Electron-Tunneling-Assisted Non-Abelian Braiding of Rotating Majorana Bound States

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It has been argued that fluctuations of fermion parity are harmful for the demonstration of non-Abelian anyonic statistics. Here, we demonstrate a striking exception in which such fluctuations are actively used. We present a theory of coherent electron transport from a tunneling tip into a Corbino geometry Josephson junction where four Majorana bound states (MBSs) rotate. While the MBSs rotate, electron tunneling happens from the tip to one of the MBSs thereby changing the fermion parity of the MBSs. The tunneling events in combination with the rotation allow us to identify a novel braiding operator that does not commute with the braiding cycles in the absence of tunneling, revealing the non-Abelian nature of MBSs. The time-averaged tunneling current exhibits resonances as a function of the tip voltage with a period that is a direct consequence of the interference between the noncommuting braiding operations. Our work opens up a possibility for utilizing parity nonconserving processes to control non-Abelian states.

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Introduction.—A braiding operation reveals the quantum statistics of identical particles [1–3]. Majorana zero-energy states bound to certain defects (e.g., vortices or edges) in topological superconductors are quasiparticles obeying non-Abelian statistics [4–8]. In an isolated system with $2N$ decoupled Majorana states, there is a 2^N-fold degenerate ground-state manifold $\{|\Psi\rangle\}$, and adiabatically moving one Majorana state around another acts as a unitary matrix on the manifold. Such unitary matrices of different braiding operations, A and B, are in general noncommutative, so that the order of operations matter,

$$AB|\Psi\rangle \neq BA|\Psi\rangle \quad \text{or} \quad (AB - BA)|\Psi\rangle \neq 0. \qquad (1)$$

Non-Abelian braiding is one of the hallmarks of topological quantum phases associated with non-Abelian statistics appearing in many contexts [3,9,10] and also represents the basic resource for executing topologically protected gates for quantum computing [1,11].

The essence of the present work is to provide transport signatures of Majorana bound states (MBSs) induced by the noncommutativity shown in Eq. (1). The envisioned system is a Corbino geometry topological Josephson junction (JJ), formed by two s-wave superconductors on a topological insulator (TI) surface [see Fig. 1(a)]. Four vortices, each hosting a MBS, rotate along the junction, and the time-dependent tunneling conductance between the junction and a metallic tip is measured [12]. A ground state of the system evolves in the fourfold degenerate ground-state manifold, governed by the rotation and the coherent electron tunneling processes. The evolution can be cast into two braiding operators [corresponding to A and B in Eq. (1)] which do not commute: one is a parity-conserving rotation and the other is a tunneling-assisted braiding. In the low bias voltage regime, the time-averaged conductance exhibits unusual peak positions, which we interpret as a direct signature of noncommutativity of the two braiding operators.

Tremendous amounts of proposals and experiments lead to great achievements in the realization [13–19], manipulation [20–25], and detection [26–37] of MBSs in

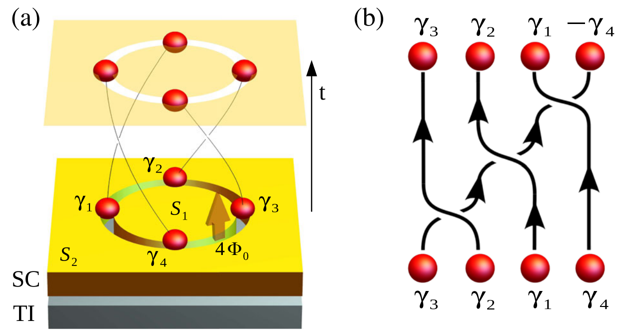

FIG. 1. (a) Schematic of a Corbino geometry Josephson junction formed by thin-film superconductors (S_1 and S_2) deposited on the surface of a topological insulator (TI). In the presence of four flux quanta $4\Phi_0$, four MBSs γ_j (red balls) appear in the junction. Majorana positions can move along the junction when applying a small voltage across the junction, allowing us to perform an adiabatic rotation. (b) Braiding depicted as worldlines of the four MBSs corresponding to the $\pi/2$ rotation shown in (a).

superconducting hybrid structures. In particular, a recent experiment exploiting a quantum anomalous Hall insulator-superconductor structure [38] boosts interest in searches for transport signatures of non-Abelian braiding [39,40]. Based on such hybrid structures, the authors of Refs. [39,40] theoretically investigated transport properties of Mach-Zehnder-like interferometers of chiral Majorana modes. The overlap or fusion of two paths of Majorana modes whose relative dynamics is determined by braiding with the other Majoranas signals a unitary evolution (which is not a phase factor) of Majorana modes.

Different to these recent studies in Refs. [39,40], we demonstrate interference involving four rotating MBSs whose braiding operations are assisted by tunneling of electrons into or out of the MBSs and thus in which the fermion parity formed by the MBSs is not conserved. Such tunneling-assisted braiding has been to the best of our knowledge not considered before, on the contrary, electron tunneling was seen detrimental for topological quantum processing [41–43]. We will show that, in our scheme, electron tunneling probes non-Abelian statistics via the tunneling conductance. Our scheme does not require control of fusions of Majorana states.

Theoretical model.—We consider a Corbino JJ deposited on the surface (x-y plane) of a three dimensional TI [Fig. 1(a)]. The circular shaped junction with a radius R is formed by thin films of inner (S_1) and outer (S_2) s-wave superconductors and contains four magnetic flux quanta, $4\Phi_0$ with $\Phi_0 = h/(2e)$, inducing a phase difference across the junction [see Eq. (4)]. The Bogoliubov–de Gennes Hamiltonian for the TI surface proximity coupled to the Corbino JJ is given by [44]

$$H_C = \frac{1}{2}\int d^2r \Phi^\dagger(\mathbf{r})\mathcal{H}_C\Phi(\mathbf{r}), \tag{2}$$

$$\mathcal{H}_C = \begin{pmatrix} \mathcal{H}_0 - \mu & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & \mu - \mathcal{H}_0 \end{pmatrix}, \tag{3}$$

and $\Phi(\mathbf{r}) = (\Phi_\uparrow, \Phi_\downarrow, \Phi_\downarrow^\dagger, -\Phi_\uparrow^\dagger)^T$ is the Nambu spinor and $\mathcal{H}_0 = v_F(\sigma_x p_x + \sigma_y p_y)$ with Pauli spin matrices $\sigma_{x,y}$ describes the surface states and μ is the chemical potential. The proximity-induced superconducting gap $\Delta(\mathbf{r})$ is

$$\Delta(\mathbf{r}) = \begin{cases} \Delta_0 e^{i\phi_1} & 0 \le r < R, \\ \Delta_0 e^{-i4\theta + i\phi_2} & r > R, \end{cases} \tag{4}$$

where ϕ_1 and ϕ_2 are spatially uniform phases in each superconducting region, and the polar-angle-dependent phase -4θ at $r > R$ is due to the presence of the four flux quanta [45]. By solving the Bogoliubov–de Gennes equation $\mathcal{H}_C\Psi(\mathbf{r}) = E\Psi(\mathbf{r})$, we find four Majorana wave functions $\Psi_{Mj}(\mathbf{r})$ with $j \in \{1, 2, 3, 4\}$, at zero energy $E = 0$. They are localized at $(r, \theta) = (R, \theta_j)$ where $\theta_j = (3\pi - 2\pi j)/4 - (\phi_1 - \phi_2)/4$, at which the local phase difference across the junction is π. Detailed calculations of the Majorana wave functions for $\mu = 0$ are given in Supplemental Material [46].

If we change $\phi_1 - \phi_2$ by 2π, the four MBSs rotate by $\pi/2$ in a clockwise direction maintaining their relative distances, as plotted in Fig. 1(a), leading to a transformation $\gamma_j \to U_c \gamma_j U_c^\dagger$,

$$\gamma_1 \to -s\gamma_2, \qquad \gamma_2 \to -s\gamma_3, \qquad \gamma_3 \to s\gamma_4, \qquad \gamma_4 \to -s\gamma_1, \tag{5}$$

where $\gamma_j = \int d^2r \Psi_{Mj}^\dagger(\mathbf{r})\Phi(\mathbf{r})$. $s = 1(-1)$ corresponds to the change of $\phi_1(\phi_2)$ by $2\pi(-2\pi)$. Graphical representation of the transformation is given in Fig. 1(b) for the $s = -1$ case. A rotation operator U_c for the transformation can be constructed as a product of three pairwise braidings $U_c = U_{41}U_{12}U_{23}$ where U_{ij} is the braiding exchange operator of γ_i and γ_j given by $U_{ij} = \exp(s\pi\gamma_i\gamma_j/4)$ [49].

The adiabatic rotation can be achieved if a dc-bias voltage V_J across the junction is much smaller than the excitation energy of the junction. For a finite V_J, $\phi_1 - \phi_2$ varies in time t as $\phi_1 - \phi_2 = \phi_0 + 2eV_J t/\hbar$ where ϕ_0 is a spontaneously chosen constant. The states $\Psi_{Mj}[\mathbf{r}, \phi_1(t), \phi_2(t)]$ then become instantaneous eigenstates of $\mathcal{H}_C[\phi_1(t), \phi_2(t)]$ at zero energy, and U_c can be considered as the time evolution operator of the MBSs from t to $t + T_J$, where $T_J = (\pi\hbar/eV_J)$ is the time needed for the $\pi/2$ rotation.

Tunneling-assisted Majorana braiding.—To explore the effect of electron tunneling, we connect a metal tip to the Corbino JJ, as depicted in Fig. 2(a). The tip is located such that an electron can tunnel onto or off the Corbino JJ through $\gamma_1(t_0)$ at $t = t_0$, and we assume that the tunnel coupling is switched on at $t = t_0$. A phase coherent time-

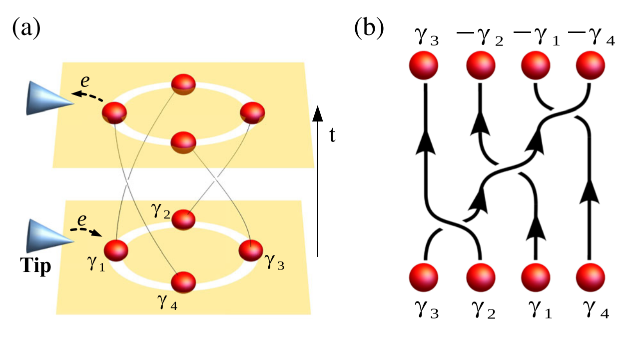

FIG. 2. (a) Time-dependent electron tunneling between the rotating MBSs and a metal tip for detecting non-Abelian statistics. (b) Tunneling-assisted braiding created by the composition of the $\pi/2$ rotation shown in (a) and electron tunneling. The tunneling effect reverses the exchange direction of a Majorana pair involving γ_1. A signature of the interference processes involving the non-Abelian braiding operations—the tunneling-assisted braiding and the braiding shown in Fig. 1(b)—is probed by the time-averaged tunneling current.

dependent tunneling event between the tip and adiabatically rotating Majorana states can occur at discrete times $t_q = t_0 + qT_J$, where $q = 0, 1, 2, \ldots$. Creation or annihilation of an electron via a Majorana state at $t = t_q$ is described by $\gamma_1(t_0)|\Psi_g(t_q)\rangle$, where $|\Psi_g(t_q)\rangle = U_c^q|\Psi_g(t_0)\rangle$ is the time-evolved initial state (being part of the ground-state manifold) of the MBSs from t_0 to t_q. Note that our proposal does not depend on the initial configuration of the ground state and other choices of Majorana states coupled to the tip at $t = t_0$. Hereafter, we will denote $\gamma_1(t_0)$ by γ_1.

The time evolution of a Majorana state from $t = t_{q'}$ to t_q at which tunneling events occur is described by the Majorana Green's function

$$M(t_q, t_{q'}) = -i\mathrm{Tr}[\rho_0\hat{\gamma}_1(t_q)\hat{\gamma}_1(t_{q'})], \tag{6}$$

where $\hat{\gamma}_1(t_q) = (U_c^\dagger)^q\gamma_1 U_c^q$ and ρ_0 is a density matrix of the Majorana state at $t = t_0$. For a more comprehensive description of the tunneling effect, we introduce a tunneling-assisted braiding operator,

$$\bar{U}_c = \gamma_1 U_c \gamma_1, \tag{7}$$

consisting of three events: changing fermion-occupation-number parity due to the tunneling at $t = t_q$, followed by an evolution for a time T_J with U_c, and then changing the parity again at $t = t_q + T_J$. The transformation governed by $\bar{U}_c$ is drawn in Fig. 2(b); comparing the cases without and with the tunneling in Figs. 1(b) and 2(b), respectively, notice that the tunneling effectively reverses the direction of the pairwise braiding when a braiding involves γ_1. Therefore, $\bar{U}_c = U_{14}U_{21}U_{23}$ can be considered—besides U_c—as another genuine braiding operator. $M(t_q, t_{q'})$ then can be presented as

$$M(t_q, t_{q'}) = -i\mathrm{Tr}[\rho_0'(\bar{U}_c)^n(U_c^\dagger)^n], \tag{8}$$

where we used the cyclic property of the trace. $\rho_0' = (U_c)^q\rho_0(U_c^\dagger)^q$ and $n = q - q'$. We find that $U_c^\dagger$ and $\bar{U}_c$ do not commute, $[\bar{U}_c, U_c^\dagger] \neq 0$. As a consequence, $M(t_q, t_{q\prime})$ is not just a sum of phase factors but involves nontrivial state changes in the ground-state manifold. We show below that the noncommuting braidings result in observable interference signatures free of the necessity of physically fusing MBSs.

Transport signatures.—To obtain the tunneling current between the tip and the JJ in the weak coupling limit, we extend the formalism of Ref. [12] to four MBSs. The Hamiltonian of the tip is $H_N = \sum_{k\sigma}\varepsilon_k c_{k\sigma}^\dagger c_{k\sigma}$, where $c_{k\sigma}$ is the electron annihilation operator in the tip with momentum k and spin σ. Since we are interested in the low-energy sector of the junction, tunneling between the tip and the MBSs is the only relevant process. Around $t = t_q$ where the coupling strength to γ_1 is maximal, we assume that the coupling increases and decreases exponentially as γ_1 approaches to and leaves from the tip, respectively, while its phase does not change significantly. Moreover, since the Majorana states are spin polarized, and couple only to electrons of the tip with their spin parallel to that of the Majorana states; electrons with opposite spin are reflected at the junction between the tip and the Corbino JJ and do not contribute to the tunneling current. Then the tunneling Hamiltonian becomes

$$H_T(t) = \sum_{k,q} e^{-\lambda|t-t_q|}V_{1k}c_k^\dagger\gamma_1 + \mathrm{H.c.}, \tag{9}$$

where λ^{-1} is the tunneling duration and V_{1k} is the coupling between the tip and γ_1. Here, we have assumed $\lambda^{-1} \ll T_J$, implying that only nearest-neighbor coupling between the tip and the MBSs is taken into account.

Using the current expression $I(t) = -edN_T/dt$ with the tip number operator $N_T = \sum_k c_k^\dagger c_k$ and lowest order perturbation theory in $H_T(t)$, the differential conductance of the time-averaged current measured after many rotation cycles of MBSs has the form

$$\frac{d\bar{I}}{dV} = \frac{e}{h}\int_{-\infty}^{\infty} d\varepsilon T(\varepsilon)[S(\varepsilon) + S(-\varepsilon)]\frac{dn_F(\varepsilon - eV)}{dV}, \tag{10}$$

where n_F is the Fermi-Dirac distribution and eV is the bias voltage. The tunneling probability $T(\varepsilon)$ and the interference term $S(\varepsilon)$ are given by

$$T(\varepsilon) = \frac{2\Gamma T_J}{\hbar}\left(\frac{2\lambda T_J}{\lambda^2 T_J^2 + \tilde{\varepsilon}^2}\right)^2, \tag{11}$$

$$S(\varepsilon) = \mathrm{Re}\left\{\frac{1}{2} + i\sum_{n=1}^{Q} e^{in\tilde{\varepsilon}}M(t_Q, t_{Q-n})\right\}. \tag{12}$$

Here, $\tilde{\varepsilon} = \varepsilon/(\hbar T_J^{-1})$ and the integer $Q \gg 1$, which will go to infinity later. $\Gamma = 2\pi\rho|V_{1k}|^2$ where ρ is the tip density of states. We assumed a wideband approximation where ρ and V_{1k} are energy independent and we neglected the contributions proportional to $e^{-\lambda T_J/2}$; note that these small contributions do not change the positions of conductance peaks. The details for the calculation of $\bar{I}$ are given in [46]. In the limit $Q \to \infty$, we obtain

$$\frac{d\bar{I}}{dV} = \frac{e^2}{h}\frac{\pi\hbar}{8T_J k_B T}\sum_l T(\varepsilon_l)\mathrm{sech}^2\left(\frac{eV - \varepsilon_l}{2k_B T}\right), \tag{13}$$

which shows peaks at $\varepsilon_l = (\hbar/4T_J)(2\pi l - \alpha)$ where l is an integer and $\alpha = \pi$ arising from a 2π rotation of the four MBSs. This perturbative calculation is valid for $T(\varepsilon_0)\hbar/(8T_J) \ll k_B T \ll E_g$ where E_g is the excitation energy of the junction.

The $d\bar{I}/dV$ in Eq. (13) is plotted in Fig. 3 for realistic parameters. It shows peaks at $eV = \varepsilon_l$. This is our main result. The peak positions are determined by T_J and α, but

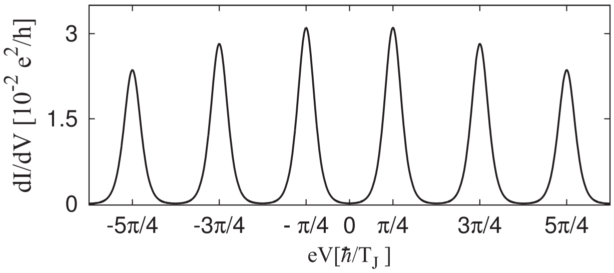

FIG. 3. Plot of the time averaged differential conductance given in Eq. (13) with parameters $\hbar T_J^{-1} = 0.1$ meV $= 10^{-1}\hbar\lambda = 10k_BT = 10\Gamma$. The conductance peak spacing $h/(4T_J)$ is a consequence of the non-Abelian state evolution within the degenerate ground-state manifold.

are independent of system details such as the initial Majorana state at $t = t_0$ and the tunneling strength Γ. Note that the periodicity T_J of the system Hamiltonian in Eqs. (2) and (9) does not coincide with the periodicity of the ground state $4T_J$ from the fact that $U_c^4 = \bar{U}_c^4 = \mathbb{1}$. It is a consequence of the nontrivial state evolution within the ground-state manifold of 4 MBSs requiring a matrix structure. As shown below, the $4T_J$ periodicity and the noncommutativity between U_c and $\bar{U}_c$ result in peaks in $d\bar{I}/dV$ separated by $h/(4T_J)$ and not by h/T_J associated with the frequency of appearance of MBSs beneath the tip. The results are the same for the case of an anticlockwise rotation of four MBSs.

Non-Abelian statistics.—In order to clearly unveil such a link between the interference effect and the non-Abelian matrix structure, we analyze the term $S(\varepsilon)$ in the occupation number basis $\{|n_1n_2\rangle\}$, where $n_1, n_2 = 0, 1$ are occupation numbers for fermionic operators $f_1 = [(\gamma_1 + i\gamma_2)/2]$ and $f_2 = [(\gamma_3 + i\gamma_4)/2]$, see [46] for more details on the occupation number representation. As Eq. (13) is independent of the initial condition, the specific form of the initial density matrix (ρ_0 or ρ_0') is unimportant. Substituting Eq. (8) into Eq. (12) leads to $S(\varepsilon) = \mathrm{Re}\{\mathrm{Tr}[\rho_0'\hat{S}(\varepsilon)]\}$, where

$$\hat{S}(\varepsilon) = \frac{1}{2} + \sum_{n=1}^{Q} e^{in\tilde{\varepsilon}}(\bar{U}_c)^n(U_c^\dagger)^n. \tag{14}$$

Note that the operations $U_c^\dagger$ and $\bar{U}_c$ do not commute, and thus the sum *cannot* be treated as a simple geometric series: $\sum_{n=1}^{\infty} e^{in\tilde{\varepsilon}}(\bar{U}_c)^n(U_c^\dagger)^n \not\rightarrow \sum_{n=1}^{\infty} e^{in\tilde{\varepsilon}}e^{in\varphi}$. The operator $\hat{S}(\varepsilon)$ comes from the overlap between the following two processes of temporal length QT_J: In process I, an electron tunnels from the tip to γ_1 at $t_0 + (Q-n)T_J$, and in process II, the tunneling happens at $t_0 + QT_J$. Here, $e^{in\tilde{\varepsilon}}$ is the dynamical phase factor gained for the time interval nT_J. The interference between terms of different n determines the peak positions of the conductance.

Let us assume that an even parity state, a mixture of $|00\rangle$ and $|11\rangle$, is prepared at $t = t_0$; the case of an odd parity state is obtained in a similar way. In the limit $Q \to \infty$, Eq. (14) for an even parity is given by

$$\hat{S}(\varepsilon)|_{\text{even}} = \frac{1}{2} + (-si\tau_z e^{i\tilde{\varepsilon}} - i\tau_y e^{i2\tilde{\varepsilon}} + si\tau_x e^{i3\tilde{\varepsilon}} - e^{i4\tilde{\varepsilon}}) \times \sum_{m=0}^{\infty} e^{im(4\tilde{\varepsilon}+\pi)}, \tag{15}$$

where $\tau_{x,y,z}$ are Pauli matrices acting in the space of the even parity states, $|00\rangle$ and $|11\rangle$. In the second line, the summation is classified into four categories in each of which the Pauli matrix (including the identity matrix) is factored out, manifesting the interference with period $4T_J$. Using Eqs. (14) and (15) yields $S(\varepsilon) + S(-\varepsilon) = \sum_m \exp[im(4\tilde{\varepsilon} + \pi)] \sim \sum_l \delta[4\tilde{\varepsilon} + \pi(2l+1)]$, where m, l are integers. Together with Eq. (10) we obtain our final result Eq. (13). We note that the period of $4T_J$ cannot be obtained by corresponding braiding operators that would commute, see [46]. We also note that this non-Abelian interference effect cannot be envisaged in a system with two MBSs where noncommuting braiding operations do not occur [12].

We remark that the suggested test of non-Abelian braiding statistics needs only a local measurement of MBSs that are at zero energy so that the way we fuse the four MBSs into the two fermions f_1 and f_2 is actually arbitrary. The period $4T_J$ also does not depend on a specific initial state (if the time-average is performed after times $t \gg T_J$) but is only a consequence of the noncommuting matrix structure of U_c and $\bar{U}_c$. The extracted information of the state changes is due to interference that is generated because the MBSs rotate in the Corbino geometry JJ. This is fundamentally different compared to other braiding schemes which use the selective switching on and off of couplings between the Majorana bound states and the readout of the non-Abelian state changes is done without physically moving the MBSs [22,50]. In our scheme the rotation induces a *dynamical coupling* between the MBSs as we discuss in detail in the Supplemental Material [46] employing the Floquet picture. There we consider also the zero temperature case to all orders in the tunneling from the tip to the MBSs.

Discussion and conclusion.—We have demonstrated that a non-Abelian state evolution can be identified in tunneling conductance measurements between four rotating MBSs in a Corbino geometry topological Josephson junction and a metal tip. Unitary evolutions of the MBSs acting on even and odd parity subspaces, which are separable if the fermion parity is conserved, are intertwined by electron tunneling, inducing parity-conserving and tunneling-assisted braiding operators. Coherent interference between different orders of round trips of Majorana states governed by the parity-conserving and tunneling-assisted braiding operators yields a time-averaged conductance exhibiting peaks with a period of $h/(4T_J)$ as a function of bias voltage between the metal tip and the Josephson junction, whereas the period of the Hamiltonian is T_J. This constitutes a clear signature of non-Abelian state evolution of four MBSs.

We explicitly showed that these results have their origin in the noncommutativity of the parity-conserving and tunneling-assisted braiding operators and are therefore independent on the way we fuse the MBSs into fermions which is fundamentally different from other recent proposals that use time-dependent couplings between the MBSs or Coulomb interaction to lift their degeneracies [11,22,50]. Here, an effective coupling between MBSs is induced dynamically by the rotation which only requires a dc-Josephson voltage applied between the two superconductors.

We expect that other kinds of exotic zero modes such as MBSs in time-reversal invariant topological superconductors [51–57] and parafermions [58–66] could be analyzed with our time-dependent tunneling scheme to manifest the quantum statistics of the corresponding modes.

The experimental realization may be challenging, but within reach of current experiments. Assuming the proximity-induced superconducting gap $\Delta_0 = 1$ meV that can be achieved, for example, in thin films of Nb or NbN [67,68], the excitation energy gap of Josephson vortices of the junction can be estimated by $E_g = \Delta_0\sqrt{4\xi/R} \sim 0.9$ meV for the radius of the junction $R = 5\xi$ [12,69], where ξ is the superconducting coherence length. We require a coherent and adiabatic rotation of the MBSs so that T_J (the time taken for the $\pi/2$ rotation) should satisfy $\hbar/E_g(= 0.7\ \mathrm{ps}) \ll T_J \ll t_{\mathrm{qp}}(\gtrsim\mu\mathrm{s})$ where t_{qp} is the quasiparticle poisoning time [70,71]. At the same time, the temperature should be much smaller than the separation between the conductance peaks $h/(4T_J)$. MBSs can be spaced unequally apart in the presence of inhomogeneities in the junction. However, they do not affect the rotation time T_J due to the periodicity of the system Hamiltonian and corresponding interference traces on the time scale of $4T_J$ due to non-Abelian evolution would remain. We believe that the Corbino geometry topological Josephson junction can also be realized in heterostructures of a thin-film topological insulator and a superconductor [72] or Pb/Co/Si(111) two-dimensional topological superconductor [73].

Our findings provide a new way of looking at braiding experiments, by actively using parity switching events by tunneling, instead of avoiding them. This may define a new way to build non-Abelian operations for topological qubits utilizing coherent fluctuations of the fermion parity. Such a change of fermion parity could be achieved on demand during a definite time using charge pumps based on quantum dots in the single electron regime [74] coupled to the setup. Quantum dots could already be coupled to MBSs in experiment [31].

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