Stochastic Discrete Time Crystals: Entropy Production and Subharmonic Synchronization

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Discrete time crystals are periodically driven systems that display spontaneous symmetry breaking of time translation invariance in the form of indefinite subharmonic oscillations. We introduce a thermodynamically consistent model for a discrete time crystal and analyze it using the framework of stochastic thermodynamics. In particular, we evaluate the rate of energy dissipation of this many-body system of interacting noisy subharmonic oscillators in contact with a heat bath. The mean-field model displays the phenomenon of subharmonic synchronization, which corresponds to collective subharmonic oscillations of the individual units. The 2D model does not display synchronization but it does show a time-crystalline phase, which is characterized by a power-law behavior of the number of coherent subharmonic oscillations with system size. This result demonstrates that the emergence of coherent oscillations is possible even in the absence of synchronization.

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Time crystals [1,2] are a phase of matter first proposed by Shapere and Wilczek [3,4]. They are closed equilibrium systems with a time-independent Hamiltonian that show oscillations in time, which corresponds to a spontaneous breaking of time translation symmetry. The name is chosen in analogy to crystals, which display spontaneous symmetry breaking of spatial translation symmetry due to the emergence of a periodic arrangement of atoms in space. Shortly after this proposal, general proofs that time crystals could not be realized in closed many-body quantum systems with short-range interactions were provided [5,6]. The debate about a possible realization of a time crystal is still active [7–13].

Compared to this initial proposal by Shapere and Wilczek, a different kind of breaking of time translation symmetry happens in discrete time crystals (DTCs) [14–18]. These are nonequilibrium quantum systems with a time-periodic Hamiltonian, for which breaking of time translation symmetry is manifested in the occurrence of subharmonic oscillations with a period longer than the period of the Hamiltonian. These discrete time crystals are not in contact with a heat bath; hence, they do not dissipate energy. They typically rely on disorder and localization to avoid a stationary state of infinite temperature, which would not support time crystalline order [15,16,19]. Interestingly, DTCs have been realized in experiments [20–25].

For such DTCs, coupling the system to an external reservoir can destroy the DTC phase [26]. Nevertheless, open systems in contact with an external reservoir allow for a broader range of mechanisms, which do not rely on disorder and localization, that do lead to a DTC phase [27–34]. In fact, the onset of subharmonic oscillations in dynamical systems under periodic driving has been known for quite some time [35]. However, the amount of energy dissipated by a DTC as an open system has not been evaluated yet.

In this Letter, we introduce a thermodynamically consistent model for a classical stochastic many-body DTC in contact with a heat bath. Our model falls within the theoretical framework of stochastic thermodynamics [36–38]. As one consequence, we can evaluate the rate of entropy production, which quantifies how much energy the system dissipates. We show that the average of this quantity and its fluctuations can be used to identify the transition to a DTC phase.

The mechanism that leads to subharmonic oscillations in our model is different from DTCs in open systems that have been proposed so far. We consider a many-body system for which each isolated unit displays a finite number of coherent subharmonic oscillations that fades away after some time due to noise [39]. By introducing interactions between these units, we show that for an interaction strength above a certain critical value the number of coherent subharmonic oscillations diverges in the thermodynamic limit, which is a signature of a DTC phase.

The mean-field version of our model displays a phenomenon that we call subharmonic synchronization. Standard synchronization is a fundamental phenomenon in physics, whereby coupled oscillators display collective oscillations [40]. For the subharmonic synchronization observed here, periodically driven oscillators display collective subharmonic oscillations. Recently a deterministic model that displays subharmonic synchronization has been proposed in Ref. [41]; our model corresponds to the first stochastic model with thermal noise that displays subharmonic synchronization.

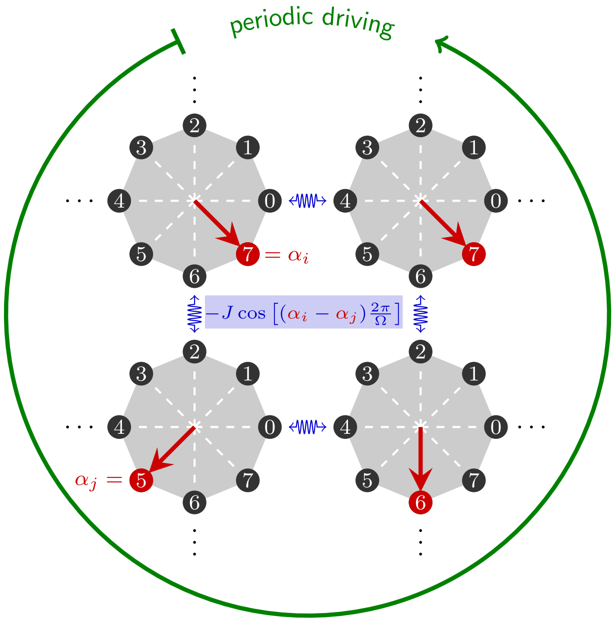

FIG. 1. Sketch of the 2D model. The clocks with eight states represent the units in our model. They are driven by an external periodic protocol that leads to subharmonic oscillations. Units interact with its nearest neighbors, which can lead to a DTC phase where the number of subharmonic coherent oscillations diverges in the thermodynamic limit.

A surprising result is obtained with the 2D version of the model. It does not show subharmonic synchronization, similar to related models for standard synchronization that do not display synchronization in 2D [42,43]. However, we show that the 2D model still exhibits a DTC phase, characterized by a number of coherent subharmonic oscillations that grows as a power law with system size. This result has broader implications for synchronization beyond DTCs, as it demonstrates that the emergence of coherent oscillations is possible even in the absence of synchronization.

Each unit of our model, which is illustrated in Fig. 1, is a clock with $\Omega \geq 3$ states labeled by $\alpha = 0, 1, \ldots, \Omega - 1$. For a single unit, the transition rate from state α to $\alpha + 1$ is

$$w_\alpha^+(t) = k e^{E_\alpha(t) - B_\alpha(t)}, \tag{1}$$

while the transition rate from state α to $\alpha - 1$ is

$$w_\alpha^-(t) = k e^{E_\alpha(t) - B_{\alpha-1}(t)}, \tag{2}$$

where the parameter k sets the timescale. The time-periodic energy of state α is $E_\alpha(t)$, the time-periodic energy barrier between states α and $(\alpha + 1)$ modΩ is $B_\alpha(t)$. Boltzmann's constant k_B and temperature T are set to $k_B = T = 1$ throughout. For $t \in [0, \tau]$, where τ is the period, the energy and energy barriers are given by

$$E_\alpha(t) = [\ln(c)/\Omega][(\alpha + \lfloor \Omega t/\tau \rfloor - 1)\,\mathrm{mod}\Omega] \tag{3}$$

and

$$B_\alpha(t) = [\ln(c)/\Omega][\Omega - 1 + (\alpha + \lfloor \Omega t/\tau \rfloor)\mathrm{mod}\Omega], \tag{4}$$

where c is a positive constant.

The model for a single unit has been analyzed in Ref. [39]. In a particular limit, where energy differences and energy barriers diverge, a single unit works as a clock that displays indefinite subharmonic oscillations with a period $(\Omega - 1)\tau$. For finite values of the transition rates, thermal fluctuations destroy the coherence of the subharmonic oscillations; i.e., two-point correlation functions in time display subharmonic oscillations that decay exponentially [39].

In the present model individual units that would display a finite number of coherent oscillations if they were alone interact. The time-independent interaction energy of this many-body system with N such units is

$$V_{\vec{\alpha}} = -\mathcal{J} \sum_{i=1}^{N} \sum_{j} \cos[2\pi(\alpha_i - \alpha_j)/\Omega]/2, \tag{5}$$

where the vector $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ represents the state of the many-body system. For the mean-field variant, the sum in j is over all units from $j = 1$ to $j = N$ and $\mathcal{J} = J/N$. For the 2D variant, the sum in j is over four nearest neighbors, $\mathcal{J} = J$, and we consider periodic boundary conditions. The full time-periodic Hamiltonian of the model is

$$\sum_{i=1}^{N} E_{\alpha_i}(t) + V_{\vec{\alpha}}, \tag{6}$$

where $E_{\alpha_i}(t)$ is given by Eq. (3).

We have performed continuous-time Monte Carlo simulations of this model using the Gillespie algorithm [44]. For a detailed definition of the model with the particular choice of transition rates we used, see Ref. [45]. The parameters of the model are set to $k = 40$, $c = 10^4$ and the period is $\tau = 1$. The number of states of each unit is $\Omega = 8$. The basic phenomenon we investigate is whether, for an interaction strength J above a certain critical value, subharmonic oscillations become coherent in the thermodynamic limit, which is a signature of the onset of a DTC phase. Changing the parameters that have been fixed leads to results that are quantitatively different but have the same physical features discussed below.

The following observables characterize this system. First, the order parameter for the synchronization of the different clocks reads [40]

$$r(t) \equiv N^{-1} \left| \sum_{i=1}^{N} e^{2\pi i \alpha_i(t)/\Omega} \right|, \tag{7}$$

where $\alpha_i(t)$ is the state of unit i at time t. Since we are interested in subharmonic oscillations, we consider the

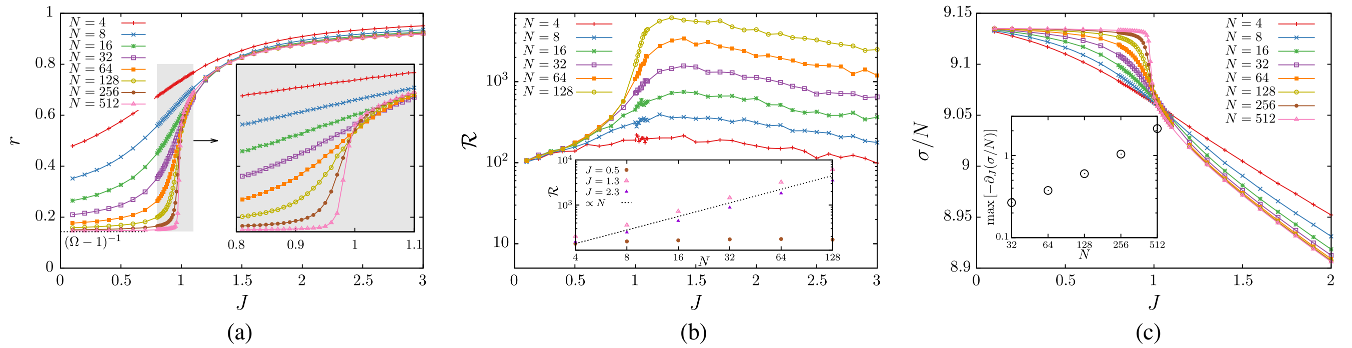

FIG. 2. Observables for the mean-field variant as functions of the interaction strength J. (a) Order parameter r indicates the onset of subharmonic synchronization above the critical point. (b) Number of coherent oscillations $\mathcal{R}$ diverge in thermodynamic limit below the critical point. (c) Rate of entropy production per unit σ/N; its first derivative diverges at criticality in the thermodynamic limit, as shown in the inset.

stroboscopic time n, with $r_n \equiv r(n\tau)$. The quantity r_n as a function of n reaches a stationary value denoted by r. If the clocks do not synchronize, then r approaches $(\Omega - 1)^{-1}$ rather than going to 0, which is related to the fact that only $(\Omega - 1)$ states of the Ω states are part of the subharmonic oscillations [45]. If the subharmonic clocks synchronize, then $r > (\Omega - 1)^{-1}$.

Second, the number of coherent oscillations is quantified by the correlation function $C(t)$, which is the density of clocks in state $\alpha_i = 0$ at time t given that at time 0 all clocks $i = 1, 2, \ldots, N$ are in state $\alpha_i = 0$. The stroboscopic quantity $C_n = C(n\tau)$ has oscillations that decay exponentially in n. The period of oscillations and the decay time are written n_{osc} and n_{dec}, respectively. The number of coherent subharmonic oscillations is defined as

$$\mathcal{R} \equiv 2\pi n_{\mathrm{dec}}/n_{\mathrm{osc}}. \tag{8}$$

The factor 2π in this definition is related to the fact that $\mathcal{R}$ can be defined as the ratio of the imaginary and real parts of an eigenvalue of the fundamental matrix [39].

Third, the amount of energy dissipation is quantified by the thermodynamic rate of entropy production [36]. This observable quantifies the energetic cost of the DTC; it is the rate of work exerted on the system due to the periodic driving. For a stochastic trajectory with total time $\mathcal{T}$ exhibiting M jumps, $\vec{\alpha}^{(1)} \to \vec{\alpha}^{(2)} \to \cdots \to \vec{\alpha}^{(M)}$, the entropy change is

$$X_\sigma \equiv \sum_{m=1}^{M-1} \ln\left(w_{\vec{\alpha}^{(m)}\to\vec{\alpha}^{(m+1)}}/w_{\vec{\alpha}^{(m+1)}\to\vec{\alpha}^{(m)}}\right), \tag{9}$$

where $w_{\vec{\alpha}^{(m)}\to\vec{\alpha}^{(m+1)}}$ is the transition rate from state $\vec{\alpha}^{(m)}$ to state $\vec{\alpha}^{(m+1)}$. The average rate of entropy production is $\sigma \equiv \langle X_\sigma \rangle/\mathcal{T}$, where the brackets denote an average over stochastic trajectories. Fluctuations of the entropy production are quantified by the Fano factor, $F_\sigma \equiv (\langle X_\sigma^2 \rangle - \langle X_\sigma \rangle^2)/\langle X_\sigma \rangle$. Both quantities, σ and F_σ, are formally defined in the limit $\mathcal{T} \to \infty$.

We first analyze the mean-field model. The results for the order parameter r are shown in Fig. 2(a). They indicate that $r > (\Omega - 1)^{-1}$ for $J > J_c \simeq 0.975$ in the thermodynamic limit. This mean-field model displays the novel phenomenon of subharmonic synchronization. It differs from related models for standard synchronization without periodic drive [42,43,46]. In these models, each unit is a biased random walk on a circle with Ω states. The bias is generated by a

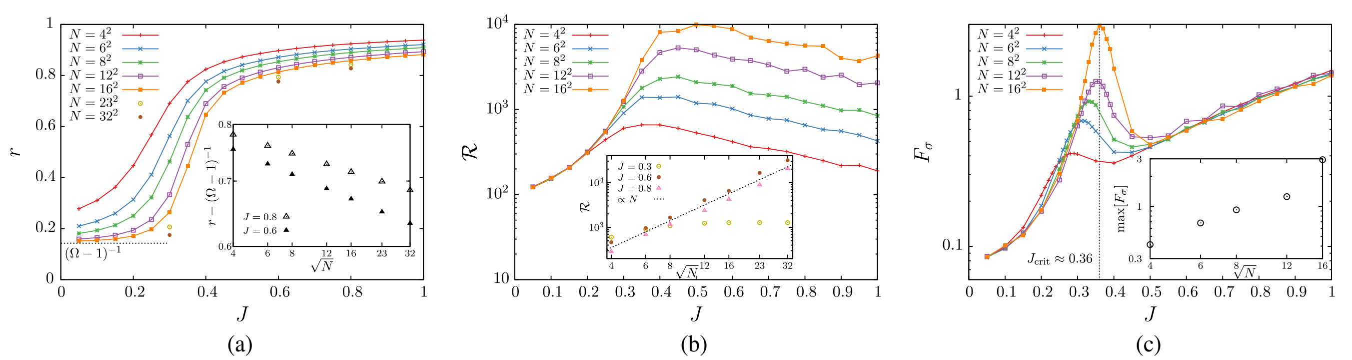

FIG. 3. Observables for the 2D variant as functions of the interaction strength J. (a) Order parameter r, which goes to zero in the thermodynamic limit for all J. (b) Number of coherent oscillations $\mathcal{R}$ grows as power law. (c) Fano factor F_σ increases with system size.

fixed thermodynamic force such as the free energy of adenosine triphosphate hydrolysis. In contrast, in our model each unit is a periodically driven clock that displays noisy subharmonic oscillations.

The scaling of the number of coherent oscillations $\mathcal{R}$ with the system size N is shown in Fig. 2(b). Below the critical point $\mathcal{R}$ saturates. Hence, subharmonic oscillations do not last indefinitely in the thermodynamic limit but fade away after some transient. Above the critical point, $\mathcal{R}$ diverges with system size as a power law, with an exponent compatible with 1. For $J \geq J_c$, subharmonic oscillations thus become indefinite in the limit $N \to \infty$, which corresponds to a DTC phase. In Fig. 2(b) it is also possible to observe that $\mathcal{R}$ becomes smaller for large values of the interaction strength J. This result can be explained with the observation that in a limit of J infinitely larger than the time-dependent part of the energy, we would have a standard equilibrium model with no subharmonic oscillations.

Since our model for a DTC is thermodynamically consistent, we can evaluate how much energy this DTC dissipates using Eq. (9). In Fig. 2(c), we show the rate of entropy production per unit σ/N as a function of the interaction strength J for different values of N. The maximum of the first derivative of σ/N with respect to J as a function of N seems to follow a power law, which indicates that this derivative diverges in the limit $N \to \infty$. With the values of N that were accessible with our simulations, we were not able to determine the exponent reliably. Our result indicates that in the thermodynamic limit there is a discontinuity in the rate of entropy production per unit, σ/N.

For the 2D model we obtain results that are qualitatively different from the results for the mean-field model. As shown in Fig. 3(a), even for regions where the order parameter r seems to be larger than $(\Omega - 1)^{-1}$ in a finite system, $r - (\Omega - 1)^{-1}$ decays to zero as a power law with system size. Hence, there is no subharmonic synchronization in the 2D model, with $r \to (\Omega - 1)^{-1}$ for any value of J in the thermodynamic limit.

Interestingly, even though the 2D model does not display subharmonic synchronization, it does still display a DTC phase. For large enough J, the number of coherent oscillations grows as a power law with system size, as shown in Fig. 3(b). Similar to the mean-field version, the exponent is approximately 1, independent of the value of J.

For the rate of entropy production in the 2D model, we cannot identify any nonanalytical behavior of σ/N or its first derivative at criticality within our numerics [45]. Nonanalytical behavior of higher order derivatives cannot be ruled out. However, we can observe a signature of a phase transition in the fluctuations of the entropy production, as quantified by the Fano factor F_σ. As shown in Fig. 3(c), the maximum of F_σ grows with system size, which indicates that the Fano factor might diverge at a possible critical point in the thermodynamic limit. We could determine neither the exponent associated with this divergence reliably nor the critical point with our numerics. Additionally, a divergence of this Fano factor at criticality has been observed in models for biochemical oscillations [47].

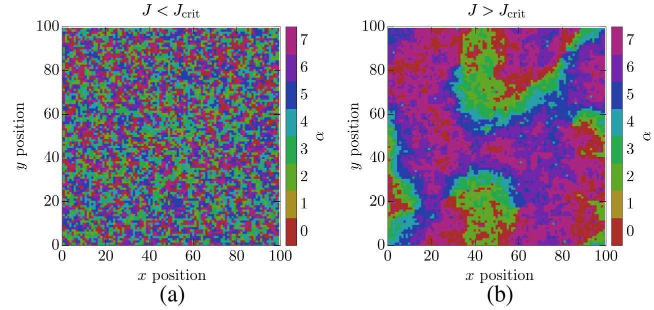

FIG. 4. Long-range order in the 2D model. The axes represent the spatial coordinates of the 2D lattice. Snapshot (a) corresponds to the model below the critical point and snapshot (b) corresponds to the DTC phase above the critical point. The colors represent the state of a unit $\alpha_i = 0, 1, \ldots, 7$. The parameters are $N = 100^2$, $J = 0.2$ for (a) and $J = 0.8$ for (b). These pictures were taken for the stroboscopic time $n = 1000$.

Further evidence of a possible phase transition in the 2D model is shown in Fig. 4, which contains snapshots of the state of the system. This picture shows that there is long-range order above the critical point with the formation of islands with a certain orientation. This result is similar to the observation of a "Kosterlitz-Thouless-type" phenomenon in a 2D model of interacting noisy oscillators (without periodic driving) [42,43]. Hence, our 2D model for a DTC is a many-body system with spatial dimensions and short-range interactions that displays long-range order, which are crucial characteristics of a DTC (see, for instance, Ref. [27]).

In summary, we have introduced a paradigmatic model for a stochastic many-body system in contact with a heat bath that displays a DTC phase. The phenomenon of subharmonic synchronization, whereby periodically driven oscillators show synchronized subharmonic oscillations, was found within the mean-field version.

For the 2D variant, there is no synchronization. However, there is a rich phenomenology with a DTC phase, which is characterized by a power-law behavior of the number of coherent subharmonic oscillations. The emergence of indefinite coherent oscillations might be possible even in the absence of synchronization. This result goes beyond DTCs and is potentially relevant for synchronization and biochemical oscillations. In this context, future work should explore the role of disorder in the period of the drive and in the energy landscape of the individual units.

As a first step toward the thermodynamics of DTCs, we have calculated the entropy production and shown how it and the associated Fano factor can be used as indicators for a transition to a DTC phase. While the original idea of time crystals can be linked with perpetual motion, such a phenomenon can be ruled out in an open system with a consistent second law such as the one analyzed here.

Our results offer *inter alia* the possibility to build a concrete theoretical model for a subharmonic heat engine that breaks time translation symmetry. It would be interesting to investigate whether power and efficiency of such a heat engine are bounded by the relations that have been found for cyclic and steady state heat engines [48–51].

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