

Genuine Network Multipartite Entanglement

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(Received 24 February 2020; accepted 22 October 2020; published 9 December 2020)

The standard definition of genuine multipartite entanglement stems from the need to assess the quantum control over an ever-growing number of quantum systems. We argue that this notion is easy to hack: in fact, a source capable of distributing bipartite entanglement can, by itself, generate genuine k -partite entangled states for any k . We propose an alternative definition for genuine multipartite entanglement, whereby a quantum state is *genuinely network k -entangled* if it cannot be produced by applying local trace-preserving maps over several $(k - 1)$ -partite states distributed among the parties, even with the aid of global shared randomness. We provide analytic and numerical witnesses of genuine network entanglement, and we reinterpret many past quantum experiments as demonstrations of this feature.

DOI: [10.1103/PhysRevLett.125.240505](https://doi.org/10.1103/PhysRevLett.125.240505)

The existence of multipartite quantum states that cannot be prepared locally is at the heart of many communication protocols in quantum information science, including quantum teleportation [1], dense coding [2], entanglement-based quantum key distribution [3], and the violation of Bell inequalities [4,5]. Most importantly, for the last two decades, the ability to entangle an ever-growing number of photons or atoms has been regarded as a benchmark for the experimental quantum control of optical systems [6–9].

Since any multipartite quantum state where two parts share a singlet can be regarded as “entangled,” another, more demanding notion of entanglement was required to assess the progress of quantum technologies. The accepted answer was genuine multipartite entanglement [10–12]. Genuine multipartite entanglement has since become a standard for quantum many-body experiments [6–9,13]. But, is it a universal measure?

In this Letter, we argue the opposite and present an alternative and stronger definition, *genuine network multipartite entanglement*, which we formulate in terms of quantum networks [14]. First, we define and compare the two approaches. Next, we present general criteria to detect genuine network entanglement and discuss the tightness of the bounds so obtained. Finally, we single out past experiments in quantum optics that can be reinterpreted as stronger demonstrations of genuine network entanglement.

Multipartite entanglement.—An n -partite quantum state can be identified with a bounded Hermitian positive semidefinite operator ρ acting on a composite Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ such that $\text{tr}(\rho) = 1$. Each factor \mathcal{H}_i with $i = 1, \dots, n$ represents the local Hilbert space of the i th party. For a subset $S \subseteq \{\mathcal{H}_i\}_i$, we denote by

$\rho_{(S)} = \text{tr}_{\bar{S}}(\rho)$ the density matrix of the reduced state on the subsystems S , where \bar{S} is the complement of S . We say that an n -partite state is fully separable if it can be written as a convex mixture of product states as follows:

$$\rho = \sum_j w_j \rho_1^j \otimes \cdots \otimes \rho_n^j, \quad \sum_j w_j = 1, \quad (1)$$

where the $\{\rho_i^j\}$ are normalized density matrices and the weights w_j are nonnegative. If ρ does not admit a decomposition of the form of Eq. (1), we say that it is entangled. The problem with the definition of full separability is that any technology capable of entangling, say, the first two particles could claim the generation of “entangled states” composed of arbitrarily many particles. Indeed, the reader can check that any state $\hat{\rho}$ of the form

$$\hat{\rho} = |\phi^+\rangle\langle\phi^+| \otimes \rho_{(\mathcal{H}_3, \dots, \mathcal{H}_n)}, \quad |\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad (2)$$

does not admit a decomposition of the form of Eq. (1).

In order to address this issue, an extended definition of multipartite separability was proposed [10–12]. Intuitively, a state is k -partite entangled if, in order to produce it, one must create k -partite entangled states and distribute them among the n parties in such a way that no party receives more than one subsystem. More formally, we say that an n -partite state is separable with respect to a partition $S_1 | \dots | S_s$ of $\{\mathcal{H}_1, \dots, \mathcal{H}_n\}$ if it can be expressed as

$$\rho = \sum_j w_j \rho_{(S_1)}^j \otimes \cdots \otimes \rho_{(S_s)}^j. \quad (3)$$

An n -partite state is *genuinely k -partite entangled* (or has entanglement depth k) if it cannot be expressed as a convex combination of quantum states, each of which is separable with respect to at least one partition $S_1|S_2|\dots$ of $\{1, \dots, n\}$ with $|S_\ell| \leq k-1$, for all ℓ . Using this definition, the state $\hat{\rho}$ in Eq. (2) is certainly genuinely 2-entangled. However, $\hat{\rho}$ is *not* genuinely 3-entangled so long as its marginal $\hat{\rho}_{(\mathcal{H}_3, \dots, \mathcal{H}_n)}$ is fully separable.

This notion of multipartite entanglement is easy to cheat, as we show next. For simplicity, we consider a tripartite scenario ($n=3$) and rename the Hilbert spaces \mathcal{A} , \mathcal{B} , and \mathcal{C} ; we split \mathcal{A} into three local subsystems \mathcal{A}' , \mathcal{A}'' , and \mathcal{A}''' and do the same for \mathcal{B} and \mathcal{C} . Now, let $\rho_{\mathcal{A}'\mathcal{B}'\mathcal{C}'} = |\phi^+\rangle\langle\phi^+|_{\mathcal{A}'\mathcal{B}'} \otimes |0\rangle\langle 0|_{\mathcal{C}'}$, and similarly $\rho_{\mathcal{A}''\mathcal{B}''\mathcal{C}''} = |\phi^+\rangle\langle\phi^+|_{\mathcal{B}''\mathcal{C}''} \otimes |0\rangle\langle 0|_{\mathcal{A}''}$ while $\rho_{\mathcal{A}'''\mathcal{B}'''\mathcal{C}'''} = |\phi^+\rangle\langle\phi^+|_{\mathcal{C}'''\mathcal{A}'''} \otimes |0\rangle\langle 0|_{\mathcal{B}'''}$. Following the same discussion as for $\hat{\rho}$, each of these three states individually is genuinely 2-entangled but not genuinely 3-entangled. However, if we consider those three states *collectively* (i.e., distributed at the same time), then the resulting state $\rho_{ABC} = \rho_{\mathcal{A}'\mathcal{B}'\mathcal{C}'} \otimes \rho_{\mathcal{A}''\mathcal{B}''\mathcal{C}''} \otimes \rho_{\mathcal{A}'''\mathcal{B}'''\mathcal{C}'''}$ is genuinely 3-entangled when considering the partition $\mathcal{A}|\mathcal{B}|\mathcal{C}$. Accordingly, the established definition of genuine k -partite entanglement is unstable under parallel composition (i.e., under simultaneous distribution of states).

In fact, enough copies of the state ρ_{ABC} enable the distribution of *any* tripartite state using the standard quantum teleportation protocol [1]. Any definition of genuine tripartite entanglement that regarded states like ρ_{ABC} as *not* genuinely tripartite entangled and, at the same time, were stable under composition and local operations and classical communication (LOCC) would thus be necessarily void.

In this Letter, we introduce the concept of *genuine network k -entanglement*, an alternative operational definition of multipartite entanglement that is stable under composition and where ρ_{ABC} is not genuinely tripartite entangled. The drawback, as will be evident from the definition, is that nongenuine network entanglement is not closed under LOCC but under the subset of LOCC transformations known as local operations and shared randomness (LOSR) [15,16]. This set of operations has been argued to be more relevant than LOCC for the study of Bell nonlocality [17,18]. Note that LOSR is a natural set of operations when the different parties in a network are separated in space and do not hold a quantum memory.

Genuine network entanglement.—We explain our definition using an adversarial approach. Eve is a vendor selling a source of tripartite quantum states to three honest scientists: Alice, Bob, and Charlie. Eve pretends that her device produces a valuable entangled tripartite state ρ_{ABC} . Unbeknown to the scientists, the source sold to them is actually composed of cheaper components: quantum sources that produce the bipartite entangled states $\sigma_{\mathcal{A}'\mathcal{B}''}$, $\sigma_{\mathcal{C}'\mathcal{A}''}$, $\sigma_{\mathcal{B}'\mathcal{C}''}$ (see Fig. 1). Alice receives the \mathcal{A}' , \mathcal{A}''

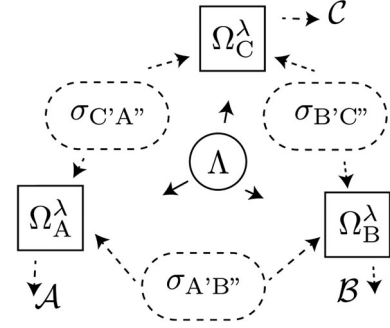


FIG. 1. Network producing a nongenuine network 3-entangled state; quantum resources and spaces are denoted using dotted lines, while classical variables are drawn using solid lines.

subsystems of the states $\sigma_{\mathcal{A}'\mathcal{B}''}$, $\sigma_{\mathcal{C}'\mathcal{A}''}$. Those can in principle interact within Alice’s experimental setup, giving rise to a new quantum system \mathcal{A} , which is what Alice eventually probes. Similarly, Bob (Charlie) will have access to system \mathcal{B} (\mathcal{C}), whose state is the result of a deterministic interaction between systems \mathcal{B}' , \mathcal{B}'' (\mathcal{C}' , \mathcal{C}''). In addition, we provide Eve with unlimited shared randomness Λ to jointly influence the local operations acting on systems $\mathcal{A}'\mathcal{A}''$, $\mathcal{B}'\mathcal{B}''$, and $\mathcal{C}'\mathcal{C}''$. It is worth noting that we do not make any assumption on the dimensionality of the “hidden” states $\sigma_{\mathcal{A}'\mathcal{B}''}$, $\sigma_{\mathcal{C}'\mathcal{A}''}$, $\sigma_{\mathcal{B}'\mathcal{C}''}$. Even if the systems \mathcal{A} , \mathcal{B} , \mathcal{C} accessible to Alice, Bob, and Charlie are a qubit each, the Hilbert space dimension of the hidden systems might well be infinite.

By performing local tomography on the state ρ_{ABC} , can Alice, Bob, and Charlie certify that the state produced by Eve’s network is indeed a valuable tripartite quantum state?

The family of states that they try to rule out can be defined formally. Let Λ be a classical random variable with distribution $P_\Lambda(\lambda)$ sent to the three labs (for example, through radio broadcast). Denoting by $\mathcal{B}(\mathcal{H})$ the set of bounded operators on the Hilbert space \mathcal{H} , we describe the deterministic operation at Alice’s by a family of linear maps $\{\Omega_{\mathcal{A}}^\lambda\}_\lambda$, where each $\Omega_{\mathcal{A}}^\lambda$ has type

$$\Omega_{\mathcal{A}}^\lambda : \mathcal{B}(\mathcal{A}' \otimes \mathcal{A}'') \rightarrow \mathcal{B}(\mathcal{A})$$

and each $\Omega_{\mathcal{A}}^\lambda$ is completely positive and trace preserving. For completeness, the other maps correspond to $\Omega_{\mathcal{B}}^\lambda : \mathcal{B}(\mathcal{B}' \otimes \mathcal{B}'') \rightarrow \mathcal{B}(\mathcal{B})$ and $\Omega_{\mathcal{C}}^\lambda : \mathcal{B}(\mathcal{C}' \otimes \mathcal{C}'') \rightarrow \mathcal{B}(\mathcal{C})$, so that the state ρ_{ABC} is

$$\rho_{ABC} = \sum P_\Lambda(\lambda) [\Omega_{\mathcal{A}}^\lambda \otimes \Omega_{\mathcal{B}}^\lambda \otimes \Omega_{\mathcal{C}}^\lambda](\sigma), \quad (4)$$

where $\sigma = \sigma_{\mathcal{A}'\mathcal{B}''} \otimes \sigma_{\mathcal{B}'\mathcal{C}''} \otimes \sigma_{\mathcal{C}'\mathcal{A}''}$.

The valuable states, those *genuinely network 3-entangled*, are those that cannot be written the way described by Eq. (4). It is easy to see that the set of states of the form of Eq. (4) is closed under tensor products and LOSR transformations. That is, the set of network 2-entangled states is a self-contained class within the

resource theory of LOSR entanglement [15,16]. This property has obvious implications for the monotonicity of any network 3-entanglement measure. Think for instance of the robustness of entanglement [19]. We could define its network 3-entanglement generalization as the minimum amount of network 2-entangled noise $R(\rho_{ABC})$ that one must add to a tripartite quantum state ρ_{ABC} to make it network 2-entangled. Closure under LOSR implies that $R(\rho_{ABC})$ is monotonically decreasing under LOSR operations. From our motivating discussion, though, it follows that $R(\rho_{ABC})$ can be arbitrarily increased by means of LOCC protocols.

Note that, in the considered adversarial scenario, rather than the state $\sigma_{A'B''} \otimes \sigma_{C'A''} \otimes \sigma_{B'C''}$, Eve could also distribute to Alice, Bob, and Charlie arbitrary convex combinations of states of the form $\sigma_{A'B''}^{(i)} \otimes \sigma_{C'A''}^{(i)} \otimes \sigma_{B'C''}^{(i)}$ for some values of i . Since the dimensionality of the primed spaces is unbounded, though, this strategy can be simulated with the operations allowed by Eq. (4). Indeed, it suffices to distribute the tensor product of the states $\sigma_{A'B''}^{(i)} \otimes \sigma_{C'A''}^{(i)} \otimes \sigma_{B'C''}^{(i)}$ and embed the index i within the hidden variable Λ (whose dimension is also unbounded). The index i would then signal in which pair of Hilbert spaces at party Z 's the map Ω_Z^i is to be applied.

The definition of genuine network entanglement can be straightforwardly extended to the n -partite case.

Definition.—A multipartite quantum state is *genuinely network k -entangled* if it cannot be generated by distributing entangled states among subsets of maximum $k-1$ parties, and letting the parties apply local trace-preserving maps, those maps being possibly correlated through global shared randomness.

Witnesses of genuine network entanglement.—The certification of ρ_{ABC} being genuinely network 3-entangled is complicated, as the dimensions of the Hilbert spaces A', \dots, C'' are in principle unbounded. To classify the degree of a state's network multipartiteness, we must somehow determine if the state can come about from a particular quantum causal process. The study of quantum causal processes has experienced great progress [14,20–23], and many techniques have recently been developed [22,24,25]. Herein, we adapt the inflation technique for causal inference [22,26] in order derive witnesses for genuine network entanglement.

As a starter, we consider a three qudit state ρ_{ABC} and quantify its proximity to the Greenberger–Horne–Zeilinger (GHZ) state [27] via the fidelity

$$F_{\text{GHZ}_d} \equiv \langle \text{GHZ}_d | \rho_{ABC} | \text{GHZ}_d \rangle, \quad (5)$$

where $|\text{GHZ}_d\rangle = \sum_{i=1}^d (|iii\rangle / \sqrt{d})$.

If ρ_{ABC} is of the form of Eq. (4), then there exists a random variable Λ , quantum states $\sigma_{A'B''}$, $\sigma_{B'C''}$, and $\sigma_{C'A''}$ and families of completely positive and trace-preserving (CPTP) maps $\{\Omega_{A'}^\lambda\}$, $\{\Omega_{B'}^\lambda\}$, and $\{\Omega_{C'}^\lambda\}$ that generate

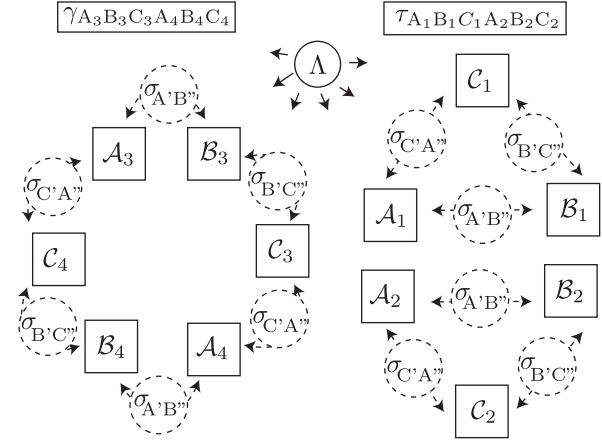


FIG. 2. Ring inflation of the triangle scenario in Fig. 1 containing copies of the state processing devices $\Omega_{A,B,C}^\lambda$; we label such copies according to their output Hilbert space A_i, B_j, C_k , where i, j, k is the index of the copy. These devices process copies of the quantum resources $\sigma_{A'B''}$, $\sigma_{B'C''}$, and $\sigma_{C'A''}$. To simplify the drawing, we omitted the indices of these copies and only indicate their original type. Note that, despite the fact that the wirings between states and CPTP maps are different than in the original scenario, every copy of a CPTP map acts on copies of the states determined by the original scenario.

ρ_{ABC} . To derive bounds on the maximum fidelity achievable by network 2-entangled states, we next imagine what states one could prepare by combining multiple realizations of the above state and channel resources. As we will see, some of the reduced density matrices of the resulting many-body *inflated states* are fully determined by the original tripartite state ρ_{ABC} . The property of ρ_{ABC} admitting a decomposition of the form of Eq. (4) will then be relaxed to that of admitting positive semidefinite inflated states satisfying said linear constraints. In the language of [26], we will be defining a nonfanout inflation of the causal scenario depicted in Fig. 1.

In this regard, consider the *ring inflation* scenario depicted in Fig. 2. If one acts on two copies of the states $\sigma_{A'B''}$, $\sigma_{B'C''}$, and $\sigma_{C'A''}$ with the maps $\{\Omega_{A'}^\lambda\}$, $\{\Omega_{B'}^\lambda\}$, and $\{\Omega_{C'}^\lambda\}$ in the ways indicated in the figure, one obtains the six-partite density matrices $\tau_{A_1 B_1 C_1 A_2 B_2 C_2}$ and $\gamma_{A_3 B_3 C_3 A_4 B_4 C_4}$. Those are essentially unknown to us, as we do not know how Eve's devices act when they are wired differently.

However, the states τ and γ are subject to several consistency constraints. To begin with, τ is symmetric under the exchange of systems $A_1 B_1 C_1$ by systems $A_2 B_2 C_2$, and so is γ under the exchange of $A_3 B_3 C_3$ by $A_4 B_4 C_4$. In addition, we observe that

$$\tau_{(A_1 B_1 C_1)} = \tau_{(A_2 B_2 C_2)} = \rho_{ABC}. \quad (6)$$

Still, we cannot say that $\tau_{A_1 B_1 C_1 A_2 B_2 C_2} = \rho_{ABC} \otimes \rho_{ABC}$ as the production of the two triangles could be classically correlated through the shared randomness Λ . However, the

state τ is separable across the $\mathcal{A}_1\mathcal{B}_1\mathcal{C}_1/\mathcal{A}_2\mathcal{B}_2\mathcal{C}_2$ partition. Both γ and τ are related to each other through the constraints

$$\gamma_{(\mathcal{A}_3\mathcal{B}_3\mathcal{A}_4\mathcal{B}_4)} = \tau_{(\mathcal{A}_1\mathcal{B}_1\mathcal{A}_2\mathcal{B}_2)} \quad (7)$$

and $\gamma_{(\mathcal{B}_3\mathcal{C}_3\mathcal{B}_4\mathcal{C}_4)} = \tau_{(\mathcal{B}_1\mathcal{C}_1\mathcal{B}_2\mathcal{C}_2)}$ and $\gamma_{(\mathcal{C}_3\mathcal{A}_4\mathcal{C}_4\mathcal{A}_3)} = \tau_{(\mathcal{C}_1\mathcal{A}_1\mathcal{C}_2\mathcal{A}_2)}$. Furthermore, τ and γ have trace one and are semidefinite positive. Finally, the reduced state $\gamma_{(\mathcal{A}_3\mathcal{B}_3\mathcal{C}_3\mathcal{B}_4)}$ is separable across the $\mathcal{A}_3\mathcal{B}_3\mathcal{C}_3/\mathcal{B}_4$ partition, and additional constraints of that type follow from cyclic symmetry.

Let us now provide some intuition as to why any state ρ_{ABC} admitting such extensions τ , γ cannot be arbitrarily close to the GHZ state. Suppose, indeed, that $F_{\text{GHZ}_d} = 1$, i.e., $\rho_{ABC} = |\text{GHZ}_d\rangle\langle\text{GHZ}_d|$ and that there exist extensions γ , τ satisfying the constraints above. A measurement in the computational basis of the sites \mathcal{A}_3 , \mathcal{B}_3 , \mathcal{C}_3 of γ will generate the random variables a_3 , b_3 , c_3 . Since $\gamma_{(\mathcal{A}_3\mathcal{B}_3)} = \rho_{(AB)} = (1/d) \sum_{i=1}^d |i, i\rangle\langle i, i|$, it must be the case that a_3 , b_3 are perfectly correlated. The same considerations hold for b_3 and c_3 . Since a_3 , b_3 , and b_3 , c_3 are pairwise perfectly correlated, so are a_3 , c_3 . Now, from the condition $\gamma_{(\mathcal{C}_3\mathcal{A}_4\mathcal{C}_4\mathcal{A}_3)} = \tau_{(\mathcal{C}_2\mathcal{A}_2\mathcal{C}_1\mathcal{A}_1)}$, we have that the distribution of c_3 and a_3 must be the same as that of c_2 and a_1 . Hence, c_2 and a_1 must be perfectly correlated. However, $\tau_{(\mathcal{A}_1\mathcal{B}_1\mathcal{C}_1)}$ is a pure state, since $\tau_{(\mathcal{A}_1\mathcal{B}_1\mathcal{C}_1)} = \rho_{ABC} = |\text{GHZ}_d\rangle\langle\text{GHZ}_d|$, and hence it must be in a product state with respect to any other system, such as \mathcal{C}_2 . It follows that a measurement in the computational basis of the sites \mathcal{A}_1 and \mathcal{C}_2 will produce two uncorrelated random variables a_1 , c_2 . We thus reach a contradiction.

The previous argument just invalidates the case $F_{\text{GHZ}_d} = 1$. A more elaborate argument (see the Supplemental Material [28] for a proof) shows that if a , b , c are the random variables resulting from measuring ρ_{ABC} locally, then any network 2-entangled state ρ_{ABC} must satisfy

$$H(a:b) + H(b:c) - H(b) \leq S(\rho_{(A)}) + S(\rho_{ABC}) - S(\rho_{(BC)}). \quad (8)$$

Here $H(x)$, $H(x:y)$, and $S(\rho)$, respectively, denote the Shannon entropy of variable x , the mutual information between the random variables x , y , and the von Neumann entropy of state ρ . The condition stated in Eq. (8) is clearly violated if $\rho_{ABC} \approx |\text{GHZ}_d\rangle\langle\text{GHZ}_d|$ and the measurements are carried in the computational basis.

Another constraint satisfied by states satisfying Eq. (4), expressed in terms of the GHZ fidelity, is

$$F_{\text{GHZ}_d} \leq \frac{2d(3d + \sqrt{2d-1})}{1 - 2d + 9d^2}. \quad (9)$$

Remarkably, in order to derive Eqs. (8) and (9), it is not necessary to invoke the existence of the six-partite states τ ,

γ , but that of their reduced density matrices $\tau_{(\mathcal{A}_1\mathcal{B}_1\mathcal{C}_1\mathcal{C}_2)}$, $\gamma_{(\mathcal{A}_3\mathcal{B}_3\mathcal{C}_3)}$. As shown in the Supplemental Material [28], both expressions, Eqs. (8) and (9), can be generalized to detect genuine network k -entanglement.

For $d = 2$, Eq. (9) establishes that any tripartite state with $F_{\text{GHZ}_2} > \frac{4}{33}(6 + \sqrt{3}) \approx 0.9372$ is genuinely network 3-entangled. As it turns out, this inequality is not tight: it can be improved to $F_{\text{GHZ}_2} > \frac{1+\sqrt{3}}{4} \approx 0.6803$ by means of semidefinite programming applied to the ring inflation.

The variables in the corresponding program are trace-one positive semidefinite matrices $\tau_{\mathcal{A}_1\mathcal{B}_1\mathcal{C}_1\mathcal{A}_2\mathcal{B}_2\mathcal{C}_2}$ and $\gamma_{\mathcal{A}_3\mathcal{B}_3\mathcal{C}_3\mathcal{A}_4\mathcal{B}_4\mathcal{C}_4}$ of size 64×64 , subject to linear constraints of the form of Eqs. (6) and (7), as well as to the permutational symmetry $1 \leftrightarrow 2$, $3 \leftrightarrow 4$. For all states $\mu_{\mathcal{X}\mathcal{Y}}$ separable across a \mathcal{X}/\mathcal{Y} partition, we add a positivity under partial transposition constraint $(\mu_{\mathcal{X}\mathcal{Y}})^{\text{PT}} \succeq 0$ [29].

This applies to τ across the $\mathcal{A}_1\mathcal{B}_1\mathcal{C}_1/\mathcal{A}_2\mathcal{B}_2\mathcal{C}_2$ partition, and to reduced states of γ for the partitions $\mathcal{A}_3\mathcal{B}_3\mathcal{C}_3/\mathcal{B}_4$, $\mathcal{B}_3\mathcal{C}_3\mathcal{A}_4/\mathcal{C}_4$, $\mathcal{C}_3\mathcal{A}_4\mathcal{B}_4/\mathcal{A}_3$.

The bound $F_{\text{GHZ}_2} > \frac{1+\sqrt{3}}{4}$ is obtained by maximizing $\langle\text{GHZ}_2|\rho_{ABC}|\text{GHZ}_2\rangle$ subject to the constraints above—a typical instance of a semidefinite program—using the optimization toolbox CVX [30] and the solver MOSEK [31].

We also employed the semidefinite optimization procedure using as reference the W state [32], $|W\rangle \equiv \frac{|001\rangle + |010\rangle + |100\rangle}{\sqrt{3}}$, concluding that any 3-qubit state ρ_{ABC} with $\langle W|\rho_{ABC}|W\rangle > 0.7602$ is genuinely network 3-entangled.

Armed with these witnesses, we find that several past experiments in quantum optics can be interpreted as demonstrations of genuine network tripartite entanglement [33–36]. Indeed, in all those experiments, the fidelity of the prepared states with respect to GHZ or W states is greater than the bounds derived above for network bipartite states. The prepared states are thus certified to contain genuine network tripartite entanglement.

Robustness to detection inefficiency.—In many experimental setups, the carriers transmitting the quantum information are often unobserved due to low detector efficiencies. The standard prescription in such a predicament consists in discarding the experimental data gathered when not all detectors click. Coming back to our adversarial setup, this postselection of measurement results opens a loophole that Eve can in principle exploit to fool Alice, Bob, and Charlie. It is possible to contemplate this contingency in the calculations above and thus bound the detection efficiency needed for certifying genuine network entanglement under postselection.

Let p indicate the fraction of experimental data preserved by postselection, i.e., the probability that all three detectors click. If ρ_{ABC}^p is the state reconstructed after postselection, then all that can be said about the true tripartite quantum state ρ_{ABC} before the postselection took place is that

$$\rho - p \times \rho_{ABC}^p \succeq 0. \quad (10)$$

As before, linear optimizations over the set of postselected states ρ_{ABC}^p can be conducted via semidefinite programming. In such instances, one continues to relate the inflated states τ and γ to the true (albeit unknown) tripartite state ρ_{ABC} , and Eq. (10) is merely added as an extra constraint. We find critical postselection probabilities beyond which one can still certify genuine network tripartite entanglement via GHZ fidelity ($p_c \approx 0.685$) or W fidelity ($p_c \approx 0.765$).

Conclusions.—In this Letter, we have argued that the standard definition of genuine multipartite entanglement is not appropriate to assess the quantum control over an ever-growing number of quantum systems. We proposed an alternative definition, *genuine network multipartite entanglement*, that captures the potential of a source to distribute entanglement over a number of spatially separated parties. We provided analytic and numerical tools to detect genuine network tripartite entanglement and also indicated how the definition can be adapted to situations where there may be local postselections on each party’s lab. Furthermore, the construction can be adapted to detect genuine network n -partite entanglement for any n .

While quite general, our numerical methods to detect genuine network entanglement demand considerable memory resources to the point that we were not able to derive new entanglement witnesses for tripartite qutrit states in a normal computer. In addition, there exist significant gaps between the bounds we derived on GHZ and W state fidelities via semidefinite programming relaxations and the lower bounds obtained using standard variational techniques [37,38]. Using such algorithms, we were not able to give lower bounds to the GHZ and W fidelities larger than 0.5170 and 2/3, respectively. A topic for future research is thus to develop better techniques for the characterization of genuine network multipartite entanglement.

We thank Antonio Acín, Jean-Daniel Bancal, T. C. Fraser, Yeong-Cherng Liang, David Schmid, and Robert W. Spekkens for useful discussions. M. N. was supported by the Austrian Science fund (FWF) stand-alone project P 30947. The work of A. P. K. was supported by Fundació Obra Social “la Caixa” (LCF/BQ/ES15/10360001), the ERC (CoG QITBOX, and the European Union’s Horizon 2020 research and innovation program, Grant Agreement No. 648913), the Spanish MINECO (FIS2016-80773-P and Severo Ochoa SEV-2015-0522), Fundació Cellex, and Generalitat de Catalunya (SGR 1381 and CERCA Program). This research was supported by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science, and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation, and Trade. This publication was made

possible through the support of a grant from the John Templeton Foundation. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation.

Note added.—After completing this manuscript, we became aware of the work of [39,40], whose authors consider a scenario very similar to that depicted in Fig. 1. Crucially, they restrict the maps $\Omega_{A,B,C}^\lambda$ to be unitary transformations, acting on convex combinations of bipartite states. The restriction to unitary maps not only allows upper-bounding the dimensionality of the source states $\sigma_{A'B'}$, $\sigma_{C'A'}$, $\sigma_{B'C'}$, but it also severely constrains the resulting set of states Δ_C . As shown in [39], tripartite qubit states in Δ_C cannot be genuinely tripartite entangled. This contrasts with the GHZ fidelity greater than 1/2 reported above, achievable by states of the form of Eq. (4).

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