Editors' Suggestion

Hidden Variable Model for Universal Quantum Computation with Magic States on Qubits

Michael Zurel,[1,2] Cihan Okay,[1,2] and Robert Raussendorf[1,2]
[1]Department of Physics and Astronomy, University of British Columbia, Vancouver, British Columbia V6T 1Z1, Canada
[2]Stewart Blusson Quantum Matter Institute, University of British Columbia, Vancouver, British Columbia V6T 1Z4, Canada

(Received 14 May 2020; revised 3 November 2020; accepted 10 November 2020; published 23 December 2020)

We show that every quantum computation can be described by a probabilistic update of a probability distribution on a finite phase space. Negativity in a quasiprobability function is not required in states or operations. Our result is consistent with Gleason's theorem and the Pusey-Barrett-Rudolph theorem.

DOI: 10.1103/PhysRevLett.125.260404

It is often pointed out that the fundamental objects in quantum mechanics are amplitudes, not probabilities [1,2]. This fact notwithstanding, here we construct a description of universal quantum computation—and hence of all quantum mechanics in finite-dimensional Hilbert spaces—in terms of a probabilistic update of a probability distribution. In this formulation, quantum algorithms look structurally akin to classical diffusion problems.

While this seems implausible, there exists a well-known special instance of it: quantum computation with magic states (QCM) [3] on a single qubit. Compounding two standard one-qubit Wigner functions, a hidden variable model can be constructed in which every one-qubit quantum state is positively represented [4]. This representation is furthermore covariant under all one-qubit Clifford unitaries and "positivity preserving" under all one-qubit Pauli measurements. The update under such operations preserves the probabilistic character of the model, and hence QCM on one qubit can be classically simulated by a probabilistic update of a probability function on eight elements (see Fig. 1 for illustration).

The prevailing view on the one-qubit example is that it is an exception and that for multiple qubits negativity will inevitably creep into any quasiprobability function of any computationally useful quantum state, rendering classical simulations inefficient [5]. This hypothesis is informed by the study of Wigner functions in finite-dimensional state spaces, which establishes Wigner function negativity as a necessary computational resource, i.e., there can be no quantum speedup without negativity [6–20]. A quantum optics notion of quantumness—negativity of Wigner functions [21,22]—and a computational notion—hardness of classical simulation—thus align.

The viewpoint just summarized requires correction. As we show in this Letter, the one-qubit case is not an exception; rather it is an example illustrating the general case. *Every* quantum state on any number of qubits can be represented by a probability function, and the update of this probability function under Pauli measurement is also probabilistic. This is the content of Theorem 1 below. We emphasize that the states and operations are *both* represented positively, not just one or the other.

In Theorem 2, we apply this to quantum computation with magic states, showing that universal quantum computation can be classically simulated by the probabilistic update of a probability distribution.

This looks all very classical, and therein lies a puzzle. In fact, our Theorem 2 is running up against a number of no-go theorems: Theorem 2 in [23] and the Pusey-Barrett-Rudolph (PBR) theorem [24] say that probability representations for quantum mechanics do not exist, and [9–13] show that negativity in certain Wigner functions is a precondition for speedup in quantum computation. Further, does not Gleason's theorem prove that the proper representation of a state in quantum mechanics is density matrices rather than probability distributions?

As we explain in the discussion part of this Letter, there is no contradiction here with those works. Rather, the above-quoted theorems make stronger assumptions than we

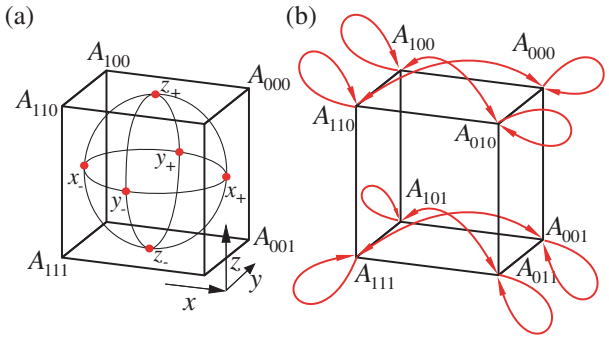

FIG. 1. One-qubit model. (a) The state space Λ_1 is a cube with eight vertices corresponding to the phase point operators $A_\alpha = [I + (-1)^{s_x}X + (-1)^{s_y}Y + (-1)^{s_z}Z]/2$, with $\alpha = (s_x, s_y, s_z) \in \mathbb{Z}_2^3$. The physical one-qubit states lie on or in the Bloch sphere that is contained in Λ_1 and touches the boundary of Λ_1 at six points corresponding to the six one-qubit stabilizer states. (b) Update of the phase point operators A_α under measurement of the Pauli observable Z. Each red arrow represents a transition probability of $1/2$.

do and establish no-go theorems because of that. However, for describing universal quantum computation—hence all quantum mechanics in finite-dimensional Hilbert spaces—our weaker assumptions suffice.

The remainder of this Letter is organized as follows. First, we define our setting and state our main results, Theorems 1 and 2. Then we prove them, and thereafter discuss the above questions.

Setting and Results.—We focus on systems of n qubits for any $n \in \mathbb{N}$ (the statement below applies to qudits in an analogous manner) and consider arbitrary quantum states evolving under sequences of Pauli measurements. The measurements need not commute, and the sequences may be arbitrarily long. This setting comprises universal quantum computation with magic states.

Below we devise a probability representation for this setting. The representation lives on a finite generalized phase space, and its purpose is to correctly reproduce the joint measurements statistics for all quantum states and all sequences of Pauli measurements.

We denote the n-qubit Pauli operators by $T_a = i^{\phi(a)} X(a_X) Z(a_Z)$, $\forall\, a = (a_X, a_Z) \in \mathbb{Z}_2^n \times \mathbb{Z}_2^n =: E_n$, with $X(a_X) := \bigotimes_{k=1}^{n} (X_k)^{a_X[k]}$, etc. The phases $\phi : E_n \to \mathbb{Z}_4$ are free to choose, subject to the constraint that all T_a are Hermitian. The projectors onto the eigenspaces of Pauli observables are $\Pi_{a,s} := [I + (-1)^s T_a]/2$.

The state space Λ_n of our probabilistic model is defined as follows. We denote by $\mathrm{Herm}_1(2^n)$ the set of Hermitian operators on n-qubit Hilbert space $H = \mathbb{C}^{2^n}$ with the property that $\mathrm{Tr}(X) = 1$ for all $X \in \mathrm{Herm}_1(2^n)$, and by $\mathcal{S}_n$ the set of all n-qubit pure stabilizer states [25–27].

Then, we define the polytope Λ_n as

$$\Lambda_n := \{X \in \mathrm{Herm}_1(2^n) | \mathrm{Tr}(|\sigma\rangle\langle\sigma| X) \geq 0,\ \forall\, |\sigma\rangle \in \mathcal{S}_n\} \quad (1)$$

(also see [28]). The elements $X \in \Lambda_n$ are the "states" (though not necessarily proper quantum states) that behave "well" under all sequences of Pauli measurements; namely, the probabilities for the outcome sequences are all nonnegative and add up to unity.

Λ_n is defined as the intersection of a finite number of halfspaces. Furthermore, it is bounded [see Supplemental Material (SM) [29], Section VI)]. Therefore, by the Minkowski-Weyl theorem [30,31], Λ_n can equivalently be described as the convex hull of finitely many extreme points (vertices). Denote by $\mathcal{A}_n$ the set of vertices of Λ_n, and the vertices by $A_\alpha \in \mathcal{A}_n$. These are our generalized phase point operators, and the corresponding index set $\{\alpha\} =: \mathcal{V}_n$ is the generalized phase space.

We now have the following result.

Theorem 1.—For all numbers of qubits $n \in \mathbb{N}$, (i) each n-qubit quantum state ρ can be represented by a probability function $p_\rho : \mathcal{V}_n \to \mathbb{R}_{\geq 0}$,

$$\rho = \sum_{\alpha \in \mathcal{V}_n} p_\rho(\alpha) A_\alpha. \quad (2)$$

(ii) For the state update under Pauli measurements, it holds that

$$\Pi_{a,s} A_\alpha \Pi_{a,s} = \sum_{\beta \in \mathcal{V}_n} q_{\alpha,a}(\beta, s) A_\beta. \quad (3)$$

For all $a \in E_n$, $\alpha \in \mathcal{V}_n$, the $q_{\alpha,a} : \mathcal{V}_n \times \mathbb{Z}_2 \to \mathbb{R}_{\geq 0}$ are probability functions.

(iii) Denote by $P_{\rho,a}(s)$ the probability of obtaining outcome s for a measurement of T_a on the state ρ. Then, the Born rule $P_{\rho,a}(s) = \mathrm{Tr}(\Pi_{a,s}\rho)$ takes the form

$$\mathrm{Tr}(\Pi_{a,s}\rho) = \sum_{\alpha \in \mathcal{V}_n} p_\rho(\alpha) Q_a(s|\alpha), \quad (4)$$

where $Q_a(s|\alpha)$ is given by

$$Q_a(s|\alpha) := \sum_{\beta \in \mathcal{V}_n} q_{\alpha,a}(\beta, s). \quad (5)$$

Hence, $0 \leq Q_a(s|\alpha) \leq 1$ for all a, s, α.

The above theorem describes a hidden variable model (HVM) [32–36]. For any fixed number of qubits, any quantum state can be described by a probability function with finitely many elements. This property distinguishes it from the HVM of Beltrametti and Bugajski [36], which also applies to all quantum states but requires an infinite state space. A further distinguishing property is the probabilistic state update under a dynamical process: the Pauli measurement.

Theorem 1 is illustrated in Fig. 1 through the example of a single qubit and in the SM, Section V, for two qubits.

Because of its capability to describe Pauli measurements, the above HVM has bearing on a model of universal quantum computation, namely quantum computation with magic states (QCM) [3]. QCM is closely related to the circuit model (see SM, Sec. IV, for background). The difference is that in QCM the set of operations is restricted to a sequence of Clifford unitaries interspersed with Pauli measurements. These operations are applied to an initial "magic" state. The only property of QCM relevant for the present discussion is its quantum computational universality [3,27,37].

To apply the above probabilistic representation to QCM, we need to consider all its operational primitives—the Pauli measurements, the Clifford unitaries, and the magic states. Magic states (like all other quantum states) and Pauli measurements are positively represented by our HVM, cf. Theorem 1. This leaves the Clifford gates. The easiest way of dealing with them is to observe that they are redundant, i.e., no computational power is lost if we consider sequences of Pauli measurements only. The reason is that the Clifford unitaries may be propagated past all measurements,

thereby conjugating the Pauli measurements into (other) Pauli measurements. After forward propagation, the unitaries can be dropped since they do not affect the statistics of the (now earlier) measurements (see, e.g., [10,12]).

With the Pauli measurements as the only essential dynamical element, QCM matches the setting described in Theorem 1. This leads to the following result.

Algorithm 1. Classical procedure to simulate a single run of a given QCM.

1. Sample from the probability distribution p_ρ. Obtain a phase space point $\alpha_0 \in \mathcal{V}_n$.
2. For all Pauli measurements T_{a_t}, $a_t \in E_n$, from $t = 1$ to $t = t_{\max}$, sample from $q_{\alpha,a=a_t}$ to obtain the new phase space point $\beta \in \mathcal{V}_n$ and measurement outcome s. Output s as the outcome for the measurement of T_{a_t}, update the phase space point $\alpha_{t-1} \to \alpha_t = \beta$, and increment $t \to t+1$.

Theorem 2.—For any $n \in \mathbb{N}$ and all n-qubit quantum states ρ the classical Algorithm 1 for sampling the outcomes of any sequence of Pauli measurements on ρ agrees with the predictions of quantum mechanics.

Thus, the HVM of Theorem 1 describes all of universal quantum computation and hence arbitrarily closely approximates all quantum mechanical dynamics in finite-dimensional Hilbert spaces.

Theorem 2 does not imply that the classical simulation Algorithm 1 is efficient. Intuition derived from previous classical simulation algorithms for quantum computation [9,13,38,39] suggests that it is inefficient. However, at present we can neither prove the efficiency nor the inefficiency of this algorithm.

Proofs.—We now turn to the proofs of Theorems 1 and 2. The proof of Theorem 1 requires a lemma.

Lemma 1.—The set Λ_n has the following properties.

(1) Λ_n contains all n-qubit quantum states; i.e., for all n-qubit density operators ρ it holds that $\rho \in \Lambda_n$.

(2) Λ_n is closed under Pauli measurement, i.e., for all $\Pi_{a,s}$ it holds that

$$X \in \Lambda_n \wedge \mathrm{Tr}(\Pi_{a,s} X) > 0 \Rightarrow \frac{\Pi_{a,s} X \Pi_{a,s}}{\mathrm{Tr}(\Pi_{a,s} X)} \in \Lambda_n.$$

Proof of Lemma 1.—All quantum states ρ satisfy the conditions $\mathrm{Tr}(|\sigma\rangle\langle\sigma|\rho) \geq 0$, for all n-qubit stabilizer states $|\sigma\rangle$ (as well as all other pure states), and $\mathrm{Tr}(\rho) = 1$; hence all quantum states ρ are in Λ_n.

Regarding Property 2, we observe that for all stabilizer states $|\sigma\rangle \in \mathcal{S}_n$ and all Pauli observables T_a, it holds that

$$\Pi_{a,s}|\sigma\rangle\langle\sigma|\Pi_{a,s} = c|\sigma'\rangle\langle\sigma'|, \text{ where } |\sigma'\rangle \in \mathcal{S}_n, \ c \geq 0. \quad (6)$$

Namely, $c = 1$ if $(-1)^s T_a|\sigma\rangle = |\sigma\rangle$, $c = 0$ if $(-1)^s T_a|\sigma\rangle = -|\sigma\rangle$, and $c = 1/2$ otherwise [27]. Combining Eq. (6) and the definition of Λ_n, Eq. (1), $\mathrm{Tr}(|\sigma\rangle\langle\sigma|\Pi_{a,s} X \Pi_{a,s}) = \mathrm{Tr}[(\Pi_{a,s}|\sigma\rangle\langle\sigma|\Pi_{a,s})X] = c\,\mathrm{Tr}(|\sigma'\rangle\langle\sigma'|X) \geq 0$. Therefore, whenever $\mathrm{Tr}(\Pi_{a,s}X) > 0$, the postmeasurement state $X'_{a,s} := \Pi_{a,s} X \Pi_{a,s}/\mathrm{Tr}(\Pi_{a,s}X)$ also has the property that

$$\mathrm{Tr}(|\sigma\rangle\langle\sigma|X'_{a,s}) \geq 0, \quad \forall\ a \in E_n, \quad \forall\ s \in \mathbb{Z}_2, \quad \forall\ |\sigma\rangle \in \mathcal{S}_n.$$

Furthermore, $\mathrm{Tr}(X'_{a,s}) = 1$. Therefore, $X'_{a,s} \in \Lambda_n$.■

Proof of Theorem 1.—With Property 1 in Lemma 1, any n-qubit quantum state ρ is in Λ_n. Hence it can be expressed as a convex combination of the vertices A_α, as in Eq. (2). Taking the trace of Eq. (2) yields $\sum_\alpha p_\rho(\alpha) = 1$, i.e., p_ρ is a probability function. This proves the first statement of Theorem 1.

With Property 2 of Lemma 1, for all phase point operators A_α and all projectors $\Pi_{a,s}$ with $\mathrm{Tr}(\Pi_{a,s}A_\alpha) > 0$, it holds that $\Pi_{a,s}A_\alpha\Pi_{a,s}/\mathrm{Tr}(\Pi_{a,s}A_\alpha) \in \Lambda_n$. Therefore,

$$\Pi_{a,s}A_\alpha\Pi_{a,s} = \sum_{\beta\in\mathcal{V}_n} q_{\alpha,a}(\beta,s)A_\beta,$$

with $q_{\alpha,a}(\beta,s) \geq 0$ for all $\beta \in \mathcal{V}_n$, and $s \in \mathbb{Z}_2$. Now fixing α, a and adding the corresponding equations for $s = 0$ and $s = 1$, and then taking the trace, we find

$$\sum_{s\in\mathbb{Z}_2}\sum_{\beta\in\mathcal{V}_n} q_{\alpha,a}(\beta,s) = 1. \quad (7)$$

Hence, $q_{\alpha,a}: \mathcal{A}_n \times \mathbb{Z}_2 \to \mathbb{R}_{\geq 0}$ is a probability distribution for all $\alpha \in \mathcal{V}_n$, $a \in E_n$. This demonstrates Eq. (3).

Regarding $Q_a(s|\alpha)$ as defined in Eq. (5), since the $q_{\alpha,a}(\beta,s)$ are all positive, it holds that $Q_a(s|\alpha) \geq 0$ for all a, s, α. Furthermore, with Eq. (7) it follows that $Q_a(0|\alpha) + Q_a(1|\alpha) = 1$ for all a, α, and therefore

$$0 \leq Q_a(s|\alpha) \leq 1, \quad \forall\ a, s, \alpha.$$

Combining Eq. (2) and the already established Eq. (3),

$$\begin{aligned}\mathrm{Tr}(\Pi_{a,s}\rho) &= \sum_{\alpha\in\mathcal{V}_n} p_\rho(\alpha)\mathrm{Tr}(\Pi_{a,s}A_\alpha\Pi_{a,s}) \\ &= \sum_{\alpha\in\mathcal{V}_n} p_\rho(\alpha)\sum_{\beta\in\mathcal{V}_n} q_{\alpha,a}(\beta,s) \\ &= \sum_{\alpha\in\mathcal{V}_n} p_\rho(\alpha)Q_a(s|\alpha).\end{aligned}$$

This proves the formulation Eq. (4) of the Born rule.■

Proof of Theorem 2.—Consider a Pauli measurement T_a on input state ρ. Using the classical simulation algorithm, the conditional probability of obtaining outcome s given the state $\alpha \in \mathcal{V}_n$ is given by Eq. (5). Therefore, the probability of obtaining outcome s given a measurement

of T_a on state ρ as predicted by the classical simulation algorithm is

$$P^{(\mathrm{Sim})}_{\rho,a}(s) = \sum_{\alpha\in\mathcal{V}_n} p_\rho(\alpha) Q_a(s|\alpha). \tag{8}$$

The outcome probability predicted by the Born rule, $P^{(\mathrm{QM})}_{\rho,a}$, is given by Eq. (4).

Comparing Eq. (8) and Eq. (4), we see that the classical simulation algorithm reproduces the outcome probabilities predicted by the Born rule for a single Pauli measurement.

Now we turn to the postmeasurement state ρ'. Quantum mechanics predicts it to be

$$\rho'^{(\mathrm{QM})} = \frac{\Pi_{a,s}\rho\Pi_{a,s}}{\mathrm{Tr}(\Pi_{a,s}\rho)}.$$

Here the numerator is

$$\begin{aligned}\Pi_{a,s}\rho\Pi_{a,s} &= \sum_{\alpha\in\mathcal{V}_n} p_\rho(\alpha)\Pi_{a,s}A_\alpha\Pi_{a,s} \\ &= \sum_{\alpha\in\mathcal{V}_n} p_\rho(\alpha)\sum_{\beta\in\mathcal{V}_n} q_{\alpha,a}(\beta,s)A_\beta,\end{aligned}$$

and so

$$\rho'^{(\mathrm{QM})} = \frac{\sum_\alpha p_\rho(\alpha)\sum_\beta q_{\alpha,a}(\beta,s)A_\beta}{\sum_\alpha p_\rho(\alpha)Q_a(s|\alpha)}. \tag{9}$$

Using the classical simulation algorithm, the probability of obtaining outcome s and state β given a measurement of T_a on state ρ is $P_{\rho,a}(\beta,s) = P_{\rho,a}(\beta|s)P_{\rho,a}(s)$. But $P_{\rho,a}(\beta,s) = \sum_\alpha p_\rho(\alpha)P_a(\beta,s|\alpha) = \sum_\alpha p_\rho(\alpha)q_{\alpha,a}(\beta,s)$ and $P_{\rho,a}(\beta|s) = p_{\rho'}(\beta)$. Therefore, the postmeasurement state according to the classical simulation algorithm is

$$\begin{aligned}\rho'^{(\mathrm{Sim})} &= \sum_{\beta\in\mathcal{V}_n} p_{\rho'}(\beta)A_\beta = \sum_{\beta\in\mathcal{V}_n}\frac{P_{\rho,a}(\beta,s)}{P_{\rho,a}(s)}A_\beta \\ &= \sum_{\beta\in\mathcal{V}_n}\frac{\sum_\alpha p_\rho(\alpha)q_{\alpha,a}(\beta,s)}{\sum_\alpha p_\rho(\alpha)Q_a(s|\alpha)}A_\beta.\end{aligned} \tag{10}$$

This agrees with Eq. (9) above. Therefore, the classical simulation algorithm also reproduces the postmeasurement state predicted by quantum mechanics for a single Pauli measurement.

Now let $\rho(t)$ denote the state before the tth measurement. Then the above shows that the classical simulation algorithm correctly reproduces the Born rule probabilities $P_{\rho,a_t}(s_t|s_1,s_2,\ldots,s_{t-1})$ as well as the postmeasurement state $\rho(t+1)$. Therefore, by induction the simulation algorithm correctly reproduces the outcome probabilities predicted by the Born rule for any sequence of Pauli measurements.■

Discussion.—We now return to the questions posed in the introduction.

(i) Existence of a probability representation. It is stated in Theorem 2 of [23] that "a quasiprobability representation of quantum theory must have negativity in either its representation of states or measurements (or both)."

This appears to contradict Theorem 1. However, there is no contradiction here, only a difference in assumptions. Through the definitions made prior to it, the above quoted theorem refers to frame representations. This requires, in particular, that the quasiprobability representation for every quantum state is unique. Clearly, our probability distribution p does not satisfy this condition.

(ii) Contextuality. Given the history of the subject [6–16,18,40–42], an interesting question is whether the present HVM is contextual or noncontextual. The Kochen-Specker notion of contextuality does not apply, because the present value assignments are not deterministic. This leaves us with Spekkens' notion [43], [44] to consider. In this regard, our HVM is preparation contextual and measurement-noncontextual. The former reflects the aforementioned nonuniqueness of p.

(iii) Negativity vs efficiency of simulation. Negativity in quasiprobability representations has been identified as a cause for slowing down the classical simulation of quantum systems by sampling. A general result has been obtained in [5] stating that a quantum system described by a quasiprobality function W with negativity $\mathcal{M} = \|W\|_1$ can be simulated by sampling at a multiplicative cost that scales like $\mathcal{M}^2$.

There are simulation schemes for QCM on qudits [9], on rebits [10], and on qubits [12–14], where negativity is the only source for the computational hardness of classical simulation. Negativity is therefore singled out as a precondition for quantum speedup.

We do not contradict the results [9–14] but now find that they are an artifact of the particular quasiprobability functions chosen. Our result lies at the opposite end of the spectrum. There is no negativity but, presumably, still computational hardness.

The absence of negativity notwithstanding, there also is continuity with prior works. The probability distribution p satisfies the four criteria of the Stratonovich-Weyl (SW) correspondence [45] (also see [46]; see SM, Section I for details). It is thus very closely related to the original Wigner function [47] and to previously defined discrete Wigner functions for finite-dimensional systems. From the SW perspective, the only condition p doesn't satisfy is uniqueness.

Furthermore, the phase point operators identified in the multiqubit setting of [13] (also see [48]) are special cases of the phase point operators discussed here (see SM, Section IV). And thus, the present approach provides a broader and yet conceptually simpler framework for the classical simulation of quantum computation by sampling, subsuming earlier ones as special cases.

(iv) The PBR theorem. The hidden variable model presented here is ψ-epistemic [49]. The PBR theorem [24] asserts that (with certain assumptions) no ψ-epistemic model can reproduce the predictions of quantum mechanics. Our result does not contradict the PBR theorem for two reasons. First, we consider only sequences of Pauli measurements rather than general measurements (this is sufficient for universal quantum computation). Second, our model does not satisfy the assumption of preparation independence required for the theorem to hold. That is, in general, $p_{\rho_1\otimes\rho_2} \neq p_{\rho_1} \cdot p_{\rho_2}$.

The assumption of preparation independence is less relevant for quantum computation with magic states, where, in the language of resource theories, the free sector is formed by stabilizer states and stabilizer operations, not local states and local operations. Further, the memory lower bound of Karanjai, Wallman, and Bartlett [16] shows that a classical simulation algorithm like that of Algorithm 1 is incompatible with this assumption.

(v) Gleason's theorem. Gleason's theorem [50] says that in Hilbert spaces H of dimension 3 or greater, the only way to assign probabilities $p(h)$ to all subspaces of $h \subset H$, represented by corresponding projectors Π_h, is via $p(h) = \mathrm{Tr}(\Pi_h \rho)$ for some valid density matrix ρ.

That is, the only consistent way to assign probabilities to measurement outcomes is the Born rule involving density matrices. Our Theorem 1 does not contradict this; rather it reproduces the Born rule, cf. Eq. (4).

However, Gleason's theorem is sometimes interpreted as a mathematical proof that density operators are the fundamental notion of state in quantum mechanics. In short, density operators are for quantum mechanics, probability distributions for classical statistical mechanics. Theorem 1 escapes this interpretation. It shows that every quantum state *can* be described by a probability distribution, and yet the Born rule for measurement is reproduced. This is possible because we have restricted measurement to Pauli observables. Note though that this restriction does not affect the universality of quantum computation with magic states!

To summarize, in this Letter we have constructed a probability function over a finite set capable of positively representing all quantum states on any number of qubits, as well as their update under all Pauli measurements, local and nonlocal. All prior quasiprobability representations invoked in the discussion of quantum computation with magic states, such as the Wigner function for qudits [6,9] or for rebits [10], and the quasiprobability over stabilizer states [14], require negativity to represent universal quantum computation.

In view of the seeming classicality of the hidden variable model for universal quantum computation constructed here, an important open question is this: Where is quantumness hiding? In this regard, we propose the polytopes Λ_n and in particular the algebraic structure of their extremal vertices as a subject for further study.

This work is supported by NSERC. We thank Andreas Döring (R. R.) and Bill Unruh (M. Z., R. R.) for discussions. M. Z. and C. O. contributed equally to this work.

[1] A. Peres, *Quantum Theory: Concepts and Methods* (Springer, New York, 1995).
[2] R. P. Feynman, R. B. Leighton, and M. Sands, *Probability Amplitudes*, The Feynman Lectures on Physics Vol. 3 (Addison-Wesley, Redwood City, 1989).
[3] S. Bravyi and A. Kitaev, Phys. Rev. A **71**, 022316 (2005).
[4] J. J. Wallman and S. D. Bartlett, Phys. Rev. A **85**, 062121 (2012).
[5] H. Pashayan, J. J. Wallman, and S. D. Bartlett, Phys. Rev. Lett. **115**, 070501 (2015).
[6] D. Gross, Computational power of quantum many-body states and some results on discrete phase spaces, Ph.D. Thesis, Imperial College London, 2005.
[7] E. F. Galvão, Phys. Rev. A **71**, 042302 (2005).
[8] C. Cormick, E. F. Galvão, D. Gottesman, J. P. Paz, and A. O. Pittenger, Phys. Rev. A **73**, 012301 (2006).
[9] V. Veitch, C. Ferrie, D. Gross, and J. Emerson, New J. Phys. **14**, 113011 (2012).
[10] N. Delfosse, P. A. Guerin, J. Bian, and R. Raussendorf, Phys. Rev. X **5**, 021003 (2015).
[11] N. Delfosse, C. Okay, J. Bermejo-Vega, D. E. Browne, and R. Raussendorf, New J. Phys. **19**, 123024 (2017).
[12] R. Raussendorf, D. E. Browne, N. Delfosse, C. Okay, and J. Bermejo-Vega, Phys. Rev. A **95**, 052334 (2017).
[13] R. Raussendorf, J. Bermejo-Vega, E. Tyhurst, C. Okay, and M. Zurel, Phys. Rev. A **101**, 012350 (2020).
[14] M. Howard and E. T. Campbell, Phys. Rev. Lett. **118**, 090501 (2017).
[15] M. Heinrich and D. Gross, Quantum **3**, 132 (2019).
[16] A. Karanjai, J. J. Wallman, and S. D. Bartlett, arXiv:1802.07744v1.
[17] H. Zhu, Phys. Rev. Lett. **116**, 040501 (2016).
[18] L. Kocia and P. Love, Phys. Rev. A **96**, 062134 (2017).
[19] J. B. DeBrota, C. A. Fuchs, and B. C. Stacey, Phys. Rev. Research **2**, 013074 (2020).
[20] J. B. DeBrota and B. C. Stacey, Phys. Rev. A **102**, 032221 (2020).
[21] R. L. Hudson, Rep. Math. Phys. **6**, 249 (1974).
[22] A. Kenfack and K. Życzkowski, J. Opt. B **6**, 396 (2004).
[23] C. Ferrie, Rep. Prog. Phys. **74**, 116001 (2011).
[24] M. Pusey, J. Barrett, and T. Rudolph, Nat. Phys. **8**, 475 (2012).
[25] D. Gottesman, in *Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics* (International Press, Cambridge, MA, 1999), pp. 32–43.
[26] S. Aaronson and D. Gottesman, Phys. Rev. A **70**, 052328 (2004).
[27] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[28] A. Heimendahl, M.Sc. Thesis, University of Cologne, 2019.
[29] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.260404 for background on QCM, proof details and additional information.

[30] G. M. Ziegler, *Lectures on Polytopes* (Springer-Verlag, New York, 1995).
[31] V. Chvatal, *Linear Programming* (W. H. Freeman and Company, New York, 1983).
[32] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[33] J. Bell, Physics **1**, 195 (1964).
[34] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967), https://www.jstor.org/stable/24902153?seq=1.
[35] D. Bohm, Phys. Rev. **85**, 166 (1952).
[36] E. G. Beltrametti and S. Bugajski, J. Phys. A **29**, 247 (1996).
[37] A. Yao, in *Proceedings of the 34th Annual IEEE Symposium on Foundations of Computer Science* (IEEE, New York, 1993), pp. 352–361.
[38] G. Vidal, Phys. Rev. Lett. **91**, 147902 (2003).
[39] M. J. Bremner, R. Jozsa, and D. J. Shepherd, Proc. R. Soc. A **467**, 459 (2011).
[40] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Nature (London) **510**, 351 (2014).
[41] M. S. Leifer and O. J. E. Maroney, Phys. Rev. Lett. **110**, 120401 (2013).
[42] N. D. Mermin, Rev. Mod. Phys. **65**, 803 (1993).
[43] R. W. Spekkens, Phys. Rev. A **71**, 052108 (2005).
[44] R. Kunjwal and R. W. Spekkens, Phys. Rev. Lett. **115**, 110403 (2015).
[45] R. L. Stratonovich, Zh. Eksp. Teor. Fiz. **31**, 1012 (1956) [Sov. Phys. JETP **4**, 891 (1957)].
[46] C. Brif and A. Mann, J. Phys. A **31**, L9 (1998).
[47] E. Wigner, Phys. Rev. **40**, 749 (1932).
[48] W. M. Kirby and P. J. Love, Phys. Rev. Lett. **123**, 200501 (2019).
[49] N. Harrigan and R. W. Spekkens, Found. Phys. **40**, 125 (2010).
[50] A. M. Gleason, Indiana University mathematics Journal **6**, 885 (1957).