$SU(1,1)$ Echoes for Breathers in Quantum Gases

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Though the celebrated spin echoes have been widely used to reverse quantum dynamics, they are not applicable to systems whose constituents are beyond the control of the $su(2)$ algebra. Here, we design echoes to reverse quantum dynamics of breathers in three-dimensional unitary fermions and two-dimensional bosons and fermions with contact interactions, which are governed by an underlying $su(1,1)$ algebra. Geometrically, $SU(1,1)$ echoes produce closed trajectories on a single or multiple Poincaré disks and thus could recover any initial states without changing the sign of the Hamiltonian. In particular, the initial shape of a breather determines the superposition of trajectories on multiple Poincaré disks and whether the revival time has period multiplication. Our work provides physicists with a recipe to tailor collective excitations of interacting many-body systems.

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It is notoriously difficult to reverse quantum many-body dynamics, which requires changing the signs of all terms in the Hamiltonian simultaneously and reversing the dynamics of all particles in a synchronized means. Nevertheless, the well-established spin echoes [1] have been widely used to overcome dephasing in spin systems, laying the foundation of many modern technologies, ranging from the nuclear magnetic resonance to the central spin problem in condensed matter systems [2–4].

The study of collective excitations has been a main theme in ultracold atoms and related topics [5–9]. Breathing modes (or breathers) of interacting fermions and bosons have provided physicists with valuable information about superfluidity and hydrodynamics in the past two decades [10–15]. However, it is in general a grand challenge to recover the initial state once collective excitations are generated. The standard spin echoes do not apply to these breathers, whose relevant degrees of freedom do not obey the $su(2)$ algebra. A crucial question then arises. Could we reverse many-body dynamics of breathers in interacting bosons and fermions?

In this work, we implement the $SU(1,1)$ group to design echoes to reverse collective excitations of quantum gases. If the initial state is an eigenstate of a harmonic trap, $SU(1,1)$ echoes can be geometrized using a single Poincaré disk and guarantee that the initial state returns at $2nT$, where n is an integer and T is the period of repeated drivings. When the initial state is not an eigenstate of a harmonic trap, multiple Poincaré disks are required to describe the dynamics. The interference between trajectories on these Poincaré disks determines whether the revival time is $2T$ or longer, the latter corresponding to period multiplication. When incommensurate frequencies exist in the dynamics, the revival time extends to infinity. These results shed light on remarkable phenomena observed in a recent experiment by Dalibard's group at ENS [16].

Following the seminal work by Pitaevskii and Rosch [17], breathers in quantum gases have been extensively studied [18–26]. However, the fundamentally important role of initial shapes was not noted until the ENS experiment [16], which found that the period of a triangular breather agrees with well-known results in harmonic traps when the quantum anomaly is negligible. In sharp contrast, an initial disk shape leads to an unprecedented period multiplication, quadrupling that of a triangle. Such an observation is readily beyond understandings built upon previous works [17–26]. More strikingly, other shapes do not have regular periodicities in experimentally accessible timescales, though the underlying Hamiltonian of breathers naturally defines a period. This remarkable ENS experiment remains unexplained as of now. Here, we show that these extraordinary behaviors of breathers originate from an intrinsic property of representing the $SU(1,1)$ group. In particular, the underlying algebra and the geometric representation of $SU(1,1)$ echoes allow us to infer how initial shapes of breathers lead to distinct superpositions of Poincaré disks and consequently, the revival times.

Generators of $SU(1,1)$ satisfy

$$[K_1, K_2] = -iK_0, \quad [K_2, K_0] = iK_1, \quad [K_0, K_1] = iK_2. \qquad (1)$$

Geometrically, $SU(1,1)/U(1)$ corresponds to a Poincaré disk [27], where each point on the disk is an $SU(1,1)$ coherent state, as shown in Fig. 1. Such a coherent state characterized by a complex number $|z| < 1$ is written as

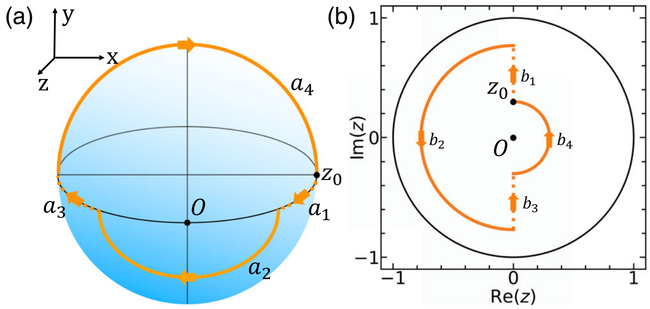

FIG. 1. (a) A spin echo on a Bloch sphere. z_0 denotes the initial state. a_1 and a_3 represent rotations about the y axis. a_2 and a_4 represent π pulses about the z axis. (b) An $SU(1,1)$ echo on a Poincaré disk. b_1 and b_3 represent boosts, which are induced by the same Hamiltonian H, along a radial direction. b_2 and b_4 represent π rotations about the origin.

$$|k,z\rangle = (1-|z|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{\Gamma(n+1)\Gamma(2k)}} z^n |k,n\rangle, \quad (2)$$

where $\Gamma(x)$ is the gamma function. k is determined by the Casimir operator $C = K_0^2 - K_1^2 - K_2^2$, $C|k,n\rangle = k(k-1)|k,n\rangle$. A single Poincaré disk is characterized by a unique k. n is obtained from $K_0|k,n\rangle = (k+n)|k,n\rangle$.

$SU(1,1)$ has been widely applied in multiple disciplines [28–40]. However, it has not been implemented to study echoes until very recently. We have found that the dynamical instability of a BEC induced by quenching the scattering length, which corresponds to a particular realization of the $SU(1,1)$ group, could be reversed by a family of $SU(1,1)$ echoes [41]. The same representation of the $SU(1,1)$ group has also been considered for studying periodically driven BECs in Refs. [42,43]. In such a particular representation [41–43], k is either a positive integer or half integer. This is similar to spin systems, whose Casimir operator is equivalent to the angular momentum. In both cases, the integral or half-integral k guarantees an echo has a single period. In sharp contrast, breathers considered here correspond to a distinct representation that has a continuous spectrum of k. Such fundamental difference provides breathers with much richer phenomena ranging from an arbitrary multiplication of the period to dynamics with noncommensurate frequencies.

K_0 is the Hamiltonian of trapped BECs [17],

$$K_0 = \frac{1}{2}\left[\sum_i -\frac{1}{2}\nabla_i^2 + \frac{1}{2}r_i^2 + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j)\right],$$
$$K_1 = \frac{1}{2}\left[\sum_i -\frac{1}{2}\nabla_i^2 - \frac{1}{2}r_i^2 + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j)\right],$$
$$K_2 = \frac{1}{4i}\sum_i (\mathbf{r}_i \cdot \nabla_i + \nabla_i \cdot \mathbf{r}_i). \quad (3)$$

We have chosen $l_{ho} = \sqrt{\hbar/(m\omega_0)}$ as the unit length and $\hbar\omega_0$ as the unit energy. In three dimensions, both noninteracting systems and unitary fermions satisfy the commutators in Eq. (1). In the latter case, $V(\mathbf{r}_i - \mathbf{r}_j)$ should be understood as $V(\mathbf{r}_i - \mathbf{r}_j) = \tilde{V}(\mathbf{r}_i - \mathbf{r}_j)\delta_{\sigma_i \neq \sigma_j}$, where $\sigma_i = \uparrow, \downarrow$, and produces a divergent scattering length. In two dimensions, fermions and bosons with contact interactions, $V(\mathbf{r}_i - \mathbf{r}_j) \sim g\delta(\mathbf{r}_i - \mathbf{r}_j)$, also have the $SU(1,1)$ symmetry, as $\delta_{2D}(\lambda\mathbf{r}) = \lambda^{-2}\delta_{2D}(\mathbf{r})$. In single-component bosons, interactions exist between any pair of particles.

$SU(1,1)$ echoes arise from the identity,

$$e^{-i(\varphi_1 K_1 + \varphi_2 K_2)} e^{-i\pi K_0} e^{-i(\varphi_1 K_1 + \varphi_2 K_2)} e^{i\pi K_0} = \mathcal{I}, \quad (4)$$

where $\mathcal{I}$ is the identity operator, φ_1 and φ_2 are two arbitrary real numbers. On the Poincaré disk, $e^{-i\pi K_0}$ is a rotation of π about the origin, and $e^{-i(\varphi_1 K_1 + \varphi_2 K_2)}$ is a boost changing $|z|$. A simple echo is illustrated in Fig. 1(b). Starting from a given initial state, $\mathcal{U}_1 = e^{-i\varphi_1 K_1}$ moves it along a diameter and is followed by a rotation $\mathcal{U}_0 = e^{-i\pi K_0}$. Using Eq. (4), we conclude $(\mathcal{U}_0\mathcal{U}_1)^2 = e^{-i2\pi K_0}$, i.e., a rotation of 2π about the origin, and thus the initial state is recovered. This echo applies to any initial states on the Poincaré disk and any $\varphi_1 K_1 + \varphi_2 K_2$.

Whereas our results apply to any eigenstates of a harmonic trap, we first choose the ground state of the Hamiltonian, $H_0 = 2K_0$, as the initial state as an example to demonstrate our scheme. To implement an $SU(1,1)$ echo, the trapping frequency is suddenly changed to $\omega_1 = \kappa\omega_0$ at $t = 0$, where κ is an arbitrary real or imaginary number. In the latter case, it corresponds to an inverted harmonic trap. When $t = t_1$, the original harmonic trap is restored and the system evolves for another time period t_0. Then the above two steps are repeated. Such dynamics are governed by the Hamiltonians

$$H_1 = (1+\kappa^2)K_0 + (1-\kappa^2)K_1, \qquad nT < t < nT + t_1,$$
$$H_0 = 2K_0, \qquad nT + t_1 < t < (n+1)T, \quad (5)$$

where n is a non-negative integer, and $T = t_0 + t_1$ defines a period. The propagator, $(\mathcal{U}_0\mathcal{U}_1)^2 = (e^{-iH_0 t_0} e^{-iH_1 t_1})^2$, from $t = nT$ to $t = (n+2)T$ can be rewritten as

$$e^{-i(\zeta_1 + 2t_0)K_0} e^{-i\eta_1 K_1} e^{-i(2\zeta_1 + 2t_0)K_0} e^{-i\eta_1 K_1} e^{-i\zeta_1 K_0}, \quad (6)$$

where $\zeta_1 = \arctan\{[(1+\kappa^2)/2\kappa]\tan\kappa t_1\}$ and $\eta_1 = 2\mathrm{arcsinh}\{[(1-\kappa^2)/2\kappa]\sin(\kappa t_1)\}$. We have used the Baker-Campbell-Hausdorff decomposition,

$$e^{-i(\xi_0 K_0 + \xi_1 K_1 + \xi_2 K_2)} = e^{-i\zeta K_0} e^{-i\eta(K_1\cos\phi + K_2\sin\phi)} e^{-i\zeta K_0}, \quad (7)$$

where $\tan\zeta = [(\xi_0/\xi)\tan(\xi/2)]$, $\cos\phi = [\xi_1/(\sqrt{\xi_1^2 + \xi_2^2})]$, $\sinh(\eta/2) = [(\sqrt{\xi_1^2 + \xi_2^2})/\xi]\sin(\xi/2)$, and $\xi^2 = \xi_0^2 - \xi_1^2 - \xi_2^2$. To deliver an echo, it is required that $\pi = 2(t_0 + \zeta_1)$, or, equivalently,

$$t_0 = \frac{\pi}{2} - \zeta_1 = \frac{\pi}{2} - \arctan\left(\frac{1+\kappa^2}{2\kappa}\tan\kappa t_1\right). \tag{8}$$

Under this condition, $(\mathcal{U}_0\mathcal{U}_1)^2 = e^{-i2\pi K_0}$. As any eigenstate of a harmonic trap is also an eigenstate of C with an eigenvalue $k(k-1)$, only an extra overall phase shows up and the system returns to the initial state after two driving periods. Once H_0 and H_1 are fixed, tuning t_0 to deliver an $SU(1,1)$ echo is an analog of adjusting the duration of the pulse to create a π rotation on a Bloch sphere in spin echoes. There also exist echoes allowing the initial state to return in a longer time, say $t = 3nT$ (Supplemental Material [44]).

The expectation value of the potential energy, $E_{\rm pot} = \langle\frac{1}{2}\sum_i r_i^2\rangle$ in the time interval $nT + t_1 < t < (n+1)T$ can be written as $E_{\rm pot} = \langle K_0 - K_1\rangle$. Using properties of $SU(1,1)$ coherent states, $\langle k,z|K_0|k,z\rangle = k[(1+|z|^2)/(1-|z|^2)]$, $\langle k,z|K_1|k,z\rangle = 2k\{\mathrm{Re}(z)/[1-|z|^2]\}$, we obtain

$$E_{\rm pot} = k[1+|z|^2 - 2\,\mathrm{Re}(z)]/(1-|z|^2). \tag{9}$$

$E_{\rm pot}$ in the time interval $nT < t < nT + t_1$ simply multiplies the above equation by κ^2. Apparently, $E_{\rm pot}$ is periodic with period $2T$. For the system prepared in the ground state of H_0 with ground state energy E_g, we have $k = E_g/2$. Results above are valid for any eigenstates of the initial Hamiltonian, hold for any finite temperatures at thermal equilibrium, and k in Eq. (9) should be understood as $\langle K_0\rangle_{\rm thermal} = \mathrm{Tr}(K_0 e^{-\beta H_0})/\mathrm{Tr}(e^{-\beta H_0})$, where β is the inverse temperature.

It is useful to consider 2D bosons as an example. Whereas it is difficult to compute the exact many-body state in the quantum dynamics controlled by the $su(1,1)$ algebra, in the weakly interacting regime, such dynamics is well captured by a Gross-Pitaevskii (GP) equation,

$$i\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\nabla^2}{2} + \frac{\kappa(t)^2 r^2}{2} + gN|\Psi(\mathbf{r},t)|^2\right)\Psi(\mathbf{r},t), \tag{10}$$

where N is the number of bosons, $g = 4\pi a_0$ with a_0 being the dimensionless scattering length. We use an imaginary time evolution to obtain the ground state of the initial Hamiltonian, H_0. We then let the condensate evolve based on the GP equation, in which the Hamiltonian is determined by Eq. (5). We trace both the overlap between $\Psi(\mathbf{r},t)$ and $\Psi(\mathbf{r},0)$, $F(t) = |\int d\mathbf{r}\Psi^*(\mathbf{r},0)\Psi(\mathbf{r},t)|$, and the absolute value of the potential energy, $|E_{\rm pot}|$. Figure 2 shows a few typical choices. (I), κ is real, corresponding to a harmonic trap whose frequency could be different from the initial one. (II), $\kappa = 0$, corresponding to turning off the harmonic trap. (III), κ is purely imaginary, meaning an inverted harmonic trap. For a generic $H = \sum_{i=0,1,2}\xi_i K_i$, $\vec{\xi} = \{\xi_0, \xi_1, \xi_2\}$ defines an external field with a strength, $\xi^2 = \xi_0^2 - \xi_1^2 - \xi_2^2$. For instance, in Eq. (5), we have $\xi = 2\kappa$. In (I), $\xi^2 > 0$, and the system follows a closed loop on the Poincaré disk. In (II), ξ vanishes. Without a confining potential in the real space, the trajectory on the Poincaré disk eventually becomes tangent with the boundary circle. In (III), ξ becomes purely imaginary. While a deconfining potential pushes BECs to expand in the real space, on the Poincaré disk, the trajectory becomes an open

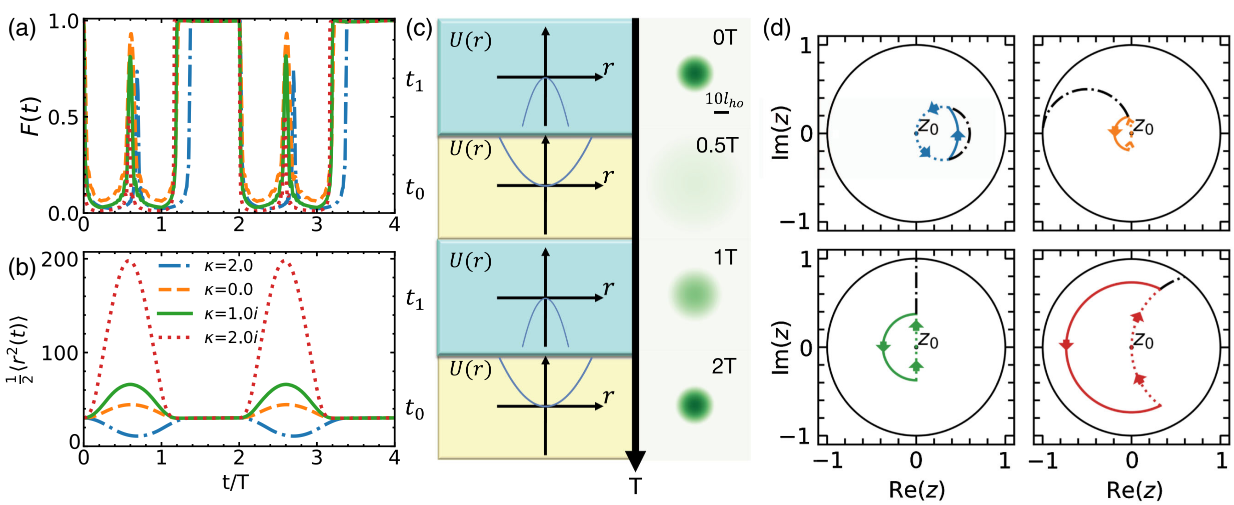

FIG. 2. (a)–(b) $F(t)$ and $\langle r^2(t)\rangle/2$ of 2D BECs. The initial state is the ground state of K_0. $\kappa = 2$, 0 correspond to a modified and vanishing harmonic trap in the time interval, $nT < t < nT + t_1$, respectively. $\kappa = i, 2i$ correspond to inverted harmonic traps. $Ng = 25600$, $\omega_0 = 20 \times 2\pi$Hz and $t_1 = \pi/8$. t_0 is determined by Eq. (8). (c) Left panel: harmonic traps in different time intervals. Right panel: snapshots of densities at different times for $\kappa = 2i$. (d) Trajectories on the Poincaré disk. Dotted and solid lines are evolutions governed by H_1 and H_0, respectively. Dot-dashed lines show the trajectories if only H_1 is applied.

path. Though quantum dynamics governed by H_1 alone in (I)–(III) are distinct, $SU(1,1)$ echoes always lead to revivals. Figure 2 clearly shows that both $F(t)$ and $|E_{\mathrm{pot}}|$ are periodic functions of t with a period of $2T$.

The initial state could also be a superposition of multiple eigenstates of C such that multiple Poincaré disks are required. We consider an arbitrary propagator $\mathcal{U}$ in the $SU(1,1)$ group acting on $|\Psi\rangle = \sum_{n,k} c_{nk}|k,n\rangle = \sum_k |\psi_k\rangle$, where $|\psi_k\rangle = \sum_n c_{nk}|k,n\rangle$ and $\langle\psi_{k'}|\psi_k\rangle \sim \delta_{k,k'}$. Here we have suppressed other quantum numbers for the same k, n. As $\mathcal{U}|\Psi\rangle = \sum_k \mathcal{U}|\psi_k\rangle$, and $\mathcal{U}|\psi_k\rangle$ corresponds to an evolution on a single Poincaré disk, the dynamics thus correspond to superpositions of trajectories on multiple Poincaré disks. If an echo, $(\mathcal{U}_0\mathcal{U}_1)^2 = e^{-2i\pi K_0}$, acts on the initial state for m times, where m is an integer, we obtain

$$e^{-2i\pi m K_0}|\Psi\rangle = \sum_{n,k} c_{nk} e^{-2i\pi m k}|k,n\rangle. \tag{11}$$

Replacing the sum in Eq. (11) by an integral over k, this initial state shall, in general, include incommensurate k's and thus lacks a finite periodicity. Here we consider systems with well-defined periodicities and apply summation of discrete k's. Since $e^{-2i\pi m k}$ is independent of n, the return probability $P(m) = |\langle\Psi(0)|\Psi(2mT)\rangle|^2$ becomes $P(m) = |\sum_k \tilde{P}_k e^{-2i\pi m k}|^2$, where $\tilde{P}_k = \sum_n |c_{nk}|^2$, $\sum_k \tilde{P}_k = 1$. It is apparent that $P(m) = 1$ only if $e^{-2i\pi m k} = e^{i\phi_0}$ for all k's with a nonzero c_{nk}, where $\phi_0 \in [0, 2\pi)$ is independent of k. This is certainly satisfied if the initial state includes a single state, $|k_0, n_0\rangle$. It is also clear that $e^{-2i\pi m k} = e^{i\phi_0}$ will never be satisfied if $|\Psi\rangle$ includes incommensurate k's. Whereas such a scenario is impossible in previous works [41–43], in breathers with a continuous spectrum of k, dynamics controlled by incommensurate k's may arise.

If k's in Eq. (11) are commensurate, i.e., all k's are represented by $k = k_0 + p/Q$, where k_0 is a given reference with a nonzero c_{nk_0}, $p \in \mathbb{Z}$, $Q \in \mathbb{N}_+$, and p and Q are co-prime numbers, we have $P(Q) = 1$, and the system evolves back to its initial state after $2Q$ periods. Therefore, different superpositions of $|k,n\rangle$ in the initial state may lead to distinct revival times after $SU(1,1)$ echoes are applied. If $Q > 1$, period multiplication emerges in the dynamics. Figure 3 shows examples corresponding to $Q = 1$ and $Q = 4$. Consequently, revival times are $2T$ and $8T$, i.e., period quadruples in the latter case.

Applying the above analysis to breathers of different initial shapes, we observe that the triangle and the disk correspond to $Q = 1$ and $Q = 4$, respectively. The initial state is chosen as the ground state of a flat-box potential with an infinite potential wall. Such an initial state is no longer an eigenstate of K_0, and it is useful to fully implement the $su(1,1)$ algebra to consider the dynamics. After turning off the flat-box potential, the system evolves based on Eq. (5). $F(t)$ of a triangle satisfies $F(t) = F(t + 2T)$. As the initial state is not an eigenstate of K_0, it must be a superposition of multiple $|k,n\rangle$ with differences between k's being integers, i.e., $Q = 1$ as shown in Fig. 3(a). The exact number of Poincaré disks can be, in principle, determined by considering a particular Hamiltonian, $\tilde{H} \equiv C$, $e^{-i\tilde{H}t}|\Psi\rangle = \sum_{nk} c_{nk} e^{-ik(k-1)t}|n,k\rangle = \sum_k e^{-ik(k-1)t}|\psi_k\rangle$. A Fourier transform of $F(t) = \sum_k e^{-ik(k-1)t}\langle\psi_k|\psi_k\rangle$ to the frequency space unfolds how many k's are involved and their corresponding weights.

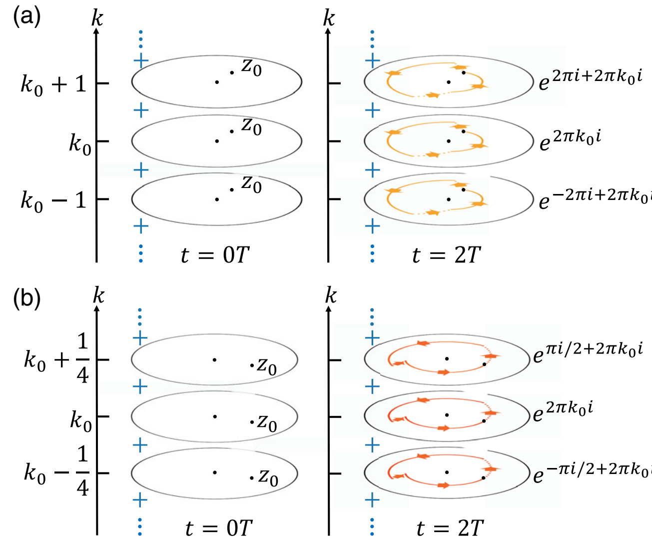

FIG. 3. (a) At $t = 2T$, trajectories on different disks accumulate the same phase. The system returns to the initial state. (b) Trajectories on different disks acquire relative phases. It takes the system $8T$ to return to the initial state.

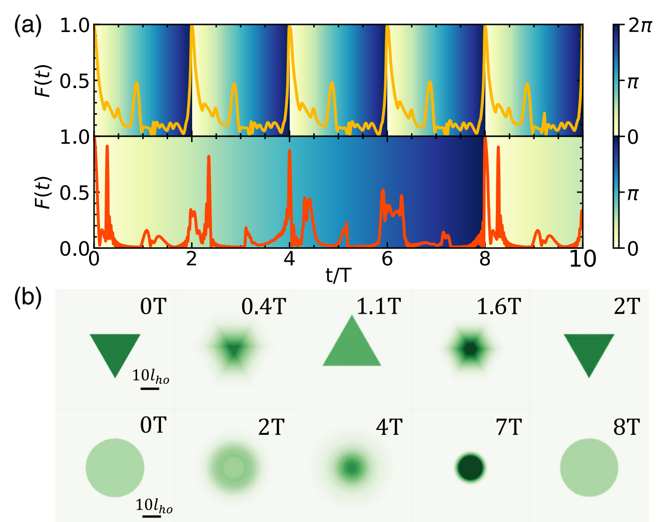

FIG. 4. (a) $F(t)$ of breathers with an initial triangular (top) and disk (bottom) shapes, respectively. $\omega_0 = 40 \times 2\pi$ Hz, $t_1 = \pi/8$, and $\kappa = 0.5i$. $Ng = 25600$ (12800) is used for the triangle (disk). The background color represents the time-dependent relative phase between different Poincaré disk. (b) Density distributions of BECs at different times.

Nevertheless, such calculations are not essential here, since our echoes apply to any superpositions in Eq. (11), regardless of the exact number of Poincaré disks involved.

The results of a disk's shape are distinct. Figure 4 shows that the revival time of the disk is $8T$. We conclude that the superposition in the initial state must be similar to Fig. 3(b). The quench dynamics in the ENS experiment has a propagator, $e^{-iK_0 t}$, corresponding to $SU(1,1)$ echoes where $t_1 = 0$. Such quench dynamics has a periodicity of $2T$ and $8T$ for the triangle and the disk, respectively [16]. This also confirms that the triangle and the disk corresponds to a superposition of multiple Poincaré disks with $Q = 1$ and $Q = 4$, respectively. We have not found other shapes, such as a square, which return to the initial states within timescales of our numerical simulations, similar to results of the quench dynamics [16]. We conclude that these shapes are described by either incommensurate k's or commensurate k's corresponding to a very large Q, which lead to revival times not observable in relevant timescales of numerics and experiments.

In experiments, it is the exact many-body state that evolves under the control of $SU(1,1)$ echoes. Results of the GP equation are expected to provide us with a good approximation in the weakly interacting limit. Nevertheless, the precise form of the many-body state corresponding to a given initial shape of the breather remains an interesting open question worthy of future studies. In contrast, c_{nk} can be straightforwardly obtained in few-body systems. For instance, in a two-body problem, eigenvalues of the Casimir operator are directly related to the angular momenta such that the initial shape of the breather allows one to directly predict the revival time (Supplemental Material [44]). Similar to spin echoes, $SU(1,1)$ echoes could be implemented to detect symmetry breaking perturbations, such as an extra external potential in experiments (Supplemental Material [44]).

Our results are obtained by an algebraic method independent on the representation and apply to any systems with the $SU(1,1)$ symmetry. We hope that our work will stimulate more interest from different disciplines to use geometric approaches to control quantum dynamics in few-body and many-body systems.

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[44] Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.253002 for discussions on the echoes with periods of $3T$, the relation between initial shapes and revival times of two unitary fermions, and the detection of the symmetry breaking perturbations using $SU(1,1)$ echoes.