Thermodynamic Uncertainty Relation for General Open Quantum Systems

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We derive a thermodynamic uncertainty relation for general open quantum dynamics, described by a joint unitary evolution on a composite system comprising a system and an environment. By measuring the environmental state after the system-environment interaction, we bound the counting observables in the environment by the survival activity, which reduces to the dynamical activity in classical Markov processes. Remarkably, the relation derived herein holds for general open quantum systems with any counting observable and any initial state. Therefore, our relation is satisfied for classical Markov processes with arbitrary time-dependent transition rates and initial states. We apply our relation to continuous measurement and the quantum walk to find that the quantum nature of the system can enhance the precision. Moreover, we can make the lower bound arbitrarily small by employing appropriate continuous measurement.

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Introduction.—Higher precision demands more resources. Although this fact is widely accepted, it has only recently been theoretically proved. The thermodynamic uncertainty relation (TUR) [1–15] (see [16] for a review) serves as a theoretical basis for this notion, and it states that current fluctuations, quantified by a coefficient of variation, are bounded from below by thermodynamic costs, such as entropy production and dynamical activity. It predicts the fundamental limit of biomolecular processes and thermodynamic engines and it can be applied to infer the entropy production of thermodynamic systems in the absence of detailed knowledge about them [17–21].

Much progress has been made on the TUR for classical stochastic thermodynamics. Quantum analogs of the TUR have been recently carried out, but they are still at an early stage. Many existing studies on quantum TURs [22–29] are concerned with rather limited situations. In the first place, although an observable of interest in the TUR of classical stochastic thermodynamics is well defined, there is no consensus regarding specific observables that should be bounded in the TURs of quantum systems. In the present Letter, we obtain a TUR for general open quantum systems, which can be described as a joint unitary evolution of a composite system comprising a principal system and an environment. Using the composite representation, we formulate a TUR in open quantum systems as a bound for the environmental measurement by using the quantum estimation theory [30–33]. The obtained relation exhibits remarkable generality. It holds for general open quantum dynamics with any counting observable and any initial density operator. Moreover, our bound applies to any classical time-dependent Markov process and counting observable. Our TUR bounds the fluctuations in the counting observables by a quantity referred to as a survival activity, which reduces to the dynamical activity [34] of classical Markov processes in a particular limit. We apply our TUR to the continuous measurement and the quantum walk and find that the system's quantum nature can enhance the precision of the observables and that an arbitrary small lower bound of the fluctuations can be achieved by employing appropriate continuous measurement.

Results.—Let us consider a system S and an environment E. The environment comprises an orthonormal basis $\{|m\rangle\}_{m=0}^{M}$. We assume that the initial states of S and E are $|\psi\rangle$ and $|0\rangle$, respectively. Because S and E interact from $t=0$ to $t=T$ via a unitary operator U acting on $S+E$, the state of $S+E$ at $t=T$ is $|\Psi(T)\rangle = U|\psi\rangle \otimes |0\rangle$ [Fig. 1(a)]. Typically, in open quantum systems, the primary object of interest is the state of the *principal system* S after the interaction. In contrast, we here focus on the state of the *environment* E after the interaction. For example, in continuous monitoring of photon emissions in open quantum systems, photons emitted into the environment during $[0,T]$ can be equivalently obtained by measuring the environment at final time $t=T$. Therefore, the environment includes all information about the measurement records of the emitted photon.

Suppose that a measurement is performed on the environment at $t=T$ by an Hermitian operator $\mathcal{G}$ [Fig. 1(a)]. Here, $\mathcal{G}$ admits the eigendecomposition $\mathcal{G}=\sum_m g(m)|\phi_m\rangle\langle\phi_m|$, where $|\phi_m\rangle$ and $g(m)$ are the eigenvector and eigenvalue of $\mathcal{G}$, respectively. Using $|\phi_m\rangle$, the state of $S+E$ at $t=T$ can be expressed as [35]

$$|\Psi(T)\rangle = U|\psi\rangle \otimes |0\rangle = \sum_{m=0}^{M} V_m|\psi\rangle \otimes |\phi_m\rangle. \quad (1)$$

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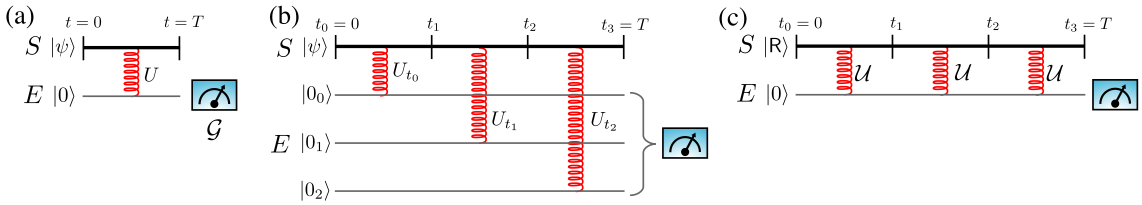

FIG. 1. Principal system S and environment E. (a) Basic model. The initial states of S and E are $|\psi\rangle$ and $|0\rangle$, respectively. The composite system $S+E$ undergoes a unitary transformation U, and E is measured by the observable $\mathcal{G}$. (b) Continuous measurement case ($N=3$). The initial states of the principal system and environment are $|\psi\rangle$ and $|0_2, 0_1, 0_0\rangle$, respectively. The initial substate $|0_k\rangle$ interacts with S within the time interval $[t_k, t_{k+1}]$ via a unitary operator U_{t_k}. The measurement record is obtained by measuring E at $t=T$. (c) Quantum walk case. The initial states of chirality (principal system) and position (environmental system) are $|\mathsf{R}\rangle$ and $|0\rangle$, respectively. The principal and environmental systems interact at each step via a unitary operator $\mathcal{U}$. The position is obtained by measuring E at step $t=T$.

Here, $V_m \equiv \langle\phi_m|U|0\rangle$ is the action on S associated with a transition in E from $|0\rangle$ to $|\phi_m\rangle$ and satisfies $\sum_{m=0}^{M} V_m^\dagger V_m = \mathbb{I}_S$, where $\mathbb{I}_S$ is an identity operator in S. Although Eq. (1) is a simple interaction model, it can describe the general open quantum dynamics starting from pure states. When tracing out E in Eq. (1), we obtain the Kraus representation $\rho(T) = \mathrm{Tr}_E[|\Psi(T)\rangle\langle\Psi(T)|] = \sum_{m=0}^{M} V_m|\psi\rangle\langle\psi|V_m^\dagger$. We hereafter assume that

$$g(0) = 0, \tag{2}$$

whose physical meaning is explained as follows. For illustrative purposes, suppose that $|\phi_m\rangle = |m\rangle$. Here, $g(0)$ is associated with $|\phi_0\rangle = |0\rangle$ in E after the interaction [Eq. (1)]. When the state of the environment after the interaction is $|0\rangle$, the environment remains unchanged before and after the interaction. For the photon-counting problem, $g(m)$ encodes the number of photons emitted into the environment. In this case, "no change" in the environment corresponds to no photon emission. Therefore, the condition of Eq. (2) is naturally satisfied by a photon-counting case. Because the condition of Eq. (2) constitutes the minimum assumption for any counting statistics, we refer to observables satisfying Eq. (2) as *counting observables*. For general open quantum dynamics and measurement of the environment, we wish to find the bound for the fluctuation of $\mathcal{G}$. Let ρ be the initial density operator of S. The mean and variance of $\mathcal{G}$ are $\langle\mathcal{G}\rangle \equiv \langle\Psi(T)|\mathbb{I}_S \otimes \mathcal{G}|\Psi(T)\rangle$ and $\mathrm{Var}[\mathcal{G}] \equiv \langle\mathcal{G}^2\rangle - \langle\mathcal{G}\rangle^2$, respectively. Using the quantum Cramér-Rao inequality [30–33], we find the following bound for the coefficient of variation of $\mathcal{G}$:

$$\frac{\mathrm{Var}[\mathcal{G}]}{\langle\mathcal{G}\rangle^2} \geq \frac{1}{\Xi}, \tag{3}$$

where

$$\Xi \equiv \mathrm{Tr}_S[(V_0^\dagger V_0)^{-1}\rho] - 1. \tag{4}$$

Equation (3) is the first main result of this Letter, and its proof is provided in the derivation section near the end. Equation (3) holds for any open quantum system as long as $V_0^\dagger V_0$ is positive definite, any counting observable $\mathcal{G}$, and any initial density operator ρ in S. Unless $V_0^\dagger V_0$ is positive definite, $(V_0^\dagger V_0)^{-1}$ is not well defined in Eq. (4), indicating that V_0 should be a full-rank matrix [36]. Equation (3) also holds for any (time-dependent) classical Markov process with any counting observable. V_0 is an operator corresponding to no change in the environment, and, therefore, the expectation of the inverse of $V_0^\dagger V_0$ quantifies activity of the dynamics. For classical Markov processes, Ξ becomes the reciprocal expectation of the survival probability, which reduces to the dynamical activity [34] in a short time limit [see Eq. (13)]. Therefore, we refer to Ξ as a *survival activity* in the present Letter. The generality of the bound implies that Ξ is a physically important quantity. When there are more than one mutually commutable counting observables $\mathcal{G}_i$, we can obtain a multidimensional variant of Eq. (3), as derived in Refs. [40,41] (see [36] for details).

We note the differences between the present TUR and related quantum TURs. Reference [24] obtained the TUR for quantum jump processes. In this case, the TUR was derived using a semiclassical approach via the large deviation principle for $T \to \infty$. Reference [26] used the classical Cramér-Rao inequality to derive a TUR in quantum nonequilibrium steady states. Their bound concerns instantaneous currents, which are defined by current operators and derived under a steady-state condition. Recently, we derived a quantum TUR for arbitrary continuous measurement satisfying a scaling condition [28]. However, the bound of Ref. [28] requires a steady-state condition under Lindblad dynamics, whereas Eq. (3) is satisfied for general open quantum dynamics.

We also comment on the relation between the quantum speed limit (QSL) [42–44] and the TUR. The QSL is concerned with the evolution speed, and quantum estimation theory has been reported to play an important role in the QSL [45–47]. While the QSL focuses on the transformation of the *principal* system, the TUR in this Letter is concerned with the evolution of the *environment*. Therefore, the QSL and TUR bound the evolution of the complementary states by thermodynamic quantities.

Quantum continuous measurement.—To observe the physical meaning of the main result, we apply Eq. (3) to continuous measurement in open quantum systems. Let us consider a Lindblad equation [48,49]:

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_{m=1}^{M} \mathcal{D}(\rho, L_m), \quad (5)$$

where H is a Hamiltonian, $\mathcal{D}(\rho,L) \equiv [L\rho L^\dagger - \{L^\dagger L,\rho\}/2]$ is a dissipator, and L_m $(1 \le m \le M)$ is an mth jump operator with M being the number of jump operators ($[\bullet,\bullet]$ and $\{\bullet,\bullet\}$ denote the commutator and anticommutator, respectively). Within a sufficiently small time interval $[t, t+\Delta t]$, one possible Kraus representation for Eq. (5) is $\rho(t+\Delta t) = \sum_{m=0}^{M} X_m \rho(t) X_m^\dagger$, where

$$X_0 \equiv \mathbb{I}_S - i\Delta t H - \frac{1}{2}\Delta t \sum_{m=1}^{M} L_m^\dagger L_m, \quad (6)$$

$$X_m \equiv \sqrt{\Delta t} L_m \quad (1 \le m \le M). \quad (7)$$

X_m satisfies the completeness relation $\sum_{m=0}^{M} X_m^\dagger X_m = \mathbb{I}_S$ up to $O(\Delta t)$.

By using the input-output formalism [50–52] (see Ref. [53] for comprehensive description), we can describe the time evolution induced by the Kraus operators of Eqs. (6) and (7) as an interaction between the system S and environment E. Let N be a sufficiently large natural number. We discretize the time by dividing the interval $[0, T]$ into N equipartitioned intervals and define $\Delta t \equiv T/N$ and $t_k \equiv k\Delta t$. We assume that the environmental orthonormal basis is $|m_{N-1}, \ldots, m_0\rangle$, where a subspace $|m_k\rangle$ interacts with S within the time interval $[t_k, t_{k+1}]$ via a unitary operator U_{t_k} [Fig. 1(b)]. When the initial states of S and E are $|\psi\rangle$ and $|0_{N-1}, \ldots, 0_0\rangle$, respectively, the state of $S+E$ at time $t = T$ is

$$\begin{aligned} |\Psi(T)\rangle &= U_{t_{N-1}} \cdots U_{t_0} |\psi\rangle \otimes |0_{N-1}, \ldots, 0_0\rangle \\ &= \sum_{\boldsymbol{m}} X_{m_{N-1}} \cdots X_{m_0} |\psi\rangle \otimes |m_{N-1}, \ldots, m_0\rangle, \end{aligned} \quad (8)$$

where $\boldsymbol{m} \equiv [m_{N-1}, \ldots, m_0]$, $X_{m_k} \equiv \langle m_k|U|0_k\rangle$ is an operator associated with the action of jumping from $|0_k\rangle$ to $|m_k\rangle$ in E, and $|m_{N-1}, \ldots, m_0\rangle$ provides the record of jump events. When the environment is measured using $|m_{N-1}, \ldots, m_0\rangle$ as a basis, the unnormalized state of the principal system is $X_{m_{N-1}} \cdots X_{m_0}|\psi\rangle$, which is referred to as a quantum trajectory conditioned on the measurement record $\boldsymbol{m} = [m_{N-1}, \ldots, m_0]$. The evolution of a quantum trajectory is given by a stochastic Schrödinger equation [54–56]: $d\rho = -i[H,\rho]dt + \sum_{m=1}^{M} (\rho \mathrm{Tr}_S[L_m \rho L_m^\dagger] - \{L_m^\dagger L_m, \rho\}/2)dt + \sum_{m=1}^{M} (L_m \rho L_m^\dagger / \mathrm{Tr}_S[L_m \rho L_m^\dagger] - \rho) d\mathcal{N}_m$, where $d\mathcal{N}_m$ is a noise increment equal to 1 when the mth jump event is detected between t and $t + dt$; otherwise, $d\mathcal{N}_m = 0$. Given the complete history of $[\mathcal{N}_m(t)]_{m=1}^{M}$, the conditional expectation is $\langle d\mathcal{N}_m(t)\rangle = \mathrm{Tr}_S[L_m \rho(t) L_m^\dagger]dt$, where $\rho(t)$ is a solution of the stochastic Schrödinger equation.

We consider a counting observable $\mathcal{G}$ in the continuous measurement, which counts the number of jump events within $[0, T]$. When expressed classically, we may write $\mathcal{G} = \sum_{m=1}^{M} G_m \mathcal{N}_m$, where $G_m \in \mathbb{R}$ is the weight of the mth jump and $\mathcal{N}_m = \int_0^T d\mathcal{N}_m$ is the number of mth jumps during $[0, T]$. Because the state of $S + E$ at time $t = T$ is given by Eq. (8), $\mathcal{G}$ can be defined quantum mechanically by $\mathcal{G} = \sum_{\boldsymbol{m}} g(\boldsymbol{m})|\boldsymbol{m}\rangle\langle\boldsymbol{m}|$. Because $\boldsymbol{m}$ is a record of jump events, $g(\boldsymbol{m})$ should be defined so that it counts and weights each jump event according to the classical definition of $\mathcal{G}$. When the environment remains unchanged before and after the interaction, $\mathcal{N}_m = 0$ for all $m \ge 1$. Therefore, $\mathcal{G}$ naturally satisfies the condition of Eq. (2). X_0 in Eq. (6) corresponds to a no-jump event within $[t, t + \Delta t]$. Because V_0 in Eq. (4) corresponds to the action associated with no-jump events within $[0, T]$, it is given by $V_0 = \lim_{N\to\infty} X_0^N$. We obtain $V_0 = e^{-T[iH+(1/2)\sum_{m=1}^{M} L_m^\dagger L_m]}$, and the survival activity is expressed as

$$\Xi = \mathrm{Tr}_S[e^{T[iH+(1/2)\sum_{m=1}^{M} L_m^\dagger L_m]} e^{T[-iH+(1/2)\sum_{m=1}^{M} L_m^\dagger L_m]} \rho] - 1. \quad (9)$$

When H and L_m depend on time, Ξ is formally given by [36]

$$\begin{aligned} \Xi = \mathrm{Tr}_S\Big[&\bar{\mathbb{T}} e^{\int_0^T dt iH(t) + \sum_{m=1}^{M} L_m^\dagger(t) L_m(t)/2} \\ &\times \mathbb{T} e^{\int_0^T dt -iH(t) + \sum_{m=1}^{M} L_m^\dagger(t) L_m(t)/2} \rho \Big] - 1, \end{aligned} \quad (10)$$

where $\mathbb{T}$ and $\bar{\mathbb{T}}$ are time-ordering and anti-time-ordering operators, respectively. Equations (9) and (10) are the second main result of the present Letter. Equation (3) with Eq. (10) is satisfied for the continuous measurement of jump events in any Lindblad equation starting from any initial density operator.

Equations (6) and (7) are not the only Kraus representations compatible with the Lindblad equation (5). A Kraus operator Y_m compatible with Eq. (5) (i.e., $\sum_m X_m \rho X_m^\dagger = \sum_m Y_m \rho Y_m^\dagger$) can be obtained by $Y_{m'} = \sum_m J_{m'm} X_m$, where $J_{m'm}$ is an arbitrary unitary operator. This unitary freedom of the Kraus operator corresponds to that in the measurement basis of the environment. As mentioned above, X_m is obtained by the measurement basis $|m\rangle$ for each time interval, that is, $X_m = \langle m|U|0\rangle$, while $Y_{m'}$ is derived via a different measurement basis $|\varphi_{m'}\rangle \equiv \sum_{m=0}^{M} (J^\dagger)_{mm'}|m\rangle$, specifically, $Y_{m'} = \langle\varphi_{m'}|U|0\rangle$. Ξ in Eq. (9) depends on how we measure the environment, that is, how we unravel the Lindblad equation. To observe the consequences of different unravellings, for simplicity, we consider a case having only one jump operator L. The Lindblad equation is invariant under the following transformation: $H \to H - (i/2)(\zeta^* L - \zeta L^\dagger)$ and $L \to L + \zeta \mathbb{I}_S$, where $\zeta \in \mathbb{C}$ is an arbitrary parameter. A physical interpretation of this transformation is presented in Refs. [57–59]. Under this transformation, Ξ becomes (for time-independent L and H) $\Xi = e^{|\zeta|^2 T}\mathrm{Tr}_S[e^{T[iH+(1/2)L^\dagger L+\zeta^* L]}e^{T[-iH+(1/2)L^\dagger L+\zeta L^\dagger]}\rho] - 1$. Therefore, for $|\zeta| \to \infty$, Ξ scales as $\Xi \sim e^{|\zeta|^2 T}$; this indicates that we can make the lower bound of Eq. (3) arbitrarily small by employing a continuous measurement with a large $|\zeta|$. This result may appear contradictory to that obtained in Ref. [28], which reported a unified lower bound valid for any continuous measurements. Note that the continuous measurements considered in Ref. [28] require a scaling condition, which is not satisfied for the measurements corresponding to the transformation above.

Classical Markov processes.—When we emulate classical Markov processes with the Lindblad equation, $[H, \sum_{m=1}^{M} L_m^\dagger L_m] = 0$ holds. In this case, from Eq. (9), we obtain

$$\Xi_{\mathrm{CL}} = \mathrm{Tr}_S[e^{T\sum_{m=1}^{M} L_m^\dagger L_m}\rho] - 1, \tag{11}$$

where the subscript "CL" is shorthand for "classical." Therefore, noncommutativeness $[H, \sum_{m=1}^{M} L_m^\dagger L_m] \neq 0$ can be a benefit of the quantum systems over their classical counterparts. We evaluate the effect of noncommutativeness in the survival activity. Assuming that T is sufficiently small, a simple calculation yields [36]

$$\Xi = \Xi_{\mathrm{CL}} + \frac{1}{2}T^2\chi + O(T^3), \tag{12}$$

where $\chi \equiv i\sum_{m=1}^{M}\mathrm{Tr}_S[[H, L_m^\dagger L_m]\rho]$ represents the expectation of the commutative relation. When $\chi > 0$, the system gains a precision enhancement due to its quantum nature.

As a corollary of the continuous measurement, we can obtain a specific expression of Ξ_{CL} for classical Markov processes. We consider a classical Markov process with N_S states $\{B_1, B_2, \ldots, B_{N_S}\}$ and a transition rate $\gamma_{ji}(t)$ corresponding to a jump from B_i to B_j at time t. Suppose that the initial probability at state B_i is given by P_i ($\sum_{i=1}^{N_S} P_i = 1$ and $P_i \geq 0$). Then, Eq. (11) is expressed as

$$\Xi_{\mathrm{CL}} = \sum_{i=1}^{N_S}\frac{P_i}{\mathcal{R}_i(T)} - 1, \tag{13}$$

where $\mathcal{R}_i(T) \equiv e^{-\int_0^T dt \sum_{j\neq i}\gamma_{ji}(t)}$ is the survival probability in which there is no jump during $[0, T]$ starting from B_i. In Eq. (13), the first term is the reciprocal expectation of the survival probability, which is an experimentally measurable quantity. For the classical Markov process, a classical representation of the counting observable $\mathcal{G}$ becomes $\mathcal{G} = \sum_{i,j,i\neq j} G_{ji}\mathcal{N}_{ji}$, where $G_{ji} \in \mathbb{R}$ is a weight for the jump from B_i to B_j and $\mathcal{N}_{ji}$ is the number of jumps from B_i to B_j during $[0, T]$. Equation (3) with Eq. (13) is satisfied for arbitrary time-dependent Markov processes and initial states. When the system activity is greater, $\mathcal{R}_i(T)$ decreases, resulting in a smaller lower bound. Indeed, for a short time limit $T \to 0$, Ξ_{CL} reduces to $\Xi_{\mathrm{CL}} \to \Upsilon$, where Υ is the dynamical activity $\Upsilon \equiv \sum_{i,j,i\neq j}\int_0^T P_i(t)\gamma_{ji}(t)dt$. Here, $P_i(t)$ is the probability of being B_i at time t. The dynamical activity quantifies the average number of jumps during $[0, T]$. In classical Markov processes, the dynamical activity has been reported to constitute the bound in the TUR [6,9,12] and the QSL [60]. For a steady-state condition, it has been reported that the fluctuations in counting observables are bounded from below by $1/\Upsilon$ [6]. However, as demonstrated numerically in Ref. [36], in some cases, $\mathrm{Var}[\mathcal{G}]/\langle\mathcal{G}\rangle^2 \geq 1/\Upsilon$ does not hold when the system is far from a steady state [36].

Quantum walk.—We apply the main result of Eq. (3) to a discrete-time one-dimensional quantum walk [61,62]. The quantum walk is defined on the chirality space spanned by $\{|\mathsf{R}\rangle, |\mathsf{L}\rangle\}$ and the position space spanned by $\{|n\rangle\}$, where n is an integer. Here, we identify the chirality and position spaces as the principal and environmental systems, respectively [Fig. 1(c)]. One-step evolution of the quantum walk is performed via the unitary operator $\mathcal{U} \equiv \mathcal{S}(\mathcal{K} \otimes \mathbb{I}_E)$, where $\mathbb{I}_E$ is an identity operator in E and $\mathcal{K}$ and $\mathcal{S}$ are the coin and conditional shift operators, respectively. For the coin operator, we employ the Hadamard gate defined by

$$\mathcal{K} = \frac{1}{\sqrt{2}}(|\mathsf{R}\rangle\langle\mathsf{R}| + |\mathsf{R}\rangle\langle\mathsf{L}| + |\mathsf{L}\rangle\langle\mathsf{R}| - |\mathsf{L}\rangle\langle\mathsf{L}|). \tag{14}$$

The conditional shift operator is given by

$$\mathcal{S} = \sum_n[|\mathsf{R}\rangle\langle\mathsf{R}| \otimes |n+1\rangle\langle n| + |\mathsf{L}\rangle\langle\mathsf{L}| \otimes |n-1\rangle\langle n|], \tag{15}$$

which increases the position when the chirality is $|\mathsf{R}\rangle$ and decreases it when the chirality is $|\mathsf{L}\rangle$. The composite system after t steps is given by $|\Psi(t)\rangle = \mathcal{U}^t|\Psi(0)\rangle$, where $|\Psi(0)\rangle$ is the initial state $|\Psi(0)\rangle = |\mathsf{R}\rangle \otimes |0\rangle$. By using the combinatorics, the amplitudes at step t can be computed [61,63,64]. At step $t = T$, the measurement is performed on the position space, where the measurement operator is defined by $\sum_n g(n)|n\rangle\langle n|$. Typically, $g(n) = n$ is employed, which corresponds to measuring the position after T steps. When $g(n)$ satisfies Eq. (2), that is, $g(n)$ is a counting observable, Eq. (3) holds. Then, we obtain

$$\Xi = \begin{cases} 2^{2u+1}\binom{u}{\frac{u}{2}}^{-2} - 1 & u \in \text{even}, \\ 2^{2u-1}\binom{u-1}{\frac{u-1}{2}}^{-2} - 1 & u \in \text{odd}, \end{cases} \tag{16}$$

where $u \equiv T/2$. Note that we consider only even T, because the amplitudes vanish for odd T. Using Stirling's approximation, $2^{2u+1}\binom{u}{u/2}^{-2} \sim \pi u$, indicating that the survival activity linearly depends on the number of steps. This is in contrast to Eq. (13) where Ξ exponentially depends on time. Although the environment confers qualitatively different information in the continuous measurement [Fig. 1(b)] and in the quantum walk [Fig. 1(c)], our result can provide the lower bounds for both systems in a unified way.

We also test the main result numerically for both classical and quantum systems to verify the bound [36].

Derivation.—We provide a brief derivation of Eq. (3) (see [36] for details). Our derivation is based on the *quantum* Cramér-Rao inequality [30–33], which has been used to derive the QSL [45–47] and the TUR [28]. Suppose that the system evolves according to Eq. (1), in which U and V_m $(0 \le m \le M)$ are parametrized by a real parameter θ as $U(\theta)$ and $V_m(\theta)$, respectively. The final state of $S + E$ depends on θ, which is expressed as $|\Psi_\theta(T)\rangle$. For arbitrary measurement operator Θ_E in E, the quantum Cramér-Rao inequality holds [31]:

$$\frac{\mathrm{Var}_\theta[\Theta_E]}{[\partial_\theta \langle \Theta_E \rangle_\theta]^2} \ge \frac{1}{\mathcal{F}_E(\theta)}, \tag{17}$$

where $\mathcal{F}_E(\theta)$ is quantum Fisher information [32,33], $\langle \Theta_E \rangle_\theta \equiv \langle \Psi_\theta(T)|\mathbb{I}_S \otimes \Theta_E|\Psi_\theta(T)\rangle$, and $\mathrm{Var}_\theta[\Theta_E] = \langle \Theta_E^2 \rangle_\theta - \langle \Theta_E \rangle_\theta^2$. From Ref. [65], $\mathcal{F}_E(\theta)$ is bounded from above by $\mathcal{F}_E(\theta) \le \mathcal{C}(\theta)$, where $\mathcal{C}(\theta) \equiv 4[\langle \psi|H_1(\theta)|\psi\rangle - \langle \psi|H_2(\theta)|\psi\rangle^2]$ with $H_1(\theta) \equiv \sum_{m=0}^{M} [\partial_\theta V_m^\dagger(\theta)][\partial_\theta V_m(\theta)]$ and $H_2(\theta) \equiv i \sum_{m=0}^{M} [\partial_\theta V_m^\dagger(\theta)] V_m(\theta)$.

To derive the main result [Eq. (3)], for $1 \le m \le M$, we consider the parametrization $V_m(\theta) \equiv e^{\theta/2} V_m$, where $\theta = 0$ recovers the original operator. Because a completeness relation should be satisfied, $V_0(\theta)$ obeys $V_0^\dagger(\theta) V_0(\theta) = \mathbb{I}_S - \sum_{m=1}^{M} V_m^\dagger(\theta) V_m(\theta) = \mathbb{I}_S - e^\theta \sum_{m=1}^{M} V_m^\dagger V_m$. For any $V_0(\theta)$ satisfying the completeness relation, there exists a unitary operator U_V such that $V_0(\theta) = U_V \sqrt{\mathbb{I}_S - e^\theta \sum_{m=1}^{M} V_m^\dagger V_m}$. Substituting $V_m(\theta)$ into $\mathcal{C}(\theta)$ (as detailed in Ref. [36]), we find

$$\mathcal{C}(\theta) = \langle \psi|(V_0^\dagger V_0)^{-1}|\psi\rangle - 1. \tag{18}$$

We next evaluate $\langle \mathcal{G} \rangle_\theta$. Because we have assumed that $g(0) = 0$ [Eq. (2)], the complicated scaling dependence of $V_0(\theta)$ on θ can be ignored when computing $\langle \mathcal{G} \rangle_\theta$. Specifically, we obtain $\langle \mathcal{G} \rangle_\theta = e^\theta \langle \mathcal{G} \rangle_{\theta=0}$. We evaluate Eq. (17) at $\theta = 0$ with $\Theta_E = \mathcal{G}$ to obtain the main result [Eq. (3)]. Although the derivation described here assumes that the initial state of S is pure (i.e., $\rho = |\psi\rangle\langle\psi|$), we can show that Eq. (3) still holds for any initial mixed state ρ in S [36].

Conclusion.—In this Letter, we have derived a TUR for open quantum systems. Because our relation holds for a general open quantum system, we expect the present study to serve as a basis for obtaining the thermodynamic bound for several quantum systems, such as quantum computation and communication.

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