Undular Diffusion in Nonlinear Sigma Models

Žiga Krajnik, Enej Ilievski, and Tomaž Prosen
Faculty for Mathematics and Physics, University of Ljubljana, Jadranska ulica 19, 1000 Ljubljana, Slovenia

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We discuss general features of charge transport in nonrelativistic classical field theories invariant under non-Abelian unitary Lie groups by examining the full structure of two-point dynamical correlation functions in grand-canonical ensembles at finite charge densities (polarized ensembles). Upon explicit breaking of non-Abelian symmetry, two distinct transport laws characterized by dynamical exponent $z = 2$ arise. While in the unbroken symmetry sector, the Cartan fields exhibit normal diffusion, the transversal sectors governed by the nonlinear analogs of Goldstone modes disclose an unconventional law of diffusion, characterized by a complex diffusion constant and undulating patterns in the spatiotemporal correlation profiles. In the limit of strong polarization, one retrieves the imaginary-time diffusion for uncoupled linear Goldstone modes, whereas for weak polarizations the imaginary component of the diffusion constant becomes small. In models of higher rank symmetry, we prove absence of dynamical correlations among distinct transversal sectors.

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Field theories provide one of the most invaluable tools in theoretical physics, with countless applications across a wide range of disciplines. One of the most renowned and best studied examples are nonlinear sigma models (NLSMs) [1–5] and extensions thereof, such as Wess-Zumino-Witten models [6–8], representing field theories of interacting fields on curved manifolds that transform as representations of non-Abelian symmetry groups. Although sigma models have played a pivotal role in the studies of Yang-Mills theories and gauge-gravity dualities [9–11], renormalization group flows [12,13], topological quantum field theories (QFTs) [5,14], and quantum criticality [15,16], their dynamical properties remain poorly understood, especially so in thermal equilibrium. One notable exception is the quantum O(3) NLSM in two space-time dimensions, a prominent example of an integrable QFT [1,3,17,18] which has attracted a considerable amount of attention in the context of low-temperature magnetization transport in Haldane antiferromagnets [5] (see [19–22]), recently revisited in [23]. Despite many efforts in the domain of quantum field theories [16,24], and recently even in classical isotropic magnets [25–28], a comprehensive understanding of dynamical properties of NLSMs in thermal equilibrium is still lacking.

Our Letter is motivated by the following fundamental question: consider G-invariant NLSMs with coset spaces $\mathcal{M} = G/H$ as their target manifolds, where isometry group G is a non-Abelian simple Lie group and isotropy subgroup $H \subset G$ identified with stability group of a continuously degenerate vacuum state. As a consequence of G invariance, the system possesses conserved Noether currents. The goal is a general classification of transport laws in thermal equilibrium states, irrespective of the coset structure, Lorentz invariance, dimensionality, and integrability. In this Letter, we make a key progress in this direction and classify dynamical two-point correlation functions in equilibrium states at generic values of background charge densities for a family of classical nonintegrable NLSMs in two space-time dimensions.

There is a widespread belief that "ergodic" (chaotic) interacting systems governed by reversible microscopic dynamical laws exhibit normal diffusion, epitomized by the celebrated Fick's second law $\phi_t = D\phi_{xx}$ [29] (unless several conservation laws are nontrivially coupled, in which case nonlinear fluctuating hydrodynamics [30] predicts a plethora of superdiffusive scaling laws [31]). Here ϕ is a real scalar field whose spatial integral is conserved under time evolution, $(d/dt)\int dx\phi = 0$. More generally, one speaks of "normal diffusion" (in thermal equilibrium) when asymptotic dynamical structure factors, reading $\langle\phi(x,t)\phi(0,0)\rangle \simeq t^{-1/z} f_G[(\lambda t)^{-1/z}x]$, are characterized by (i) dynamical exponent $z = 2$ and (ii) Gaussian stationary scaling profile $f_G(\zeta) = \exp(-\zeta^2)$, parametrized by a *real* state-dependent (diffusion) constant $D = \lambda/4$. In what follows, we shall explain how in systems with non-Abelian continuous symmetries, conserved Noether charges from the symmetry-broken sectors evade the conventional paradigm of normal diffusion.

Undular diffusion at a glance.—The theme of this Letter is an anomalous type of diffusion law we dub as "undular diffusion." To set the stage, we would first like to offer some basic intuition behind this notion. To this end, we consider a classical isotropic ferromagnet. The vacuum (minimum energy configuration) corresponds to all the spins aligning in the same direction, taking the role of a

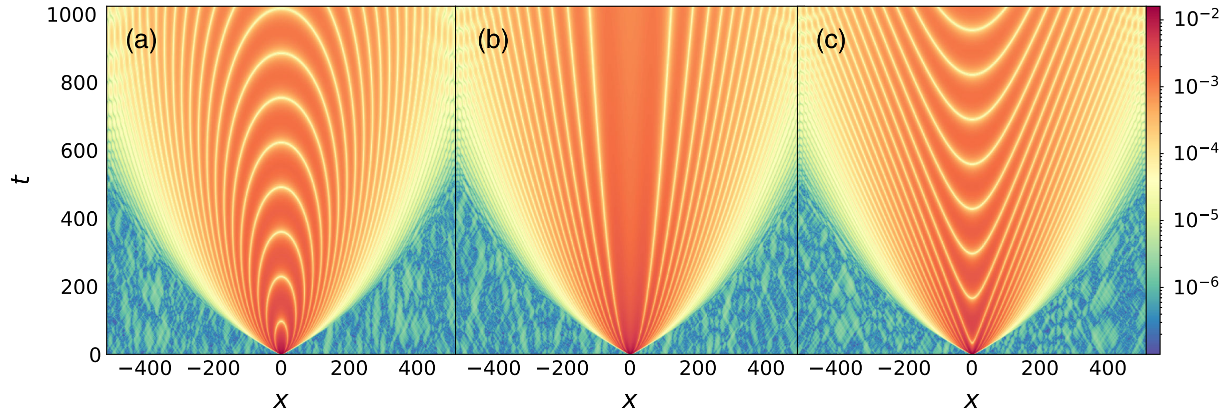

FIG. 1. Dynamical correlation functions in the transversal sector, computed in a nonintegrable space-time lattice discretization of S^2 Landau-Lifshitz field theory [34], immersed in a longitudinal magnetic field of magnitude b (pointing in the z direction). We display $\langle S^x(x,t)S^x(0,0)\rangle_\mu$ evaluated in a grand-canonical ensemble at infinite temperature and chemical potential $\mu = 5$ ($\langle S^z\rangle \approx 0.8$), shown in absolute value (time step $\tau = 1$, length $L = 1024$, average over 7.5×10^5 iterations). Three types of dynamical patterns can be discerned: (a) elliptic regime ($b = -6 \times 10^{-3}$), (b) parabolic regime of undular diffusion without a field ($b = 0$), and (c) hyperbolic regime ($b = 6 \times 10^{-3}$). The characteristic curves resemble conic sections associated with linear Goldstone modes.

local order parameter. The order parameter always picks a random polarization direction (a unit vector on a two-sphere), while the rotational symmetry of the model implies that the vacuum state is continuously degenerate and the symmetry is said to be *spontaneously* broken. It is widely known that a spontaneous breaking of continuous symmetry is accompanied by soft Nambu-Goldstone modes; in ferromagnets specifically, these are quadratically dispersing magnons which resolve small fluctuations about the symmetry-broken ferromagnetic vacuum.

Suppose we would like to understand transport properties of an isotropic ferromagnet at finite temperature. Invariance under continuous rotational symmetry implies that all the components of magnetization are globally conserved under time evolution. Transport properties of the model are most commonly extracted from the late-time relaxation of temporal correlation functions among distinct components. In canonical Gibbs states, which respect the full rotational symmetry of the model, there is no distinction between magnetization components. By invoking standard hydrodynamic arguments (based on gradient expansion of local conserved currents), one expects to find normal diffusion governed by the aforementioned Fick's law.

Consider now the grand-canonical ensemble where rotational symmetry is *explicitly* broken by inclusion of chemical potentials: one polarization direction (and thereby magnetization component) becomes distinguished, while the remaining two components are proclaimed as transversal. The question is whether such a symmetry breaking scenario "at finite density" has any effect on transport properties. One may indeed expect the difference to show up in the transversal sector; it is evident that, in the limit of strong polarization, where thermal fluctuations are dominated by fluctuations near the ferromagnetic vacuum, one should recover precessional motion governed by the spectrum of Goldstone modes, which one can interpret as a diffusion process in imaginary time. Accordingly, it is natural to anticipate that at any intermediate density (i.e., finite chemical potential) the diffusive relaxation of transversal correlators acquires an extra imaginary component, combining into a single "complex Goldstone mode." This is precisely what happens, as shown in the remainder of the Letter.

Minimal example.—We proceed by detailing out the "minimal model" of undular diffusion: the classical Landau-Lifshitz field theory [32,33] (using subscripts to designate partial derivatives)

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times \mathbf{B}, \tag{1}$$

written in terms of the unit vector (spin) field $\mathbf{S} \equiv (S^x, S^y, S^z)^T$ taking values on a two-sphere, $\mathbf{S}\cdot\mathbf{S} = 1$. We have also included an external magnetic field $\mathbf{B} = b\hat{\mathbf{e}}_z$ aligned with the vacuum polarization axis $\hat{\mathbf{e}}_z = (0,0,1)^T$ to study also "dynamical" breaking of symmetry.

To study dynamics in the symmetry-broken states, we introduce transversal complex fields $S^\pm = S^x \pm iS^y$. In Fig. 1 we display the dynamical correlator $\frac{1}{2}\mathrm{Re}\langle S^+(x,t)S^-(0,0)\rangle_\mu$, averaged with respect to the invariant grand-canonical Gibbs state at "infinite temperature" and chemical potential μ with a local probability density $\varrho_\mu^{(1)}(\mathbf{S}) = [\pi\sinh(\mu)/\mu]^{-1}\exp[\mu(1-2S^z)]$. To avoid special features related to integrability of Eq. (1), we performed our simulations on a *nonintegrable* lattice discretization (see Supplemental Material [34]).

In the absence of an external magnetic field, we encounter undular diffusion, manifesting itself in the form of a spatially undulating correlation function with a

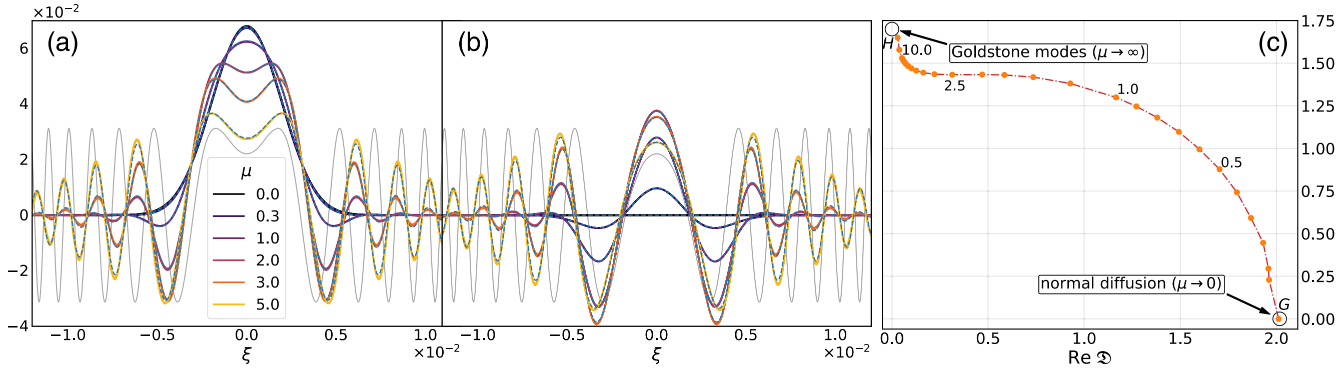

FIG. 2. Stationary asymptotic profiles of the transversal Goldstone mode in the nonintegrable space-time lattice discretization of the S^2 Landau-Lifshitz field theory [34] without a field ($b=0$), displaying (a) real and (b) imaginary components of $\lim_{t\to\infty}|t|^{1/2}\langle S^+(x,t)S^-(0,0)\rangle_\mu/2$ as a function of the scaled variable $\xi = t^{-1/2}x$ and U(1) chemical potential μ [using same parameters as in Fig. 1(b)]. Gray curves mark the prediction of the linear theory, cf. Eq. (2) (in arbitrary units). Dashed blue lines show best two-parameter fits to Eq. (9). (c) Dependence of real and imaginary components of complex diffusion constant $\mathfrak{D}$ on chemical potential μ. Symmetry points H and G designate the U(1)-invariant vacuum and SU(2)-invariant equilibrium measure, respectively (red dashed line is a guide to the eye).

characteristic diffusive (parabolic) pattern displayed in Fig. 1(b). When the rotational symmetry of the model is "dynamically broken" with an external positive (negative) magnetic field , $\int_{\mathbb{R}} S^{\pm}(x,t)dx = e^{\mp ibt}\int_{\mathbb{R}} S^{\pm}(x,0)dx$, we observe hyperbolic (elliptic) characteristics, as shown in Figs. 1(a) and 1(c).

The origin of the observed patterns is best explained by inspecting the vicinity of the ferromagnetic vacuum ($|S^z|\to 1$), where the equation of motion in the transversal sector reduces to a *linear* theory [34]

$$[i\partial_t \pm (\partial_x^2 - b)]S^{\pm}(x,t) = 0. \tag{2}$$

Its Green's function $\mathcal{G}_{\pm}(k) = \exp[\mp \omega^b_{\mathrm{mag}}(k)t]$ describes magnons with a gapped quadratic (i.e., type-II) dispersion law

$$\omega^b_{\mathrm{mag}}(k) = \mathfrak{D}_\infty k^2 + ib, \tag{3}$$

written as an imaginary-time diffusion with an imaginary diffusion constant $\mathfrak{D}_\infty \equiv \mathfrak{D}(\mu\to\infty) = i$. Characteristics associated with Eq. (2) are conic sections. Remarkably, their presence remains visible even in the nonlinear dynamic away from the vacuum (i.e., at general values of μ), as shown in Fig. 1.

In Fig. 2 we display the numerically computed stationary profiles for the transversal dynamical correlator (without the field, $b=0$) depending on chemical potential μ. When approaching the vacuum (i.e., at large μ), the profiles converge toward the prediction of the linear theory [34] (gray curves in Fig. 2). In the opposite regime, $\mu\to 0$, the profiles smoothen out into a Gaussian. In Fig. 2(c) we extract the complex diffusion constant $\mathfrak{D}(\mu)$ by fitting the scaling function [given in Eq. (9)].

The above phenomenology offers the following suggestive interpretation: at finite spin density ($\mu\neq 0$), the late-time relaxation of nonlinearly evolving fields from the symmetry-broken sector is governed by an unconventional Goldstone mode, which has acquired an extra diffusive component, characterized by a single hydrodynamic generalized Fick's law of diffusion with a *complex* diffusion constant $\mathfrak{D}(\mu)$.

To conclude, we remark that two-point correlations $\langle S^z(x,t)S^{\pm}(0,0)\rangle_\mu$ and $\langle S^{\pm}(x,t)S^{\pm}(0,0)\rangle_\mu$ both trivially vanish as consequence of the residual U(1) symmetry about $\mathbf{e}_z$. The only remaining nonzero correlator allowed by symmetry is therefore the longitudinal one, $\langle S^z(x,t)S^z(0,0)\rangle_\mu$, which, as expected, undergoes normal diffusion with real diffusion constant.

Symmetry of higher rank.—It is natural to ask if any new feature can arise in models exhibiting symmetries of higher rank. Thus, our next aim is to classify the dynamical two-point correlation functions among the Noether charges of a class of models invariant under non-Abelian groups of higher rank, comprising multiple Nambu-Goldstone modes in their spectrum. We mainly wish to discern whether enhanced symmetry can affect dynamics in the symmetry-broken sector due to interaction among distinct transversal modes.

Here we shall consider the simplest class of (nonrelativistic) continuous ferromagnets invariant under the action of unitary Lie groups $\mathrm{SU}(n+1)$, whose target spaces are complex projective manifolds $\mathcal{M}_n \equiv \mathbb{CP}^n$. The latter are naturally parametrized by complex fields $z^a(x,t)$ (alongside their conjugate counterparts $\bar{z}^a$), and for compactness we introduce the vector of affine coordinates $\mathbf{z}\equiv(z_1,\ldots,z_n)^T$ on $\mathcal{M}_n$. As a starting point, we consider the most general effective Lagrangian invariant under $\mathrm{SU}(n+1)$ in the form

$$\mathcal{L}_{\rm eff} \simeq \mathcal{L}_{\rm WZ} - \mathcal{L}^{(2)}_{\mathbb{CP}^n} + \text{higher} - \text{order terms}, \tag{4}$$

where $\mathcal{L}^{(2)}_{\mathbb{CP}^n} \equiv \sum_{a,b=1}^{n} \eta_{ab}\bar{z}^a_x z^b_x$ is the second-order term in gradient expansion parametrized by the unique G-invariant Riemann (Fubini-Study) metric on $\mathbb{CP}^n$, reading explicitly $\eta_{ab} = [(1+\mathbf{z}^\dagger\mathbf{z})\delta_{ab} - \bar{z}^a z^b]/(1+\mathbf{z}^\dagger\mathbf{z})^2$, and $\mathcal{L}_{\rm WZ} = i(1+\mathbf{z}^\dagger\mathbf{z})^{-1}(\mathbf{z}^\dagger\mathbf{z}_t - \mathbf{z}_t^\dagger\mathbf{z})$ denotes the Wess-Zumino geometric term.

To simplify our analysis, we shall discard all the higher-order terms in Eq. (4). This way, we end up with nonrelativistic classical sigma models on cosets $\mathbb{CP}^n = G/H$, with isotropy subgroup $H = \mathrm{SU}(n)\times \mathrm{U}(1)$ leaving the ferromagnetic vacuum intact (modulo a phase). Matrix-valued fields $M(x,t)$ on $\mathcal{M}_n$ are unitary matrices subjected to a nonlinear constraint $M^2 = \mathbb{1}$, in terms of which the Hamiltonian reads simply $H_{\mathbb{CP}^n} = 2\int dx \mathcal{L}^{(2)}_{\mathbb{CP}^n} = \frac{1}{4}\int dx \mathrm{Tr}(M_x^2)$. The equation of motion is given by a nonlinear partial differential equation (Landau-Lifshitz field theories of higher rank) [27]

$$M_t = \frac{1}{2i}[M, M_{xx}] + i[B, M], \tag{5}$$

where we have simultaneously adjoined the external field $H_B = \int dx \mathrm{Tr}(BM)$, which induces dynamical breaking of conservation laws associated with G.

Longitudinal and transversal fields can be inferred with respect to the Cartan-Weyl basis of Lie algebra $\mathfrak{g} = \mathfrak{su}(n+1)$ (see, e.g., [44,45] and the Supplemental Material [34] for details). Weyl generators, which are indexed by root vectors spanning the root lattice Δ of $\mathfrak{g}$, are assigned complex Weyl fields $\phi^{\pm\alpha}$. To every Cartan generator we associate a real longitudinal field ϕ^i and formally assign to it a "zero root" forming a set Δ_0. To obtain the ϕ field, one simply traces the corresponding generator times the matrix $M \in \mathcal{M}_n$.

We proceed by introducing grand-canonical Gibbs states, including generic chemical potentials coupling to the Cartan charges $\mathcal{Q}^i = \int dx \phi^i(x)$. In such a state, the original symmetry $G = \mathrm{SU}(n+1)$ gets lowered down to the residual symmetry of its maximal Abelian subgroup $T = \mathrm{U}(1)^{\times n}$. There are thus n "unbroken" longitudinal fields ϕ^i associated with the Cartan generators. On the other hand, the symmetry-broken sector comprises $n_t = \frac{1}{2}\dim(G/T) = \frac{1}{2}n(n+1)$ pairs of canonically conjugate complex "transversal" modes $\phi^{\pm\alpha}$. To define a stationary measure invariant under T, we introduce the diagonal "torus Hamiltonian" $H_{\boldsymbol{\mu}} = -\frac{1}{2}\mathrm{diag}(\mu_0, \mu_1, \ldots, \mu_n)$, parametrized by chemical potentials $\mu_i \in \mathbb{R}$ (subjected to $\mathrm{Tr}H_{\boldsymbol{\mu}} = 0$) and define an invariant normalized measure $\varrho^{(n)}_{\boldsymbol{\mu}} d\Omega^{(n)}$ ($\int_{\mathcal{M}_n} d\Omega^{(n)}\varrho^{(n)}_{\boldsymbol{\mu}} = 1$), with volume element $d\Omega^{(n)}$ on $\mathbb{CP}^n$ and density

$$\varrho^{(n)}_{\boldsymbol{\mu}}(M) = \frac{1}{\mathcal{Z}^{(n)}_{\boldsymbol{\mu}}}\exp[\mathrm{Tr}(H_{\boldsymbol{\mu}}M)], \tag{6}$$

where $\mathcal{Z}^{(n)}_{\boldsymbol{\mu}} = \int_{\mathcal{M}_n} d\Omega^{(n)} \exp[\mathrm{Tr}(H_{\boldsymbol{\mu}}M)]$ represents the partition function. One can think of Eq. (6) as the grand-canonical Gibbs measure at infinite temperature (known in symplectic geometry as an equivariant measure). Further details can be found in the Supplemental Material [34].

By direct analogy to the previous basic case of $\mathbb{CP}^1 \cong S^2$, the mixed correlators $\langle \phi^i(x,t)\phi^{\pm\alpha}(0,0)\rangle_{\boldsymbol{\mu}}$ and paired intrasectoral correlators $\langle \phi^{\pm\alpha}(x,t)\phi^{\pm\alpha}(0,0)\rangle_{\boldsymbol{\mu}}$ once again vanish as a direct corollary of the T invariance of the measure (6). Indeed, this statement remains valid even in Gibbs states at any inverse temperature β.

The new ingredient now is that models of higher rank possess additional intersectoral correlations among distinct transversal (Weyl) fields. A starting point for their analysis is the following "neutrality selection rule" for equal-time N-point correlators

$$\sum_{j\in\{1\ldots N\}}^{\sigma_j \notin \Delta_0} \sigma_j \neq \mathbf{0} \Rightarrow \langle \phi^{\sigma_1}_{\ell_1}\phi^{\sigma_2}_{\ell_2}\ldots\phi^{\sigma_N}_{\ell_N}\rangle_{\beta,\boldsymbol{\mu}} = 0, \tag{7}$$

which, in conjunction with the commutation relations, implies (see Supplemental Material [34] for proofs) the "kinematic" decoupling of transversal modes into subsectors, that is,

$$\langle \phi^{\pm\alpha}(x,t)\phi^{\gamma \nparallel \alpha}(0,0)\rangle_{\beta,\boldsymbol{\mu}} = 0. \tag{8}$$

Consequently, the only dynamical two-point correlation functions allowed by symmetry are, besides the longitudinal $\langle \phi^i(x,t)\phi^j(0,0)\rangle_{\boldsymbol{\mu}}$, the *intrasectoral* correlations $\langle \phi^{\pm\alpha}(x,t)\phi^{\mp\alpha}(0,0)\rangle_{\boldsymbol{\mu}}$.

Numerical analysis of asymptotic stationary profiles within each transversal "α sector" shows that asymptotic dynamical structure factors are accurately captured by scaling profiles of undular diffusion

$$\langle \phi^{\alpha}(x,t)\phi^{-\alpha}(0,0)\rangle_{\boldsymbol{\mu}} = \frac{\chi_{\alpha,-\alpha}}{(4\pi\mathfrak{D}_\alpha|t|)^{1/2}} e^{-x^2/(4\mathfrak{D}_\alpha|t|)}, \tag{9}$$

characterized by a complex diffusion constant $\mathfrak{D}_\alpha(\boldsymbol{\mu})$, which recombines the effects of relaxation and precessional motion into a single hydrodynamic mode. In the strong-polarization limit, we recover the frequency of the (linear) Goldstone modes, $\lim_{|\boldsymbol{\mu}|\to\infty}\mathfrak{D}_\alpha(\boldsymbol{\mu}) = i\omega_\alpha(\langle\phi^j\rangle_{\rm vac})$, whereas in the opposite regime of weak polarization $\lim_{|\boldsymbol{\mu}|\to 0}\mathfrak{D}_\alpha(\boldsymbol{\mu}) = D_\alpha \in \mathbb{R}$ [46].

Summary.—A succinct summary of our results is given in Table I. Dynamical (connected) two-point correlations functions can be grouped into three classes: (I) longitudinal correlations $\langle \phi^i(x,t)\phi^j(0,0)\rangle_{\boldsymbol{\mu}}$, with dynamical exponent $z = 2$ and Gaussian asymptotic profiles [47], (II) transversal α sectors $\langle \phi^{\pm\alpha}(x,t)\phi^{\mp\alpha}(0,0)\rangle_{\boldsymbol{\mu}}$, with dynamical exponent $z = 2$ and *undulating* asymptotic stationary profiles (exemplified for $n = 1$ in Fig. 2), and (III) (i) vanishing mixed and transversal correlations

TABLE I. Complete classification of dynamical two-point correlation functions among the Noether fields.

Sector	Correlators	Transport
Longitudinal	$\langle \phi^i(x,t)\phi^j(0,0)\rangle_\mu$	Normal diffusion
Transversal	$\langle \phi^{\pm\alpha}(x,t)\phi^{\mp\alpha}(0,0)\rangle_\mu$	**Undular diffusion**
	$\langle \phi^{\pm\alpha}(x,t)\phi^{\pm\alpha}(0,0)\rangle_\mu$	
Trivial	$\langle \phi^i(x,t)\phi^{\pm\alpha}(0,0)\rangle_\mu$	No transport
	$\langle \phi^{\pm\alpha}(x,t)\phi^{\gamma\nparallel\alpha}(0,0)\rangle_\mu$	

$\langle \phi^i(x,t)\phi^{\pm\alpha}(0,0)\rangle_\mu = \langle \phi^{\pm\alpha}(x,t)\phi^{\pm\alpha}(0,0)\rangle_\mu = 0$, and (ii) vanishing *intersectoral* correlations $\langle \phi^{\pm\alpha}(x,t)\phi^{\gamma\nparallel\alpha}(0,0)\rangle_\mu = 0$.

Properties I and II have been established based on numerical observations, while III(i) is a direct corollary of invariance under the torus subgroup T. Property III (ii) follows from the "neutrality rule" (7). Indeed, we believe I–III are generic properties of nonintegrable Hamiltonian dynamics invariant under non-Abelian compact Lie group G with G/H-valued local degrees of freedom (order parameter), averaged with respect to a polarized T-invariant ensemble. In effect, the listed properties likewise apply to dynamical two-point functions in grand-canonical Gibbs ensembles at finite temperature, which will experience an additional "smearing" effect across a length scale comparable to the thermal correlation length.

Conclusion.—Focusing on a class of nonrelativistic sigma models invariant under unitary Lie groups, we have investigated the structure of dynamical correlations among the Noether charges in an equilibrium state with broken continuous symmetry. While longitudinal correlations among the Cartan fields expectedly undergo normal diffusion, we found that dynamics in the transversal (symmetry-broken) sector is governed by unorthodox Goldstone modes that satisfy a complexified diffusion law, characterized by dynamical exponent $z = 2$ and "complex Gaussian" profiles governed by a complex diffusion constant, which we have suggestively named undular diffusion. The phenomenon is present in a generic nonintegrable (chaotic) dynamics and does not depend on the microscopic details of the model or particular lattice discretization.

The main lesson to draw is twofold: (A) the ubiquitous Fick's law of diffusion, believed to be a hallmark of chaotic reversible many-body dynamics, can indeed be violated in systems that support type-II Goldstone modes, and (B) dynamical systems invariant under non-Abelian Lie group G do not support any dynamical correlations among the conserved Noether currents from different transversal $\mathfrak{su}(2)$ sectors in grand-canonical Gibbs equilibrium states. With regard to (A), an alternative viewpoint is to argue that undular diffusion is an analytic prolongation of the Fick's law of diffusion into the complex plane.

We note that classical nonrelativistic NLSMs that appear as the leading term of the gradient expansion of G-invariant dynamics on Hermitian symmetric spaces, such as Eq. (5), are commonly found to be integrable [48,49]. A salient feature of integrable dynamics (which can be accurately captured by generalized hydrodynamics [50,51]) are stable nonlinear modes (solitons), which render longitudinal correlators ballistic (quantified by finite charge Drude weights [52–55]) with diffusive corrections [56–59], or even superdiffusive dynamics that takes place in unpolarized Gibbs states [60,61], recently examined in [23,27,28,62–67]. In performing numerical computation, we have always employed appropriate lattice discretizations to ensure that integrability is manifestly broken. We have nonetheless verified that even in integrable discretizations of $\mathbb{CP}^n$ sigma models (5) [27], dynamics of transversal models associated with internal "precessional" degrees of freedom still display undular diffusive profiles.

On general grounds, one can expect that the phenomenon survives quantization, i.e., to persist in quantum lattice ferromagnets invariant under non-Abelian Lie groups (irrespective of integrability) and to extend to higher space-time dimensions.

There are several interesting venues left to be explored, for instance, (i) develop a quantitative framework to access asymptotic stationary profiles that characterize undular diffusion, (ii) extend the analysis to other symmetry groups and coset spaces, and (iii) infer the structure of transversal dynamical correlators also in relativistic sigma models, both in the classical and quantum settings.

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