Anomalous Diffusion in Dipole- and Higher-Moment-Conserving Systems

Johannes Feldmeier,[1,2] Pablo Sala,[1,2] Giuseppe De Tomasi,[3] Frank Pollmann,[1,2] and Michael Knap[1,2]
[1]*Department of Physics and Institute for Advanced Study, Technical University of Munich, 85748 Garching, Germany*
[2]*Munich Center for Quantum Science and Technology (MCQST), Schellingstraße 4, D-80799 München, Germany*
[3]*T.C.M. Group, Cavendish Laboratory, J. J. Thomson Avenue, Cambridge CB3 0HE, United Kingdom*

(Received 7 April 2020; accepted 10 November 2020; published 10 December 2020)

The presence of global conserved quantities in interacting systems generically leads to diffusive transport at late times. Here, we show that systems conserving the dipole moment of an associated global charge, or even higher-moment generalizations thereof, escape this scenario, displaying subdiffusive decay instead. Modeling the time evolution as cellular automata for specific cases of dipole- and quadrupole conservation, we numerically find distinct anomalous exponents of the late time relaxation. We explain these findings by analytically constructing a general hydrodynamic model that results in a series of exponents depending on the number of conserved moments, yielding an accurate description of the scaling form of charge correlation functions. We analyze the spatial profile of the correlations and discuss potential experimentally relevant signatures of higher-moment conservation.

DOI: 10.1103/PhysRevLett.125.245303

Introduction.—Thermal equilibrium of generic systems is characterized by a finite number of conserved quantities, such as energy, particle number, or charge. A coarse grained nonequilibrium time evolution that irreversibly leads to such an equilibrium state is therefore expected to be dominated by transport of these quantities at late times, smoothing out inhomogeneities of the initial state [1]. This framework also extends to closed, interacting quantum systems, where the corresponding timescale of transport, separating "early" and "late" times, is usually marked by the onset of local thermalization. A phenomenological description of the ensuing dynamics can then be given in terms of a classical hydrodynamic description, with quantum properties merely entering the effective diffusion constant [2–10].

A particularly intriguing set of conserved quantities arises in the presence of a global U(1) charge together with the conservation of one or several of its higher moments, such as dipole or quadrupole moment. These *higher-moment* conserving models have attracted much attention in the context of fractons [11–18], and are realizable in synthetic quantum matter [19] or solid state systems [20,21]. In particular, the intertwined relation between the internal charge conservation law and its dipole moment has recently been shown to have significant impact on nonequilibrium properties [22–29]. In the most severe cases, for short-ranged interactions and low spin representations, the system fails to thermalize due to a strong fragmentation of the many-body Hilbert space into exponentially many disconnected sectors [23,24], giving rise to statistically localized integrals of motion [29].

In this Letter, we study late time transport in ergodic models of dipole- and even higher-moment conserving 1D systems. We avoid strong Hilbert space fragmentation by including longer-range interactions and higher spin representations. Building on the intuition of an effective classical description in the regime of incoherent transport, we employ a cellular automaton approach to numerically study the longtime dynamics (see, e.g., [30–32] for related approaches). We find anomalously slow, subdiffusive transport of the underlying charges, described by a cascade of exponents depending on the highest conserved moment. We further develop a general analytic hydrodynamic approach, valid for arbitrary conserved multipole moments that is in full agreement with our numerical results. Moreover, we discuss experimental characteristics of higher-moment conservation and the consistency of our findings with quantum dynamics.

Higher-moment conserving models.—We start by constructing generic Hamiltonians conserving arbitrary moments of the charge. These will serve as input for the definition of suitable automata dynamics, as well as the derivation of an effective hydrodynamic description. The construction of such models is best understood recursively, starting from a simple Hamiltonian of the form $\hat{H} = \hat{H}^{(0)}_{r=2} + \hat{H}_z$, with $\hat{H}^{(0)}_{r=2} = \sum_x (\hat{S}^+_x \hat{S}^-_{x+1} + \text{H.c.})$ hosting local XY-type terms of range $r = 2$ that conserve the total charge $Q^{(0)} = \sum_x \hat{S}^z_x$, and $\hat{H}_z$ containing arbitrary local terms diagonal in the $\hat{S}^z$ basis that render the model nonintegrable. The $\hat{S}^\pm_x$, $\hat{S}^z_x$ are spin operators in a given representation S. Here, an elementary term $h^{(0)}_2(x) \equiv \hat{S}^+_x \hat{S}^-_{x+1}$ can be interpreted as the creation of a dipole against some background. A new term that additionally conserves the dipole moment $Q^{(1)} = \sum_x x\hat{S}^z_x$ can then be obtained by simply multiplying this operator with its Hermitian conjugate

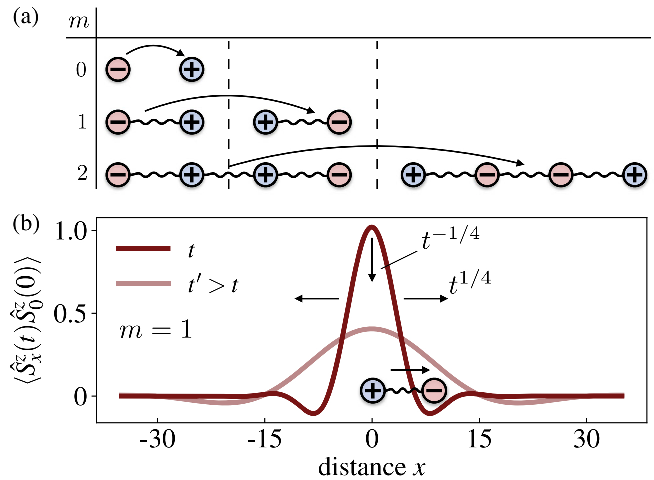

FIG. 1. Higher moment conservation laws. (a) Recursive construction: A charge-conserving move ($m = 0$) creates a local dipole, which in turn is a charge-neutral dynamical object of a dipole-conserving model ($m = 1$). This process is iterated to conserve higher moments. (b) The late time dynamics of charges exhibits subdiffusive decay, with algebraic exponents depending on the highest conserved charge moment (here, $m = 1$).

at some shifted position, e.g., $h_3^{(1)}(x) = [h_2^{(0)}(x)]^\dagger h_2^{(0)}(x+1)$, yielding $\hat{H}_3^{(1)} = \sum_x \hat{S}_x^-(\hat{S}_{x+1}^+)^2\hat{S}_{x+2}^- + \text{H.c.}$, a model that has been studied in the context of Hilbert space fragmentation [23–29].

The above recursion can be iterated to obtain models conserving arbitrary moments of the charge

$$Q^{(m)} = \sum_x x^m \hat{S}_x^z. \tag{1}$$

Formally, we consider an mth-moment conserving Hamiltonian of range r in the form $\hat{H}_r^{(m)} = \sum_x h_r^{(m)}(x) + \text{H.c.}$, whose local terms can be expanded as

$$h_r^{(m)}(x) = \bigotimes_{i=0}^{r-1} (\hat{S}_{x+i}^{\text{sgn}[\sigma_m(i)]})^{|\sigma_m(i)|}, \quad \text{with} \quad \sigma_m(i) \in \mathbb{Z}, \tag{2}$$

where by definition $\sigma_m(0) \neq 0$, $\sigma_m(r-1) \neq 0$, and $\text{sgn}[\cdot] \in \{+,-\}$ is the signum function. For the XY terms, $\sigma_0(0) = -\sigma_0(1) = 1$. Again, arbitrary terms diagonal in $\hat{S}^z$ could be added to Eq. (2) without affecting the conservation laws. Analogous to the argument above, given $h_r^{(m-1)}(x)$, we can then construct a $(r+\ell)$-range term that additionally conserves the mth moment by imposing the recursive relation

$$\sigma_m(i) = -\sigma_{m-1}(i) + \sigma_{m-1}(i-\ell), \tag{3}$$

on the exponents of the spin ladder operators. Equation (3) reflects the construction of $\hat{H}_{r+\ell}^{(m)}$ via shifting an elementary *m pole* by ℓ sites. As illustrated in Fig. 1(a), the elementary *m pole* configurations have vanishing lower moments and a spatially independent mth moment, similar to usual charges. However, their number is not conserved.

We notice that Eq. (3) can be rephrased as a discrete lattice derivative of spacing ℓ, $\sigma_m(i) = -\Delta_x[\sigma_{m-1}](i)$, which implies $\sigma_m(i) = (-\Delta_x)^m[\sigma_0](i)$. If the elementary XY terms are interpreted as a finite difference with spacing $\ell = 1$, $(-\Delta_x)[f](0) = \sum_i \sigma_0(i)f(i)$ with some lattice function $f(i)$, the exponents $\sigma_m(i)$ effectively correspond to a lattice discretization of the $(m+1)^{st}$ derivative

$$(-\Delta_x)^{m+1}[f](0) = \sum_i \sigma_m(i)f(i). \tag{4}$$

Using the spin commutation relations and Eq. (4), we see that $[Q^{(n)}, h_r^{(m)}(x)] = \sum_i \sigma_m(i)(x+i)^n = (-\Delta_x)^{m+1}[x^n] = 0$ for $n \leq m$, i.e., all moments $Q^{(n\leq m)}$ are indeed conserved. The same holds for longer-range Hamiltonians, using alternative discretization schemes of the involved derivatives. We note that this is a discretized version of the field theory construction in Ref. [16].

Cellular automaton approach.—Studying the full quantum evolution of Eq. (2) is challenging. However, we can make progress by considering a classical automaton time evolution that respects the same conservation laws of Eq. (1), which are the crucial properties concerning the late time dynamics. The discrete automaton evolution consists of a sequential application of local updates mapping z-basis product states onto z-basis product states. The updates are designed to mimic the action of the Hamiltonian by updating between local strings of spins $\mathbf{s}(x) = (s_x, \ldots, s_{x+r-1})$, with $s_i \in \{-S, \ldots, S\}$, that are connected by the local terms of Eq. (2) and thus feature the same conserved charge moments, similar to Refs. [30,32]. The time evolution can then be represented by a classically simulable circuit, see Fig. 2(a). Furthermore, by assigning a finite acceptance probability to each update, we simulate a *stochastic* automaton [33], enabling us to meaningfully study dynamics starting from fixed initial states. More details can be found in the Supplemental Material [34]. We emphasize that the details of the implementation, including its stochastic nature, are *not* essential to the hydrodynamics studied in the following.

Hydrodynamics.—To understand the dynamics of charges at late times, the main quantity studied within our numerical approach are the (infinite temperature) correlation functions

$$C^{(m)}(x,t) = \langle \hat{S}_x^z(t)\hat{S}_0^z(0)\rangle, \tag{5}$$

and particularly the return probability $C^{(m)}(0,t)$, where $\langle \ldots \rangle$ denotes an average of $\hat{S}_x^z(t)\hat{S}_0^z(0)$ over randomly chosen initial states of the automaton dynamics. For concreteness, we study a dipole conserving model with $S = 1$ including interaction terms of Eq. (2) of ranges $r = 3$ and $r = 4$ [38],

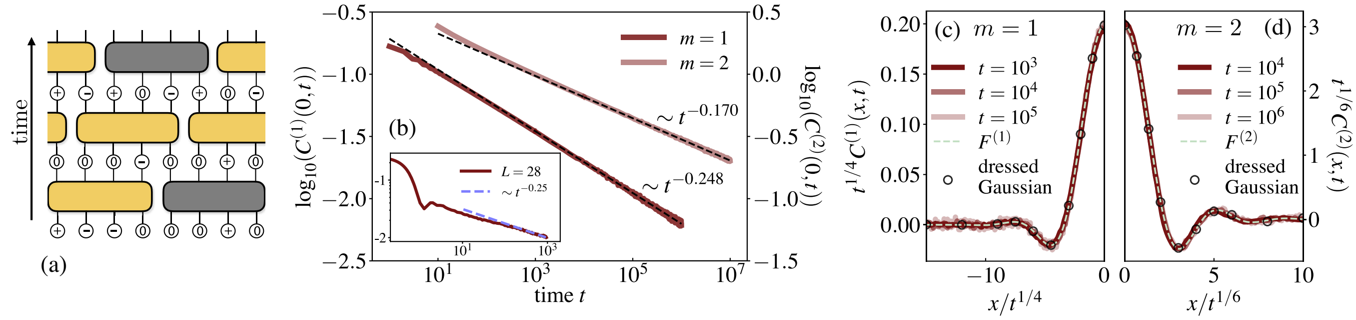

FIG. 2. Hydrodynamics. (a) Illustration of an automaton circuit, for $m = 1$ using dipole-conserving updates of range four between product states (spin representation $S = 1$). With some finite probability, updates are either applied (yellow gates) not (gray gates), yielding effectively stochastic updates. (b) Return probability $C^{(m)}(0,t)$ for dipole- and quadrupole conservation. The longtime behavior approaches an m-dependent algebraic decay $\sim t^{-1/2(m+1)}$. The numerical values of the exponents where extracted from fits over the latter three time decades (dashed lines). Inset: return probability of a small system size, dipole-conserving quantum model (spin $S = 1/2$), consistent with subdiffusive decay. (c),(d) Scaling collapse for $m = 1$ and $m = 2$ according to the long wavelength description Eq. (8). In addition to the numerical data, the fundamental solution of Eq. (8) (dashed line) and the corresponding Gaussian expansion (see main text) up to order $n = 4$ (circles) are shown. The system size is $L = 10^4$ and correlations were averaged over at least 10^3 random initial states in all panels.

as well as a quadrupole conserving model with larger spin representation $S = 4$ including ranges $r = 4$ and $r = 5$, both on system sizes up to $L = 10^4$. While generally the hydrodynamic tails for given m are expected to be universal for sufficiently ergodic systems, in numerical practice, larger r and S will provide faster convergence to this longtime behavior. As the dynamics is expected to become slower upon increasing m, i.e., the number of constraints, we choose a larger spin representation $S = 4$ in the quadrupole case to allow for a more accurate determination of algebraic exponents.

We show the results of the automaton evolution for the return probability $C^{(m)}(0,t)$ in Fig. 2(b). In contrast to ordinary diffusive systems which possess the scaling $C^{(0)}(0,t) \sim t^{-1/2}$ [1], we numerically estimate the algebraic late time decay exponents to be $C^{(1)}(0,t) \sim t^{-0.248} \approx t^{-1/4}$ in the dipole-conserving model, and $C^{(2)}(0,t) \sim t^{-0.170} \approx t^{-1/6}$ for the conservation of quadrupole moment. Consistent with these results, we observe a subdiffusive decay of $C^{(1)}(0,t)$ for a dipole-conserving $S = 1/2$ quantum spin chain as shown in the inset of Fig. 2(b) (see the Supplemental Material [34] for more details on the quantum case). This signals significant deviations from generic diffusion, induced by the higher-moment conservation laws Eq. (1).

To understand how this slow anomalous diffusion can emerge from a classical hydrodynamic description of a quantum evolution, we consider the Heisenberg evolution equation of the charge density $\hat{S}_x^z$ for the previously introduced models. This yields $(d/dt)\hat{S}_x^z = (i/\hbar)[\hat{H}_r^{(m)}, \hat{S}_x^z] = (-\Delta_x)^{m+1}\Omega_x^{(m)}$ using Eq. (4), with $\Omega_x^{(m)} = -(i/\hbar)[h_r^{(m)}(x) -$ H.c.], which takes the form of a generalized "multipole current" of m poles (in fact, this is a one-dimensional version of generalized currents appearing in fractonic systems [39]). This form of the time evolution applies to arbitrary Hamiltonians conserving the mth moment of the charge, with microscopic details only entering $\Omega_x^{(m)}$. Imposing a continuity equation for the charge density $(d/dt)\hat{S}_x^z = (-\Delta_x)J_x^{(m)}$, we obtain the form of the charge current $J_x^{(m)} = (-\Delta_x)^m \Omega_x^{(m)}$, resulting, e.g., in the familiar $J_x^{(0)} = \Omega_x^{(0)}$ for the diffusive case. To arrive at a differential equation, we consider the evolution of expectation values and go to the limit of long wavelengths ($\Delta_x \to \partial_x$) assuming large enough variation lengths in space, such that $(d/dt)\langle \hat{S}_x^z \rangle = (-\partial_x)^{m+1} \langle \Omega_x^{(m)} \rangle$.

To obtain a closed equation for the now coarse-grained charge density $\langle \hat{S}_x^z \rangle = \langle \hat{S}_x^z \rangle(t)$, we require a hydrodynamic assumption which relates the multipole current to the derivatives of the charge density (see, e.g., Ref. [40]). We therefore expand $\langle \Omega_x^{(m)} \rangle = -D(\partial_x)^{l(m)} \langle \hat{S}_x^z \rangle$, and our task is to find the *lowest* possible (i.e., most scaling relevant) $l(m) \in \mathbb{N}$ such that $D \neq 0$ is consistent with the conservation of all moments $Q^{(n \leq m)}$ (we provide a scaling analysis in the Supplemental Material [34] that shows that nonlinear terms in the expansion of $\langle \Omega_x^{(m)} \rangle$ are irrelevant).

For charge-conserving interacting quantum systems ($m = 0$), known to generically exhibit diffusive transport at late times [2–10], we should obtain Fick's law $\langle \Omega_x^{(0)} \rangle = \langle J_x^{(0)} \rangle = -D\partial_x \langle \hat{S}_x^z \rangle$, i.e., $l(0) = 1$, resulting in the usual diffusion equation for $\langle \hat{S}_x^z \rangle$. However, general solutions of the diffusion equation break higher-moment conservation. This is seen most easy for the example of a melting domain wall, which exhibits a net current of charge, violating dipole conservation.

How can we thus generalize Fick's law to higher conserved moments? We notice that in a closed system with open boundary conditions and in the absence of sinks or sources, the current $\langle\Omega_x^{(m)}\rangle_{\text{eq}} = \langle\Omega_x^{(m)}\rangle(t\to\infty) = 0$ is expected to vanish in equilibrium. Combining this condition with $\langle\Omega_x^{(m)}\rangle = -D(\partial_x)^{l(m)}\langle\hat{S}_x^z\rangle$ leads to the equilibrium charge distribution

$$\langle\hat{S}_x^z\rangle_{\text{eq}} = a_0 + a_1 x + \cdots + a_{l(m)-1}x^{l(m)-1} = \sum_{s=0}^{l(m)-1} a_s x^s. \qquad (6)$$

Equation (6) is a polynomial of degree $l(m)-1$ and contains a number $l(m)$ of independent constants a_s which characterize the equilibrium state. On the other hand, since we assumed conservation of all moments $Q^{(n\le m)}$, we know that equilibrium has to be characterized by $m+1$ independent parameters. This immediately determines $l(m) = m+1$, and the natural generalization of Fick's law is thus given by $\langle\Omega_x^{(m)}\rangle = -D(\partial_x)^{m+1}\langle\hat{S}_x^z\rangle$. This coincides with the intuition of the finite difference construction of $\Omega_x^{(m)}$: the dynamics balances out inhomogeneities of the mth derivative of the charge density.

Inserting this relation back into the evolution equation for the charge density, we finally arrive at the generalized hydrodynamic equation

$$\frac{d}{dt}\langle\hat{S}_x^z\rangle = -D(-1)^{m+1}(\partial_x)^{2(m+1)}\langle\hat{S}_x^z\rangle, \qquad (7)$$

valid for systems conserving all multipole moments up to and including m. We emphasize that our derivation not only predicts the hydrodynamic equation (7), but also the expected equilibrium distribution equation (6) in closed systems, where the corresponding constants $a_s = a_s(Q^{(n\le m)})$ are uniquely fixed by the charge moments $Q^{(n\le m)}$ of the initial state. In systems of size L they go as $|a_s| \sim \mathcal{O}(L^{-s})$, but are manifest in, e.g., observables involving macroscopic distances like $\langle\hat{S}_L^z\rangle - \langle\hat{S}_0^z\rangle$. The prediction Eq. (6) can be verified numerically in small systems by monitoring the charge distribution resulting from a fixed initial state at very late times. Figure 3(a) shows a chosen initial charge distribution in a system of size $L = 20$, as well as the late time distributions obtained from evolving the system using both dipole- and quadrupole-conserving automata. The resulting distributions are in very good agreement with the predicted polynomials of Eq. (6), validating our approach.

We further notice that while usually, the hydrodynamic description of a system conserving $m+1$ quantities is given by a set of $m+1$ coupled equations for the associated densities and currents [1,41,42], the present systems are described by a *single* equation (7) for the charge density. This is due to the hierarchical structure of the conservation laws Eq. (1) that specify all $Q^{(m)}$ in terms of the fundamental charges of the theory.

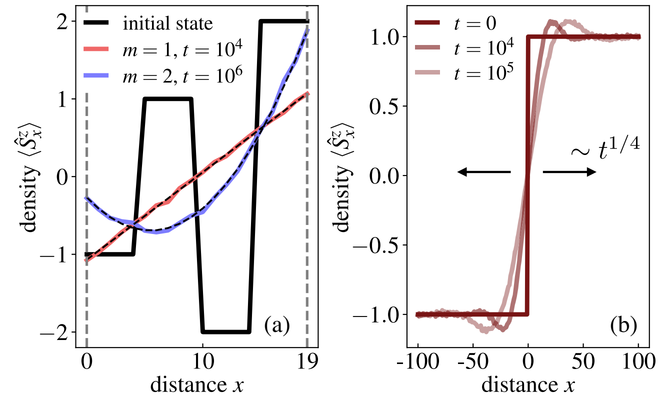

FIG. 3. Implications of higher-moment conservation. (a) In a finite size system with open boundary conditions (gray dashed lines), the charge density relaxes to an equilibrium distribution that is a polynomial of order m (here, $S = 3$). The black dashed lines are the analytical predictions from Eq. (6). (b) The melting of a domain wall in a dipole conserving system for sufficiently large spin (here, $S = 2$) appears as the cumulative distribution function of $C^{(1)}(x,t)$, with characteristic charge density oscillations.

Analytical solution.—It is worthwhile to consider the solutions of Eq. (7) in more detail. The normalized fundamental solution is of the form

$$G^{(m)}(x,t) = \frac{1}{(Dt)^{1/2(m+1)}} F^{(m)}\left[\frac{x^{2(m+1)}}{t}\right], \qquad (8)$$

where $F^{(m)}$ is a universal scaling function which can be written in terms of generalized hypergeometric functions [43]. The time evolution of a charge density profile starting from $G^{(m)}(x,0) = \delta(x)$ is described by Eq. (8), and is thus expected to coincide with the correlator $C^{(m)}(x,t)$ by standard linear response theory [1]. Figures 2(c) and 2(d) show that this is indeed the case, displaying full agreement with our numerical results for both $m = 1$, 2 upon fitting the only free parameter D. In particular, as demonstrated in Figs. 2(c) and 2(d), $C^{(m)}(x,t)$ accurately follows the scaling collapse predicted by Eq. (8).

For $m = 0$, Eq. (7) reduces to the usual diffusion equation and $C^{(0)}(x,t)$ is a Gaussian probability distribution describing the movement of an initially localized excitation through the system [1,44]. For $m \ge 1$, as shown in Fig. 2 and more generally clear from a vanishing second-moment $\langle x^2\rangle_{G^{(m)}} = \int dx x^2 G^{(m)}(x,t) = 0$, $C^{(m)}(x,t)$ *cannot* be interpreted as a probability distribution. Instead, the associated oscillations in the profile of $C^{(m)}(x,t)$ form a characteristic signature of higher-moment conservation that can potentially also be observed in quench experiments of domain wall initial states, see Fig. 3(b).

Finally, we notice that the central peak of $C^{(m)}(x,t)$ in Figs. 2(c) and 2(d) is well approximated by a

Gaussian $g(x,t) = \exp(-x^2/\sigma^2(t))/\sqrt{\pi\sigma^2(t)}$ with $\sigma(t) = (Dt)^{1/2(m+1)}$. The additional dressing density modulations can be understood heuristically if we interpret the Gaussian distribution $g(x,t)$ as describing the movement of an excitation through the system as part of m poles. Conservation of $Q^{(m>0)}$ implies that a surrounding cloud of opposite charge has to be dragged along, see Fig. 1(b). The effective length scale for this process is given by $\sigma(t)$. This intuition can be formalized by making an ansatz $C^{(m)}(x,t) = c_0 g(x,t) - c_2\{g[x+\sigma(t)] + g[x-\sigma(t)]\} \approx g(x,t)[c_0 - 4c_2 x^2/\sigma(t)^2]$. Thus, a positive charge moving to $x \pm \sigma(t)$ implies an increased likelihood of simultaneously finding a negative charge at x. Generalizing this physically motivated ansatz, we can expand $C^{(m)}(x,t) = g(x,t)\sum_n c_{2n}^{(m)}[-x^2/\sigma^2(t)]^n$. Figures 2(c) and 2(d) show excellent agreement already at low orders of the expansion, where each term provides an additional oscillation in the spatial profile of $C^{(m)}(x,t)$.

Conclusions and outlook.—We have studied the long-time dynamics of higher-moment conserving models, obtaining a generalized hydrodynamic equation relevant for fractonic systems that leads to subdiffusive decay of charge correlations. We emphasize that for dipole conservation, our results provide a postdiction of the subdiffusive scaling experimentally observed in Ref. [19], where an initially prepared k-wave density mode of interacting fermions was found to decay as $\sim\exp(-k^4 t)$ in the presence of a strong, linearly tilted potential. The linear potential couples directly to the center of mass $\sum_x x\hat{n}_x$ (see also [45]) and can thus be thought of as inducing an effective dipole conservation on long length scales. The observed decay agrees with Eq. (7) when written in Fourier space. In analogy, the present analysis suggests that effective quadrupole conservation may be obtained by application of a *harmonic* potential.

In addition to the subdiffusive decay of the return probability, we have identified oscillations in the spatial density profile both for delta and domain wall initial conditions as characteristic properties of higher-moment conservation. Such oscillations should be detectable in quantum quench experiments. Furthermore, higher-moment conservation leads to a modified scaling of the full-width-half-maximum of the Lorentzian line shape as $\sim k^{2(m+1)}$ in Fourier space, which could be detectable in scattering experiments [46]. Finally, we expect our results to reflect the longtime dynamics of closed quantum systems as indicated by investigations on small system sizes, see Supplemental Material [34]. Understanding the full impact of higher-moment conservation laws on the dynamics of quantum systems, in particular in the presence of energy conservation, remains an objective for future study.

We thank Monika Aidelsburger, Alexander Schuckert, and Adam Smith for insightful discussions. We especially thank Tibor Rakovszky for discussions and for carefully reading the manuscript. We acknowledge support from the Technical University of Munich–Institute for Advanced Study, funded by the German Excellence Initiative and the European Union FP7 under Grant Agreement No. 291763, the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy–EXC–2111–390814868, la Caixa Foundation (ID 100010434) fellowship grant for postgraduate studies, Research Unit FOR 1807 through Grants No. PO 1370/2-1, DFG TRR80 and DFG Grants No. KN1254/1-1, No. KN1254/1-2, and from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme (Grant Agreements No. 771537 and No. 851161).

Note added.—While finalizing this manuscript, we became aware of a related work by A. Gromov, A. Lucas, and R. M. Nandkishore [47] and a related work by A. Morningstar, V. Khemani, and D. Huse which appeared in the previous arXiv posting [48].

[1] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 1995).
[2] S. Mukerjee, V. Oganesyan, and D. Huse, Statistical theory of transport by strongly interacting lattice fermions, Phys. Rev. B **73**, 035113 (2006).
[3] J. Lux, J. Müller, A. Mitra, and A. Rosch, Hydrodynamic long-time tails after a quantum quench, Phys. Rev. A **89**, 053608 (2014).
[4] A. Bohrdt, C. B. Mendl, M. Endres, and M. Knap, Scrambling and thermalization in a diffusive quantum many-body system, New J. Phys. **19**, 063001 (2017).
[5] E. Leviatan, F. Pollmann, J. H. Bardarson, D. A. Huse, and E. Altman, Quantum thermalization dynamics with matrix-product states, arXiv:1702.08894.
[6] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Diffusive Hydrodynamics of Out-of-Time-Ordered Correlators with Charge Conservation, Phys. Rev. X **8**, 031058 (2018).
[7] V. Khemani, A. Vishwanath, and D. A. Huse, Operator Spreading and the Emergence of Dissipative Hydrodynamics under Unitary Evolution with Conservation Laws, Phys. Rev. X **8**, 031057 (2018).
[8] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, A Universal Operator Growth Hypothesis, Phys. Rev. X **9**, 041017 (2019).
[9] S. Gopalakrishnan and R. Vasseur, Kinetic Theory of Spin Diffusion and Superdiffusion in *XXZ* Spin Chains, Phys. Rev. Lett. **122**, 127202 (2019).
[10] A. Schuckert, I. Lovas, and M. Knap, Nonlocal emergent hydrodynamics in a long-range quantum spin system, Phys. Rev. B **101**, 020416(R) (2020).
[11] C. Chamon, Quantum Glassiness in Strongly Correlated Clean Systems: An Example of Topological Overprotection, Phys. Rev. Lett. **94**, 040402 (2005).

[12] S. Vijay, J. Haah, and L. Fu, A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations, Phys. Rev. B **92**, 235136 (2015).

[13] M. Pretko, Subdimensional particle structure of higher rank $U(1)$ spin liquids, Phys. Rev. B **95**, 115139 (2017).

[14] D. J. Williamson, Z. Bi, and M. Cheng, Fractonic matter in symmetry-enriched $U(1)$ gauge theory, Phys. Rev. B **100**, 125150 (2019).

[15] D. Bulmash and M. Barkeshli, Generalized $U(1)$ gauge field theories and fractal dynamics, arXiv:1806.01855.

[16] A. Gromov, Towards Classification of Fracton Phases: The Multipole Algebra, Phys. Rev. X **9**, 031035 (2019).

[17] Y. You, T. Devakul, S. L. Sondhi, and F. J. Burnell, Fractonic Chern-Simons and BF theories, Phys. Rev. Research **2**, 023249 (2020).

[18] M. Pretko, X. Chen, and Y. You, Fracton phases of matter, Int. J. Mod. Phys. A **35**, 2030003 (2020).

[19] E. Guardado-Sanchez, A. Morningstar, B. M. Spar, P. T. Brown, D. A. Huse, and W. S. Bakr, Subdiffusion and Heat Transport in a Tilted Two-Dimensional Fermi-Hubbard System, Phys. Rev. X **10**, 011042 (2020).

[20] Y. You and F. von Oppen, Building fracton phases by Majorana manipulation, Phys. Rev. Research **1**, 013011 (2019).

[21] H. Yan, O. Benton, L. D. C. Jaubert, and N. Shannon, Rank–2 $U(1)$ Spin Liquid on the Breathing Pyrochlore Lattice, Phys. Rev. Lett. **124**, 127203 (2020).

[22] S. Pai, M. Pretko, and R. M. Nandkishore, Localization in Fractonic Random Circuits, Phys. Rev. X **9**, 021003 (2019).

[23] P. Sala, T. Rakovszky, R. Verresen, M. Knap, and F. Pollmann, Ergodicity Breaking Arising from Hilbert Space Fragmentation in Dipole-Conserving Hamiltonians, Phys. Rev. X **10**, 011047 (2020).

[24] V. Khemani and R. Nandkishore, Local constraints can globally shatter Hilbert space: A new route to quantum information protection, Phys. Rev. B **101**, 174204 (2020).

[25] S. Moudgalya, B. A. Bernevig, and N. Regnault, Quantum many-body scars in a Landau level on a thin torus, arXiv:1906.05292.

[26] S. R. Taylor, M. Schulz, F. Pollmann, and R. Moessner, Experimental probes of Stark many-body localization, Phys. Rev. B **102**, 054206 (2020).

[27] V. Khemani, M. Hermele, and R. Nandkishore, Localization from Hilbert space shattering: From theory to physical realizations, Phys. Rev. B **101**, 174204 (2020).

[28] S. Moudgalya, A. Prem, R. Nandkishore, N. Regnault, and B. A. Bernevig, Thermalization and its absence within Krylov subspaces of a constrained Hamiltonian, arXiv:1910.14048.

[29] T. Rakovszky, P. Sala, R. Verresen, M. Knap, and F. Pollmann, Statistical localization: From strong fragmentation to strong edge modes, Phys. Rev. B **101**, 125126 (2020).

[30] M. Medenjak, K. Klobas, and T. Prosen, Diffusion in Deterministic Interacting Lattice Systems, Phys. Rev. Lett. **119**, 110603 (2017).

[31] S. Gopalakrishnan and B. Zakirov, Facilitated quantum cellular automata as simple models with non-thermal eigenstates and dynamics, Quantum Sci. Technol. **3**, 044004 (2018).

[32] J. Iaconis, S. Vijay, and R. Nandkishore, Anomalous subdiffusion from subsystem symmetries, Phys. Rev. B **100**, 214301 (2019).

[33] F. Ritort and P. Sollich, Glassy dynamics of kinetically constrained models, Adv. Phys. **52**, 219 (2003).

[34] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.245303 for additional information on the numerical method, the quantum spin chain example, and a scaling analysis, which includes Refs. [35–37]

[35] A. Weiße, G. Wellein, A. Alvermann, and H. Fehske, The kernel polynomial method, Rev. Mod. Phys. **78**, 275 (2006).

[36] S. Bera, G. De Tomasi, F. Weiner, and F. Evers, Density Propagator for Many-Body Localization: Finite-Size Effects, Transient Subdiffusion, and Exponential Decay, Phys. Rev. Lett. **118**, 196801 (2017).

[37] J. Lux, Fluctuations in and out of equilibrium: Thermalization, quantum measurements and Coulomb disorder, Ph.D. thesis, University of Cologne, 2016.

[38] This model was shown to thermalize for typical initial states in Ref. [23].

[39] J.-K. Yuan, S. A. Chen, and P. Ye, Fractonic superfluids, Phys. Rev. Research **2**, 023267 (2020).

[40] B. Doyon, Lecture notes on generalised hydrodynamics, SciPost Phys. Lect. Notes **18** (2020).

[41] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed., Course of Theoretical Physics Vol. 6 (Pergamon Press, Oxford, 1987) [Translated from the Russian by J. B. Sykes and W. H. Reid].

[42] Dieter Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (CRC Press, Boca Raton, 2018).

[43] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications Inc., New York, 1965).

[44] H. van Beijeren, Transport properties of stochastic Lorentz models, Rev. Mod. Phys. **54**, 195 (1982).

[45] S. Mandt, A. Rapp, and A. Rosch, Interacting Fermionic Atoms in Optical Lattices Diffuse Symmetrically Upwards and Downwards in a Gravitational Potential, Phys. Rev. Lett. **106**, 250602 (2011).

[46] V. J. Minkiewicz, M. F. Collins, R. Nathans, and G. Shirane, Critical and spin-wave fluctuations in nickel by neutron scattering, Phys. Rev. **182**, 624 (1969).

[47] A. Gromov, A. Lucas, and R. M. Nandkishore, Fracton hydrodynamics, Phys. Rev. Research **2**, 033124 (2020).

[48] A. Morningstar, V. Khemani, and D. A. Huse, Kinetically constrained freezing transition in a dipole-conserving system, Phys. Rev. B **101**, 214205 (2020).