Lett. **115**, 131603 (2015).

[18] J. Polchinski and V. Rosenhaus, The spectrum in the Sachdev-Ye-Kitaev model, J. High Energy Phys. 04 (2016) 001.

[19] M. Mezei and D. Stanford, On entanglement spreading in chaotic systems, J. High Energy Phys. 05 (2017) 065.

[20] Y. Huang, Y.-L. Zhang, and X. Chen, Out-of-time-ordered correlators in many-body localized systems, Ann. Phys. (Amsterdam) **529**, 1600318 (2017).

[21] Y.-L. Zhang, Y. Huang, and X. Chen, Information scrambling in chaotic systems with dissipation, Phys. Rev. B **99**, 014303 (2019).

[22] D. A. Roberts and B. Yoshida, Chaos and complexity by design, J. High Energy Phys. 04 (2017) 121.

[23] R. Prakash and A. Lakshminarayan, Scrambling in strongly chaotic weakly coupled bipartite systems: Universality beyond the Ehrenfest timescale, Phys. Rev. B **101**, 121108(R) (2020).

[24] S. Pappalardi, A. Russomanno, B. Žunkovič, F. Iemini, A. Silva, and R. Fazio, Scrambling and entanglement spreading in long-range spin chains, Phys. Rev. B **98**, 134303 (2018).

[25] Q. Hummel, B. Geiger, J. D. Urbina, and K. Richter, Reversible Quantum Information Spreading in Many-Body Systems Near Criticality, Phys. Rev. Lett. **123**, 160401 (2019).

[26] D. J. Luitz and Y. Bar Lev, Information propagation in isolated quantum systems, Phys. Rev. B **96**, 020406(R) (2017).

[27] S. Pilatowsky-Cameo, J. Chávez-Carlos, M. A. Bastarrachea-Magnani, P. Stránský, S. Lerma-Hernández, L. F. Santos, and J. G. Hirsch, Positive quantum Lyapunov exponents in experimental systems with a regular classical limit, Phys. Rev. E **101**, 010202(R) (2020).

[28] T. Xu, T. Scaffidi, and X. Cao, Does Scrambling Equal Chaos? Phys. Rev. Lett. **124**, 140602 (2020).

[29] K. Hashimoto, K.-B. Huh, K.-Y. Kim, and R. Watanabe, Exponential growth of out-of-time-order correlator without chaos: Inverted harmonic oscillator, J. High Energy Phys. 11 (2020) 068.

[30] J. Cotler, N. Hunter-Jones, J. Liu, and B. Yoshida, Chaos, complexity, and random matrices, J. High Energy Phys. 11 (2017) 048.

[31] R. Fan, P. Zhang, H. Shen, and H. Zhai, Out-of-time-order correlation for many-body localization, Sci. Bull. **62**, 707 (2017).

[32] R. de Mello Koch, J.-H. Huang, C.-T. Ma, and H. J. Van Zyl, Spectral form factor as an OTOC averaged over the Heisenberg group, Phys. Lett. B **795**, 183 (2019).

[33] C.-T. Ma, Early-time and late-time quantum chaos, Int. J. Mod. Phys. A **35**, 2050082 (2020).

[34] A. Touil and S. Deffner, Quantum scrambling and the growth of mutual information, Quantum Sci. Technol. **5**, 035005 (2020).

[35] B. Yan, L. Cincio, and W. H. Zurek, Information Scrambling and Loschmidt Echo, Phys. Rev. Lett. **124**, 160603 (2020).

[36] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).

[37] A. Peres, Stability of quantum motion in chaotic and regular systems, Phys. Rev. A **30**, 1610 (1984).
[38] R. A. Jalabert and H. M. Pastawski, Environment-Independent Decoherence Rate in Classically Chaotic Systems, Phys. Rev. Lett. **86**, 2490 (2001).
[39] T. Gorin, T. Prosen, T. H. Seligman, and M. Žnidarič, Dynamics of Loschmidt echoes and fidelity decay, Phys. Rep. **435**, 33 (2006).
[40] A. Goussev, R. A. Jalabert, H. M. Pastawski, and D. A. Wisniacki, Loschmidt echo, Scholarpedia **7**, 11687 (2012).
[41] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.030601 for additional information and the proofs of the claims appearing in the Letter, which includes Refs. [42–46].
[42] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An Introduction to Random Matrices* (Cambridge University Press, Cambridge, England, 2010).
[43] J. Eisert, M. Cramer, and M. B. Plenio, Colloquium: Area laws for the entanglement entropy, Rev. Mod. Phys. **82**, 277 (2010).
[44] R. Goodman and N. R. Wallach, *Symmetry, Representations, and Invariants* (Springer, New York, 2009).
[45] J. Watrous, Is there any connection between the diamond norm and the distance of the associated states? Theoretical Computer Science Stack Exchange, https://cstheory.stackexchange.com/q/4920, 2011.
[46] Z. Webb, The Clifford group forms a unitary 3-design, Quantum Inf. Comput. **16**, 1379 (2016).
[47] P. Zanardi, Entanglement of quantum evolutions, Phys. Rev. A **63**, 040304(R) (2001); X. Wang and P. Zanardi, Quantum entanglement of unitary operators on bipartite systems, Phys. Rev. A **66**, 044303 (2002).
[48] P. Zanardi, C. Zalka, and L. Faoro, Entangling power of quantum evolutions, Phys. Rev. A **62**, 030301(R) (2000).
[49] X. Wang, S. Ghose, B. C. Sanders, and B. Hu, Entanglement as a signature of quantum chaos, Phys. Rev. E **70**, 016217 (2004).
[50] A. Lakshminarayan, Entangling power of quantized chaotic systems, Phys. Rev. E **64**, 036207 (2001).
[51] A. J. Scott and C. M. Caves, Entangling power of the quantum Baker's map, J. Phys. A **36**, 9553 (2003).
[52] R. Pal and A. Lakshminarayan, Entangling power of time-evolution operators in integrable and nonintegrable many-body systems, Phys. Rev. B **98**, 174304 (2018).
[53] M. Ledoux, *The Concentration of Measure Phenomenon*, Mathematical Surveys and Monographs Vol. 89 (American Mathematical Society, Providence, 2001).
[54] J. Emerson, Y. S. Weinstein, M. Saraceno, S. Lloyd, and D. G. Cory, Pseudo-random unitary operators for quantum information processing, Science **302**, 2098 (2003).
[55] D. P. DiVincenzo, D. W. Leung, and B. M. Terhal, Quantum data hiding, IEEE Trans. Inf. Theory **48**, 580 (2002).
[56] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, Symmetric informationally complete quantum measurements, J. Math. Phys. (N.Y.) **45**, 2171 (2004).
[57] A. J. Scott, Tight informationally complete quantum measurements, J. Phys. A **39**, 13507 (2006).
[58] D. Gross, K. Audenaert, and J. Eisert, Evenly distributed unitaries: On the structure of unitary designs, J. Math. Phys. (N.Y.) **48**, 052104 (2007).
[59] P. Reimann, Foundation of Statistical Mechanics Under Experimentally Realistic Conditions, Phys. Rev. Lett. **101**, 190403 (2008).
[60] N. Linden, S. Popescu, A. J. Short, and A. Winter, Quantum mechanical evolution towards thermal equilibrium, Phys. Rev. E **79**, 061103 (2009).
[61] L. Campos Venuti, N. T. Jacobson, S. Santra, and P. Zanardi, Exact Infinite-Time Statistics of the Loschmidt Echo for a Quantum Quench, Phys. Rev. Lett. **107**, 010403 (2011).
[62] A. Nahum, S. Vijay, and J. Haah, Operator Spreading in Random Unitary Circuits, Phys. Rev. X **8**, 021014 (2018).
[63] C. B. Dağ, K. Sun, and L.-M. Duan, Detection of Quantum Phases Via Out-of-Time-Order Correlators, Phys. Rev. Lett. **123**, 140602 (2019).
[64] Á. M. Alhambra, J. Riddell, and L. P. García-Pintos, Time Evolution of Correlation Functions in Quantum Many-Body Systems, Phys. Rev. Lett. **124**, 110605 (2020).
[65] R. Bhatia, *Matrix Analysis*, Vol. 169 (Springer-Verlag, Berlin, 2013).
[66] Y. Huang, F. G. S. L. Brandao, and Y.-L. Zhang, Finite-Size Scaling of Out-of-Time-Ordered Correlators at Late Times, Phys. Rev. Lett. **123**, 010601 (2019).
[67] A. W. Harrow, L. Kong, Z.-W. Liu, S. Mehraban, and P. W. Shor, A separation of out-of-time-ordered correlator and entanglement, arXiv:1906.02219.
[68] J. M. Deutsch, Quantum statistical mechanics in a closed system, Phys. Rev. A **43**, 2046 (1991).
[69] M. Srednicki, Chaos and quantum thermalization, Phys. Rev. E **50**, 888 (1994).
[70] M. Rigol, V. Dunjko, and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, Nature (London) **452**, 854 (2008).
[71] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics, Adv. Phys. **65**, 239 (2016).
[72] Y. Huang, Universal eigenstate entanglement of chaotic local Hamiltonians, Nucl. Phys. **B938**, 594 (2019).
[73] T.-C. Lu and T. Grover, Renyi entropy of chaotic eigenstates, Phys. Rev. E **99**, 032111 (2019).
[74] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[75] S. Bose and V. Vedral, Mixedness and teleportation, Phys. Rev. A **61**, 040101(R) (2000).
[76] E. Lubkin, Entropy of an $n$-system from its correlation with a $k$-reservoir, J. Math. Phys. (N.Y.) **19**, 1028 (1978).
[77] A. Hamma, S. Santra, and P. Zanardi, Quantum Entanglement in Random Physical States, Phys. Rev. Lett. **109**, 040502 (2012).
[78] E. M. Fortes, I. García-Mata, R. A. Jalabert, and D. A. Wisniacki, Gauging classical and quantum integrability through out-of-time-ordered correlators, Phys. Rev. E **100**, 042201 (2019).

[79] I. García-Mata, M. Saraceno, R. A. Jalabert, A. J. Roncaglia, and D. A. Wisniacki, Chaos Signatures in the Short and Long Time Behavior of the Out-of-Time Ordered Correlator, Phys. Rev. Lett. **121**, 210601 (2018).

[80] J. Rammensee, J. D. Urbina, and K. Richter, Many-Body Quantum Interference and the Saturation of Out-of-Time-Order Correlators, Phys. Rev. Lett. **121**, 124101 (2018).

[81] I. Bengtsson and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, England, 2017).

[82] A. K. Ekert, C. M. Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, Direct Estimations of Linear and Nonlinear Functionals of a Quantum State, Phys. Rev. Lett. **88**, 217901 (2002).

[83] F. A. Bovino, G. Castagnoli, A. Ekert, P. Horodecki, C. M. Alves, and A. V. Sergienko, Direct Measurement of Nonlinear Properties of Bipartite Quantum States, Phys. Rev. Lett. **95**, 240407 (2005).

[84] C. M. Alves and D. Jaksch, Multipartite Entanglement Detection in Bosons, Phys. Rev. Lett. **93**, 110501 (2004).

[85] A. J. Daley, H. Pichler, J. Schachenmayer, and P. Zoller, Measuring Entanglement Growth in Quench Dynamics of Bosons in an Optical Lattice, Phys. Rev. Lett. **109**, 020505 (2012).

[86] R. Islam, R. Ma, P. M. Preiss, M. E. Tai, A. Lukin, M. Rispoli, and M. Greiner, Measuring entanglement entropy in a quantum many-body system, Nature (London) **528**, 77 (2015).

[87] T. Brydges, A. Elben, P. Jurcevic, B. Vermersch, C. Maier, B. P. Lanyon, P. Zoller, R. Blatt, and C. F. Roos, Probing Rényi entanglement entropy via randomized measurements, Science **364**, 260 (2019).

[88] A. Elben, B. Vermersch, C. F. Roos, and P. Zoller, Statistical correlations between locally randomized measurements: A toolbox for probing entanglement in many-body quantum states, Phys. Rev. A **99**, 052323 (2019).

[89] H.-Y. Huang, R. Kueng, and J. Preskill, Predicting many properties of a quantum system from very few measurements, Nat. Phys. **16**, 1050 (2020).

[90] A. Elben, R. Kueng, H.-Y. R. Huang, R. van Bijnen, C. Kokail, M. Dalmonte, P. Calabrese, B. Kraus, J. Preskill, P. Zoller, and B. Vermersch, Mixed-State Entanglement from Local Randomized Measurements, Phys. Rev. Lett. **125**, 200501 (2020).

[91] B. Vermersch, A. Elben, L. M. Sieberer, N. Y. Yao, and P. Zoller, Probing Scrambling Using Statistical Correlations Between Randomized Measurements, Phys. Rev. X **9**, 021061 (2019).

[92] M. K. Joshi, A. Elben, B. Vermersch, T. Brydges, C. Maier, P. Zoller, R. Blatt, and C. F. Roos, Quantum Information Scrambling in a Trapped-Ion Quantum Simulator with Tunable Range Interactions, Phys. Rev. Lett. **124**, 240505 (2020).

[93] M. A. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[94] A. Y. Kitaev, Quantum computations: Algorithms and error correction, Russ. Math. Surv. **52**, 1191 (1997).

[95] A. Y. Kitaev, A. Shen, and M. N. Vyalyi, *Classical and Quantum Computation* (American Mathematical Society, Providence, 2002).

[96] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, Cambridge, England, 2013).

[97] M. Akila, D. Waltner, B. Gutkin, and T. Guhr, Particle-time duality in the kicked Ising spin chain, J. Phys. A **49**, 375101 (2016).

[98] B. Bertini, P. Kos, and T. Prosen, Exact Correlation Functions for Dual-Unitary Lattice Models in $1+1$ Dimensions, Phys. Rev. Lett. **123**, 210601 (2019).

[99] L. Piroli, B. Bertini, J. I. Cirac, and T. Prosen, Exact dynamics in dual-unitary quantum circuits, Phys. Rev. B **101**, 094304 (2020).

[100] B. Bertini, P. Kos, and T. Prosen, Operator entanglement in local quantum circuits I: Chaotic dual-unitary circuits, SciPost Phys. **8**, 67 (2020).

[101] S. A. Rather, S. Aravinda, and A. Lakshminarayan, Creating Ensembles of Dual Unitary and Maximally Entangling Quantum Evolutions, Phys. Rev. Lett. **125**, 070501 (2020).