Editors' Suggestion

Quantum Advantage in the Charging Process of Sachdev-Ye-Kitaev Batteries

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The exactly solvable Sachdev-Ye-Kitaev (SYK) model has recently received considerable attention in both condensed matter and high energy physics because it describes quantum matter without quasiparticles, while being at the same time the holographic dual of a quantum black hole. In this Letter, we examine SYK-based charging protocols of quantum batteries with N quantum cells. Extensive numerical calculations based on exact diagonalization for N up to 16 strongly suggest that the optimal charging power of our SYK quantum batteries displays a superextensive scaling with N that stems from genuine quantum mechanical effects. While the complexity of the nonequilibrium SYK problem involved in the charging dynamics prevents us from an analytical proof, we believe that this Letter offers the first (to the best of our knowledge) strong numerical evidence of a quantum advantage occurring due to the maximally entangling underlying quantum dynamics.

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Introduction.—In the era of quantum supremacy for quantum computing [1,2], research on the potential usefulness of quantum mechanical resources (such as entanglement) in energy science has led a consistent number of authors to introduce and study "quantum batteries" (QBs). A QB [3,4] is a system composed of N identical quantum cells, where energy is stored and from which work can be extracted.

In 2013, Alicki and Fannes [3] suggested that "entangling unitary controls", i.e., unitary operations acting globally on the state of the N quantum cells, lead to better work extraction capabilities from a QB, when compared to unitary operations acting on each quantum cell separately. Hovhannisyan *et al.* [5] were the first to demonstrate that entanglement generation leads to a speed-up in the process of work extraction, thereby leading to larger delivered power. Later on, the authors of Refs. [6,7] focused on the charging (rather than the discharging) procedure and identified two types of charging schemes: (i) the *parallel* charging scheme in which each of the N quantum cells is acted upon independently of the others and (ii) the *collective* charging scheme, where global unitary operations (i.e., the entangling unitary controls of Ref. [3]) acting on the full Hilbert space of the N quantum cells are allowed. They were able to show that, in the collective charging case and for $N \geq 2$, the charging power of a QB is larger than in the parallel scheme. This collective speed-up (stemming from entangling operations) during the charging procedure of a QB has been named "quantum advantage".

In the quest for such quantum advantage and potential laboratory implementations of QBs—based, e.g., on circuit quantum electrodynamics and trapped-ion setups—the abstract concepts of "quantum cell" and "entangling operations" have been recently spelled out more explicitly [8–27]. Different prototypes of QBs have been devised: (i) Dicke models, where arrays of N qubits (i.e., the proper battery) are coupled to a harmonic energy source [9–14], (ii) deterministic spin chains [8,26,27], and (iii) disordered spin chains [16,17]. These quantum cells can be charged by switching on either direct [8,16,17] or effective [9–14] interactions between them.

The authors of Refs. [8,9] proposed two concrete implementations of the collective charging scheme, and claimed the existence of a quantum advantage over the parallel charging procedure. However, Julià-Farré *et al.* [27] noticed that the Hamiltonians adopted in Refs. [8,9] were not properly defined in the thermodynamic limit, in the sense that their average values did not display extensivity with N, but, rather, displayed a superlinear growth with N. Moreover, the same authors were able to derive a rigorous bound for the charging power, allowing to distinguish between a genuine entanglement-induced speed-up and spurious effects, given, e.g., by the lack of a well-defined thermodynamic limit. In agreement with Ref. [26], the conclusion of Ref. [27] is that *all* the many-body QB models proposed in the literature so far do not feature any genuine quantum advantage.

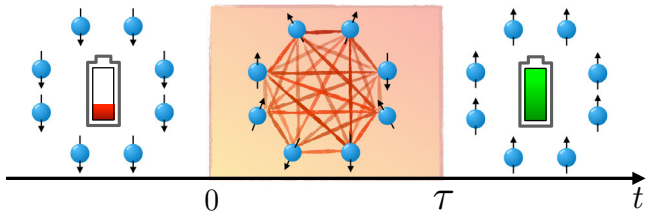

FIG. 1. The charging protocol of a QB made of N spin-1/2 units, described by the $\hat{\mathcal{H}}_0$ in Eq. (1). At time $t < 0$, the battery is fully discharged. In the time interval $0 < t < \tau$, the interacting charging Hamiltonian $\hat{\mathcal{H}}_1$ is switched on, and energy is injected via the quench. Finally, at time τ, interactions are switched off and $\hat{\mathcal{H}}_0$ is switched back on, so that the stored energy $E_N(\tau)$ is conserved thereafter.

Motivated by this literature, we propose a model of a QB which (i) is properly defined in the thermodynamic limit and (ii) unequivocally presents a genuine quantum advantage. Our implementation relies on the Sachdev-Ye-Kitaev (SYK) model [28–31], which has recently attracted a great deal of attention for its exact solvability and profound properties. The SYK model describes quantum matter with no quasiparticles. It displays fast scrambling [32,33], has a nonzero entropy density at vanishing temperature [34,35], all its eigenstates exhibit volume-law entanglement entropy [36,37], and is holographically connected to the dynamics of AdS_2 horizons of quantum black holes [29,30,38,39]. Proposals to realize the SYK Hamiltonian have been recently put forward and rely on ultracold atoms [40], graphene flakes with irregular boundaries [41], and topological superconductors [42,43].

Many-body QBs and figures of merit.—Consider a QB made of N identical quantum cells (for an illustration, see Fig. 1), which are governed by the following free and local Hamiltonian ($\hbar = 1$):

$$\hat{\mathcal{H}}_0 = \sum_{j=1}^{N} \hat{h}_j. \tag{1}$$

At time $t = 0$, the system is prepared in its ground state $|0\rangle$, physically representing the discharged battery. By suddenly switching on a suitable interaction Hamiltonian $\hat{\mathcal{H}}_1$ for a finite amount of time τ (and switching off $\hat{\mathcal{H}}_0$), one aims at injecting as much energy as possible into the quantum cells [6–8]. The time interval τ is called the charging time of the protocol. The full model Hamiltonian can thus be written as

$$\hat{\mathcal{H}}(t) = \hat{\mathcal{H}}_0 + \lambda(t)(\hat{\mathcal{H}}_1 - \hat{\mathcal{H}}_0), \tag{2}$$

where $\lambda(t)$ is a classical parameter that represents the external control exerted on the system, and which is assumed to be given by a step function equal to 1 for $t \in [0, \tau]$ and zero elsewhere. Such charging protocol is experimentally feasible, e.g., in cold-atom setups [44], where implementing sudden quenches is a standard procedure. Accordingly, denoting by $|\psi(t)\rangle$ the state of the system at time t, its total energy $E_N^{\rm tot}(t) = \langle\psi(t)|\hat{\mathcal{H}}(t)|\psi(t)\rangle$ is constant for all values of t but $t = 0$ and $t = \tau$ (the switching points).

The energy injected into the N quantum cells can be expressed in terms of the mean local energy at the end of the protocol, $E_N(\tau) = \langle\psi(\tau)|\hat{\mathcal{H}}_0|\psi(\tau)\rangle$. In writing the previous equation, we have set to zero the ground-state energy $\langle 0|\hat{\mathcal{H}}_0|0\rangle$. Other crucial figures of merit are the average charging power $P_N(\tau) = E_N(\tau)/\tau$ and its optimal value,

$$P_N(\tau^*) = \max_{\tau>0} P_N(\tau), \tag{3}$$

obtained at time τ^*. In the following, we will be mainly interested in the scaling of the optimal charging power $P_N(\tau^*)$ with the number N of quantum cells.

SYK-based charging protocols.—We assume each quantum cell to be a spin-1/2 system. In the absence of charging operations, the system is described by the noninteracting Hamiltonian (1), with $\hat{h}_j = \omega_0\hat{\sigma}_j^y/2$. Here, $\omega_0 > 0$ represents a magnetic field strength (with units of energy) and $\hat{\sigma}_j^\alpha$ ($\alpha = x, y, z$) are the Pauli matrices. The battery energy $E_N(\tau)$ will be measured in units of the energy scale ω_0. At time $t = 0$, the quantum cells are initialized in the ground state of $\hat{\mathcal{H}}_0$, $|0\rangle = \otimes_{j=1}^N |\downarrow^{(y)}\rangle_j$, where $\hat{\sigma}_j^y|\downarrow^{(y)}\rangle_j = -|\downarrow^{(y)}\rangle_j$.

For the charging Hamiltonian $\hat{\mathcal{H}}_1$, we use the complex SYK (c-SYK) [30,45,46] model Hamiltonian:

$$\hat{\mathcal{H}}_1^{\text{c-SYK}} = \sum_{i,j,k,l=1}^{N} J_{i,j,k,l}\hat{c}_i^\dagger\hat{c}_j^\dagger\hat{c}_k\hat{c}_l, \tag{4}$$

where $\hat{c}_j^\dagger$ ($\hat{c}_j$) is a spinless fermionic creation (annihilation) operator [47]. This has to be understood in its spin-1/2 representation, which is obtained by the Jordan-Wigner transformation $\hat{c}_j^\dagger = \hat{\sigma}_j^+(\Pi_{m=1}^{j-1}\hat{\sigma}_m^z)$, where $\hat{\sigma}_j^\pm \equiv (\hat{\sigma}_j^x \pm i\hat{\sigma}_j^y)/2$ [48]. The couplings $J_{i,j,k,l}$ are zero-mean Gaussian-distributed complex random variables, with variance $\langle\langle J_{i,j,k,l}^2\rangle\rangle = J^2/N^3$, satisfying $J_{i,j,k,l} = J_{k,l,i,j}^*$ and $J_{i,j,k,l} = -J_{j,i,k,l} = -J_{i,j,l,k}$. In the following, we average any quantity of interest $\mathcal{O}$ over the distribution of $\{J_{i,j,k,l}\}$, and denote by $\langle\langle\mathcal{O}\rangle\rangle$ the averaged value, i.e., $\langle\langle\mathcal{O}\rangle\rangle \equiv \int P(\{J_{i,j,k,l}\})\mathcal{O}(\{J_{i,j,k,l}\})d\{J_{i,j,k,l}\}$.

We emphasize that our choice of battery and charging Hamiltonians is such that $[\hat{\mathcal{H}}_0, \hat{\mathcal{H}}_1] \neq 0$, a condition which ensures energy injection into the QB by the charging protocol (2). Note, finally, that the Hamiltonian in Eq. (4) is invariant under particle-hole symmetry (PHS) in the thermodynamic limit $N \to \infty$. Extra terms, however, need to be added to it in order to enforce PHS at any finite N [45]:

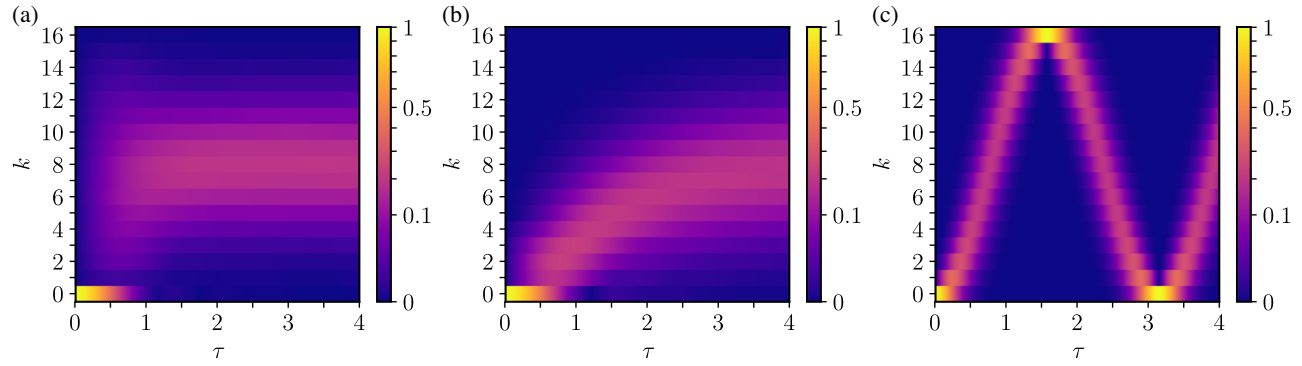

FIG. 2. Dynamics of the dimensionless population $p_k(\tau)$ of the QB energy levels as a function of time τ (in units of $1/J$) and the level index k for three different charging protocols: c-SYK (a), b-SYK with $\bar{J} = J$ (b), and parallel with $K = J$ (c). Data in (a) and (b) correspond to a single realization of disorder in the couplings $J_{i,j,k,l}$ and $\bar{J}_{i,j,k,l}$.

$$\hat{\mathcal{H}}_1^{\text{c-SYK (PHS)}} = \hat{\mathcal{H}}_1^{\text{c-SYK}} + \frac{1}{2}\sum_{i,j,k,l=1}^{N} J_{i,j,k,l} \times (\delta_{i,k}\hat{c}_j^\dagger \hat{c}_l - \delta_{i,l}\hat{c}_j^\dagger \hat{c}_k - \delta_{j,k}\hat{c}_i^\dagger \hat{c}_l + \delta_{j,l}\hat{c}_i^\dagger \hat{c}_k). \tag{5}$$

Hereafter, we will always use this version of the c-SYK model. We have, however, checked that our main findings do not qualitatively change if PHS is not enforced and Eq. (4), rather than Eq. (5), is used as charging Hamiltonian.

In the following, we will also consider charging Hamiltonians based on a bosonic version of the SYK model (b-SYK) [45]:

$$\hat{\mathcal{H}}_1^{\text{b-SYK}} = \sum_{i,j,k,l=1}^{N} \bar{J}_{i,j,k,l}\hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_k \hat{b}_l, \tag{6}$$

where $\hat{b}_j^\dagger$ ($\hat{b}_j$) creates (annihilates) a hard-core boson. The following relations are obeyed: $\{\hat{b}_j, \hat{b}_j^\dagger\} = 1$ and $[\hat{b}_i, \hat{b}_j] = 0$ for $i \neq j$. Hence, $\hat{b}_j^\dagger$ can be directly written in its spin representation as $\hat{b}_j^\dagger = \sigma_j^+$. Similarly to $J_{i,j,k,l}$, the quantities $\bar{J}_{i,j,k,l}$ in Eq. (6) are random, Gaussian-distributed variables, with variance $\langle\langle \bar{J}_{i,j,k,l}^2 \rangle\rangle = J^2/N^3$, satisfying $\bar{J}_{i,j,k,l} = \bar{J}_{k,l,i,j}^*$ and $\bar{J}_{i,j,k,l} = \bar{J}_{j,i,k,l} = \bar{J}_{i,j,l,k}$ (in order to comply with the bosonic commutation rules of the model). For PHS to hold, we enforce the site indices i, j, k, l in Eq. (6) to be all different [45]. Note that the dependence of the variance of the couplings $J_{i,j,k,l}$ and $\bar{J}_{i,j,k,l}$ on the inverse third power of N ensures that all our SYK charging Hamiltonians are well defined in the thermodynamic limit. Indeed, their average values scale extensively with N [51].

Finally, we will also examine a parallel charging protocol [6,7] based on the following Hamiltonian:

$$\hat{\mathcal{H}}_1^{\parallel} = K\sum_{j=1}^{N} \hat{\sigma}_j^x. \tag{7}$$

In this case, each of the N quantum cells is acted upon independently of the others and no entanglement is generated [27]. The charging protocol based on $\hat{\mathcal{H}}_1^{\parallel}$ will therefore serve as reference model, to be compared against c-SYK and b-SYK charging models.

Microscopy of the charging dynamics in energy space.—As an indicator of the speed of the dynamics, we start by looking at the time evolution of the energy-level occupations. Consider the spectral decomposition of Hamiltonian (1): $\hat{\mathcal{H}}_0 = \sum_{k=0}^{N} \epsilon_k \sum_i |k,i\rangle\langle k,i|$, where $\epsilon_k = k\omega_0$ denote its eigenvalues and the index i accounts for the degenerate eigenvectors. We are interested in the dynamics of the populations:

$$p_k(\tau) = \sum_i |\langle k,i|\psi(\tau)\rangle|^2. \tag{8}$$

Figure 2 displays $p_k(\tau)$ for the three charging Hamiltonians mentioned above: c-SYK [Fig. 2(a)], b-SYK [Fig. 2(b)], and parallel [Fig. 2(c)]. While in the latter two cases the charging protocol generates a dynamics that is clearly local in energy space, this is not the case for the c-SYK model. This charging model generates a nonlocal population dynamics in energy space, which manifests as a sudden macroscopic population of excited levels. Indeed, after an ultrashort "thermalization" time [52], a central band of excited energy levels appears uniformly populated. (Further details on the thermalization properties of c-SYK QBs are provided in Ref. [48].) This nonlocality is a direct realization of the global charging dynamics envisioned by the authors of Ref. [6]. Recurrences appearing in the charging dynamics highlighted in Fig. 2(c) witness the integrability of the parallel Hamiltonian in Eq. (7), which is absent in the SYK models.

Power, bounds, and quantum advantage.—Quantitative conclusions on the charging performances of SYK QBs, compared to those of other reference many-body QBs, can be drawn from the analysis of the optimal power $P_N(\tau^*)$ in Eq. (3) and its scaling with N. Specifically, a rigorous

certification of the quantum origin of the charging advantage of the c-SYK model can be achieved by considering the following bound [27]:

$$P_N(\tau) \leq 2\sqrt{\Delta_\tau \hat{\mathcal{H}}_0^2 \Delta_\tau \hat{\mathcal{H}}_1^2}, \tag{9}$$

where $\Delta_\tau \hat{\mathcal{H}}^2 \equiv (1/\tau)\int_0^\tau dt[\langle \hat{\mathcal{H}}^2\rangle_t - (\langle \hat{\mathcal{H}}\rangle_t)^2]$ and $\langle \hat{\mathcal{O}}\rangle_t \equiv \langle\psi(t)|\hat{\mathcal{O}}|\psi(t)\rangle$. Here, $\Delta_\tau \hat{\mathcal{H}}_1^2$ represents the charging speed in the Hilbert space: larger values of such quantity correspond to trivial increases of the charging speed. In contrast, $\Delta_\tau \hat{\mathcal{H}}_0^2$ is connected with the distance traveled in the Hilbert space. An enhancement of it can be linked to shortcuts in the Hilbert space: starting from a pure state and going through highly entangled states, it is possible to reduce the length of the trajectory in such space, consequently enhancing the charging power [27]. This is a genuine quantum effect, with no classical analog. Any increase of the average optimal power linked to $\Delta_\tau \hat{\mathcal{H}}_0^2$ can be considered as the smoking gun of a genuine quantum advantage, unreproducible by classical dynamics. A detailed derivation of the bound (9) is provided in Ref. [48].

If the battery Hamiltonian $\hat{\mathcal{H}}_0$ is made of a sum of local terms, as in Eq. (1), it is possible to write $\Delta_\tau \hat{\mathcal{H}}_0^2$ as $\Delta_\tau \hat{\mathcal{H}}_0^2 = \Delta_\tau^{\rm loc}\hat{\mathcal{H}}_0^2 + \Delta_\tau^{\rm ent}\hat{\mathcal{H}}_0^2$, with [27]

$$\Delta_\tau^{\rm loc}\hat{\mathcal{H}}_0^2 \equiv \frac{1}{\tau}\int_0^\tau dt \sum_i [\langle \hat{h}_i^2\rangle_t - \langle \hat{h}_i\rangle_t^2], \tag{10}$$

$$\Delta_\tau^{\rm ent}\hat{\mathcal{H}}_0^2 \equiv \frac{1}{\tau}\int_0^\tau dt \sum_{i\neq j} [\langle \hat{h}_i\hat{h}_j\rangle_t - \langle \hat{h}_i\rangle_t\langle \hat{h}_j\rangle_t]. \tag{11}$$

The quantity (10), being a sum of local terms, scales linearly with N (i.e., is extensive) by construction. On the other hand, $\Delta_\tau^{\rm ent}\hat{\mathcal{H}}_0^2$, whose explicit form can be immediately linked to correlations between sites i and j, may display a superlinear scaling with N. Because of the nonlinearity of the bound (9), which applies to a single disorder realization, averaging over disorder is not straightforward. Through the Cauchy-Schwarz inequality, though, it is possible to rewrite it as $\langle\langle P_N(\tau)\rangle\rangle \leq 2\langle\langle\sqrt{\Delta_\tau \hat{\mathcal{H}}_0^2 \Delta_\tau \hat{\mathcal{H}}_1^2}\rangle\rangle \leq 2\sqrt{\langle\langle\Delta_\tau \hat{\mathcal{H}}_0^2\rangle\rangle\langle\langle\Delta_\tau \hat{\mathcal{H}}_1^2\rangle\rangle}$, meaning that one can separately study the averaged quantities $\langle\langle\Delta_\tau \hat{\mathcal{H}}_0^2\rangle\rangle$ and $\langle\langle\Delta_\tau \hat{\mathcal{H}}_1^2\rangle\rangle$. Here we are interested in the scaling at the optimal time τ^*; thus we focus on

$$\langle\langle P_N(\tau^*)\rangle\rangle \leq 2\sqrt{\langle\langle\Delta_{\tau^*} \hat{\mathcal{H}}_0^2\rangle\rangle\langle\langle\Delta_{\tau^*} \hat{\mathcal{H}}_1^2\rangle\rangle}. \tag{12}$$

Since the battery energy is measured in units of ω_0 and time in units of $1/J$, the averaged charging power $\langle\langle P_N(\tau^*)\rangle\rangle$ is measured in units of $\omega_0 J$. Given this choice, we specify the energy scales of the b-SYK and parallel charging protocols by setting $\bar{J} = K = J$ [53].

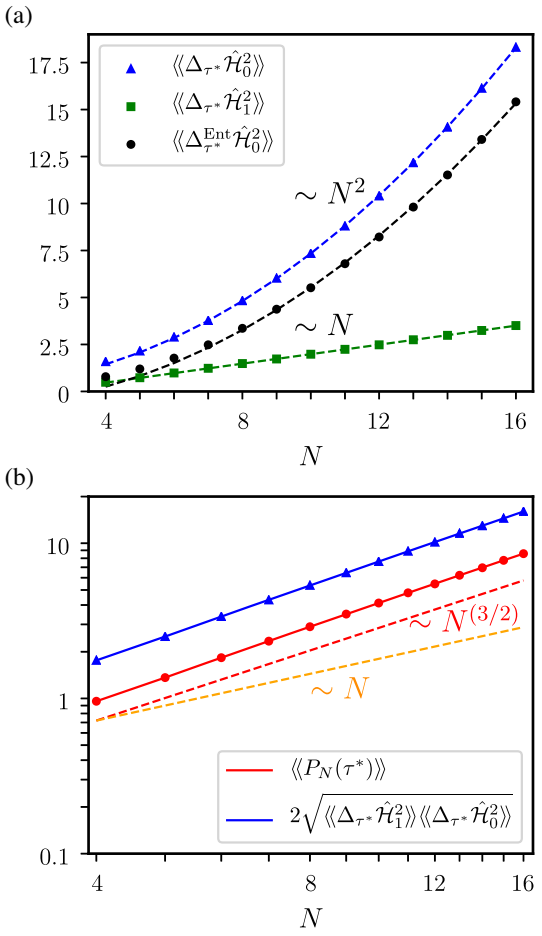

FIG. 3. (a) The relevant quantities for the bound (12), evaluated at the optimal time τ^*, and averaged over disorder: time-averaged variances $\langle\langle\Delta_{\tau^*}\hat{\mathcal{H}}_0^2\rangle\rangle$ (blue triangles, in units of ω_0^2), $\langle\langle\Delta_{\tau^*}\hat{\mathcal{H}}_1^2\rangle\rangle$ (green squares, in units of J^2), $\langle\langle\Delta_{\tau^*}^{\rm ent}\hat{\mathcal{H}}_0^2\rangle\rangle$ (black circles, in units of ω_0^2), as functions of N. Dashed curves denote linear (green) and quadratic (blue, black) fits to the numerical results. The four data points corresponding to the smallest N have been discarded from the fits. (b) The optimal power (red) $\langle\langle P_N(\tau^*)\rangle\rangle$ and the quantity in the right-hand side of Eq. (12) (blue) are plotted as functions of N, in a log-log scale and in units of $\omega_0 J$. Dashed lines correspond to power laws $\sim N^{1+k}$ ($k = 0.5$, red; $k = 0$, orange) and are plotted as guides to the eye. Data in this figure refer to the c-SYK QB model, and have been obtained after averaging over $N_{\rm dis} = 10^3$ (for $N = 4, \ldots, 10$), 5×10^2 (for $N = 11, 12$), and 10^2 (for $N = 13, \ldots, 16$) instances of disorder in the couplings $\{J_{i,j,k,l}\}$.

Figure 3(a) shows the relevant quantities for the bound (12), for a c-SYK QB. While $\langle\langle\Delta_{\tau^*}\hat{\mathcal{H}}_1^2\rangle\rangle$ is extensive in N, we observe that both $\langle\langle\Delta_{\tau^*}\hat{\mathcal{H}}_0^2\rangle\rangle$ and $\langle\langle\Delta_{\tau^*}^{\rm ent}\hat{\mathcal{H}}_0^2\rangle\rangle$ display a superlinear scaling with N, which is compatible with a quadratic growth. This means that, during the time evolution, the c-SYK charging Hamiltonian generates the maximum possible nonlocality between the quantum cells, in the form of N-partite entanglement [27]. This, together with Eq. (12), suggests a superlinear scaling with N of the optimal charging power,

$$\langle\langle P_N(\tau^*)\rangle\rangle \sim N^{1+k}, \quad \text{with} \quad k > 0, \tag{13}$$

where $k \approx 0.5$. For the first time in the literature on QB models [8–27], we are thus in a situation where the power enhancement is linked to $\Delta_\tau \hat{\mathcal{H}}_0^2$, a fact that hints at a quantum advantage (i.e., advantage over any classical battery) displayed by the c-SYK model with respect to the charging task. Further details on the comparison between quantum and classical many-body batteries are given in Ref. [48].

The left- and right-hand side members of the inequality (12) are displayed in Fig. 3(b), red and blue data, respectively. We clearly see a superlinear scaling with N ($k = 0.5$ corresponds to the red dashed straight line). We have also considered the b-SYK and parallel charging models, showing that, in both cases, all the quantities $\langle\langle \Delta_{\tau^*} \hat{\mathcal{H}}_0^2 \rangle\rangle$, $\langle\langle \Delta_{\tau^*} \hat{\mathcal{H}}_1^2 \rangle\rangle$, and $\langle\langle \Delta_{\tau^*}^{\text{ent}} \hat{\mathcal{H}}_0^2 \rangle\rangle$ are extensive in N [48]. In agreement with the results shown in Figs. 2 and 3, we thus conclude that these two QB models do not display any genuine quantum advantage.

We finally recall that optimal charging powers that scale faster than N have been found in Refs. [8,9]. Unfortunately, such superlinear scalings do not stem from $\Delta_\tau \hat{\mathcal{H}}_0^2$ but rather from $\Delta_\tau \hat{\mathcal{H}}_1^2$, and therefore have no quantum origin [27]. The fact that the Hamiltonians used in Refs. [8,9] are not properly defined in the thermodynamic limit is ultimately at the origin of the spurious super-extensive scaling of the optimal charging power. This is *explicitly* shown in Ref. [48] for Dicke QBs. In this Letter, we have bypassed this problem by choosing the appropriate scaling [28–31,45,46] with N of the variance $\langle\langle J_{i,j,k,l}^2 \rangle\rangle = J^2/N^3$ of the c-SYK coupling parameters.

Alternatively, another strategy to rule out any spurious effect on the optimal charging power is to use a "renormalization" approach that consists in dividing the charging Hamiltonian by its operator norm [48]. This procedure allows for a fair comparison between different QB models [7]. In agreement with the results illustrated above, we have found a clear increase with N of the optimal charging power only for the renormalized c-SYK Hamiltonian [48].

In the future, it will be interesting to study SYK-type models in the context of heat engines [54,55], where minimizing timescales is also of central importance.

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