Editors' Suggestion

Automatic Fine-Tuning in the Two-Flavor Schwinger Model

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(Received 3 August 2020; accepted 21 September 2020; published 26 October 2020)

I discuss the two-flavor Schwinger model both without and with fermion masses. I argue that the phenomenon of "conformal coalescence," in unparticle physics in which linear combinations of short-distance operators can disappear from the long-distance theory, makes it easy to understand some puzzling features of the model with small fermion masses. In particular, I argue that for an average fermion mass m_f and a mass difference δm, so long as both are small compared to the dynamical gauge boson mass $m = e\sqrt{2/\pi}$, isospin-breaking effects in the low-energy theory are exponentially suppressed by powers of $\exp[-(m/m_f)^{2/3}]$ even if $\delta m \approx m_f$. In the low-energy theory, this looks like exponential fine-tuning, but it is done automatically by conformal coalescence.

DOI: 10.1103/PhysRevLett.125.181601

The Schwinger model with two flavors.—In his classic paper [1] (which mostly concerns the model without flavor), Coleman briefly discusses the two-flavor model and identifies three puzzles.

> *"In the one-quark theory, everything that happened, even for strong coupling, was qualitatively understandable in terms of the basic ideas of the naive quark model, the picture of quarks confined in a linear potential. For the two-quark theory, there are three strong-coupling phenomena that I cannot understand in these terms: (i) Why are the lightest particles in the theory a degenerate isotriplet, even if one quark is 10 times heavier than the other? (ii) Why does the next-lightest particle have $I^{PG} = 0^{++}$ rather than 0^{--}? (iii) For $|\theta| = \pi$, how can an isodoublet quark and an isodoublet antiquark, carrying opposite electric charges, make an isodoublet bound state with electric charge zero?"*

I will argue that by taking proper account of the unparticle physics [2] of the massless model we can easily resolve the first two and understand why the third is not puzzling. The ideas in this paper are closely related to the analysis of diagonal color models in $1+1$ [3]. See also Refs. [4–7]. Two papers that I know of—Refs. [8,9]—address Coleman's puzzles explicitly, but I think that their suggestions are quite different from mine. I will suggest that the resolution of the first puzzle is a new mechanism for exponential fine-tuning.

The Lagrangian is

$$\mathcal{L} = \left(\sum_{j=1}^{2} \bar\psi_j (i\partial\!\!\!/ - eA\!\!\!/)\psi_j\right) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m_f \sum_{j=1}^{2}\bar\psi_j\psi_j. \quad (1)$$

I begin by discussing $m_f = 0$ and consider the mass term below.

For gauge-invariant correlators of local fields, the result of summing the perturbation theory to all orders can be found simply by making the following replacements:

$$A^\mu = \epsilon^{\mu\nu}\partial_\nu(\mathcal{B} - \mathcal{C})/m, \quad (2)$$

$$\psi_j = e^{-i(\pi/2)^{1/2}(\mathcal{C}-\mathcal{B})\gamma^5}\Psi_j, \quad (3)$$

where

$$m^2 = 2e^2/\pi \quad (4)$$

with the free-field Lagrangian

$$\mathcal{L}_f = \left(\sum_{j=1}^{2} i\bar\Psi_j \partial\!\!\!/ \Psi_j\right) - \frac{m^2}{2}\mathcal{B}^2 + \frac{1}{2}\partial_\mu\mathcal{B}\partial^\mu\mathcal{B} - \frac{1}{2}\partial_\mu\mathcal{C}\partial^\mu\mathcal{C}. \quad (5)$$

So that Ψ_j for $j = 1$ to 2 is a free doublet of fermion fields, $\mathcal{B}$ is a spinless field with mass m, and $\mathcal{C}$ is a massless ghost.

The massless model has a classical chiral $\mathrm{U}(2)\times\mathrm{U}(2)$ symmetry broken by the chiral anomaly down to $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. It is a Banks-Zaks model [10] with free-fermion behavior at distances much smaller than

$1/m$ and a low-energy sector with conformal symmetry for distances much larger than $1/m$. I will sometimes follow Coleman's lead and refer to the fermions as "quarks." There are fermion-antifermion operators transforming like the (2,2) representation of the chiral $SU(2) \times SU(2)$ symmetry. They are shown below:

$$O^j_{k12} = \psi^*_{j1}\psi_{k2} \quad \text{and} \quad O^j_{k21} = \psi^*_{j2}\psi_{k1}, \tag{6}$$

where j and k are flavor indices, $\gamma^5\psi_{k1} = \psi_{k1}$, and $\gamma^5\psi_{k2} = -\psi_{k2}$. To infinite order in perturbation theory, these flow at long distances to independent unparticle operators with dimension $1/2$. But this changes due to nonperturbative effects associated with the chiral $SU(2) \times SU(2)$ singlet operators (for later convenience, we have reversed the order of the flavors on the ψ's compared to the ψ^*'s):

$$O^z_{12} \equiv \psi^*_{11}\psi^*_{21}\psi_{22}\psi_{12} \quad \text{and} \quad O^z_{21} \equiv \psi^*_{12}\psi^*_{22}\psi_{21}\psi_{11} \tag{7}$$

for which [with the $\mathcal{O}(1)$ constant $\xi \equiv e^{\gamma_E}/2$]

$$\langle 0|TO^z_{12}(x)O^z_{21}(0)|0\rangle = \frac{(\xi m)^4}{16\pi^4}\exp[4K_0(m\sqrt{-x^2+i\epsilon})]. \tag{8}$$

Note that there is no arbitrariness here, because these composite operators do not require multiplicative renormalization for $m_f = 0$, so the position-space correlators are well defined for nonzero separation. A subtractive renormalization is required for the two-point function at zero separation. These "have zero anomalous dimension"—that is, they go to constants at long distances. They were called zero-dimension operators (ZDOPs) in Ref. [3], and I adopt that acronym here. Cluster decomposition requires that these operators have vacuum expectation values (VEVs)

$$\langle 0|O^z_{12}(0)|0\rangle = e^{i\theta}\frac{(\xi m)^2}{4\pi^2} \quad \text{and} \quad \langle 0|O^z_{21}(0)|0\rangle = e^{-i\theta}\frac{(\xi m)^2}{4\pi^2}, \tag{9}$$

where θ is a parameter that labels the vacuum state [11–13].

Conformal coalescence.—For simplicity, I will focus on the dimension $1/2$ operators with zero flavor U(1) charge:

$$O_1 \equiv \psi^*_{11}\psi_{12} \quad \text{and} \quad O_2 \equiv \psi^*_{21}\psi_{22} \tag{10}$$

and their complex conjugates

$$O^*_1 = \psi^*_{12}\psi_{11} \quad \text{and} \quad O^*_2 = \psi^*_{22}\psi_{21}. \tag{11}$$

The theory has a conserved axial isospin symmetry associated with the charges

$$\vec{I} = \frac{1}{2}\int dx^1 \bar{\psi}(x)\vec{\sigma}\gamma^5\psi(x), \tag{12}$$

where the Pauli matrices $\vec{\sigma}$ act on the flavor space and the operators O_1 and O_2 have opposite charges for the third component, $I_3 = +1$ and -1. The perturbative two-point functions are

$$\langle 0|TO_j(x)O^*_k(0)|0\rangle = \delta_{jk}\frac{\xi m}{(2\pi)^2}\exp\left[K_0\left(m\sqrt{-x^2+i\epsilon}\right)\right] \times \left(\frac{1}{-x^2+i\epsilon}\right)^{1/2}. \tag{13}$$

The ZDOPs produce nonperturbative corrections to (13). The perturbative three-point correlation function with an added ZDOP can be written as

$$\begin{aligned}\langle 0|TO^z_{21}(z)O_1(x)O_2(0)|0\rangle \\ = \langle 0|TO^z_{12}(z)O^*_1(x)O^*_2(0)|0\rangle = \frac{(\xi m)^3}{(2\pi)^4}\left(\frac{1}{-x^2+i\epsilon}\right)^{1/2} \\ \times \exp[2\kappa_0(z-x) + 2\kappa_0(z) - \kappa_0(x)],\end{aligned} \tag{14}$$

where

$$\kappa_0(x) = K_0(m\sqrt{-x^2+i\epsilon}). \tag{15}$$

The form of (14) can be understood (and indeed overdetermined) as follows. The overall counting of factors of 2π comes from the free-fermion skeleton and is, thus, the same as (8). The long-distance behavior is determined by the anomalous dimension, and, because the ZDOP has zero anomalous dimension, there is no long-distance dependence on z. The z dependence must be entirely in the K_0 terms, which are determined by the gauge coupling and which must combine to agree with (8) as $x \to 0$. The long-distance x dependence and, thus, the $1/(-x^2+i\epsilon)^{1/2}$ term must be the same as in (13). There is no contribution to the x dependence from the free-fermion skeleton, and, thus, the x dependence from the $1/(-x^2+i\epsilon)^{1/2}$ term must cancel the x dependence from the K_0's at short distances, which fixes the coefficient of $K_0(m\sqrt{-x^2+i\epsilon})$ in the exponential. The power of (ξm) is equal to sum of the coefficients of the K_0's in the exponential.

Now cluster decomposition can be applied to (14) just as it can in (8). We can pull the ZDOP away to infinity and replace it by its VEV, Eq. (9); then the exponential in (14) goes to 1 and what remains is a nonperturbative contribution to the two-point functions of the dimension $1/2$ operators. Thus, [note that if $x \to 0$ in (16), this reduces to (9)]

$$\langle 0|TO_1(x)O_2(0)|0\rangle = e^{-i\theta}\frac{(\xi m)}{(2\pi)^2}\exp[-K_0(m\sqrt{-x^2+i\epsilon})]\left(\frac{1}{-x^2+i\epsilon}\right)^{1/2},$$
$$\langle 0|TO_1^*(x)O_2^*(0)|0\rangle = e^{i\theta}\frac{(\xi m)}{(2\pi)^2}\exp[-K_0(m\sqrt{-x^2+i\epsilon})]\left(\frac{1}{-x^2+i\epsilon}\right)^{1/2}. \tag{16}$$

The ZDOP VEV has given us a nonperturbative contribution to the two-point function that is fixed by the calculable three-point function. It is amusing that we can calculate this exactly. In general, we might have to include the contributions from n-point functions with more ZDOPs, but in this example, these do not give any new contributions. But there are more surprises in store. Define

$$O_g \equiv e^{i\theta/2}O_1 + ge^{-i\theta/2}O_2^* \quad \text{and}$$
$$O_g^* \equiv e^{-i\theta/2}O_1^* + ge^{i\theta/2}O_2, \tag{17}$$

where $g = \pm 1$.

Then

$$\langle 0|TO_{\pm1}(x)O_{\pm1}(0)|0\rangle = \langle 0|TO_{\pm1}(x)O_{\mp1}(0)|0\rangle = \langle 0|TO_{\pm1}(x)O_{\mp1}^*(0)|0\rangle = 0. \tag{18}$$

The first two terms in (18) must vanish because of axial isospin symmetry. The vanishing of the third term follows because the parameter g is the multiplicative quantum number for a θ-dependent G parity that is conserved by perturbative and nonperturbative interactions:

$$e^{i\theta/2}O_1 \leftrightarrow e^{-i\theta/2}O_2^* \quad \text{and} \quad e^{i\theta/2}O_2 \leftrightarrow e^{-i\theta/2}O_1^*. \tag{19}$$

The only nonzero two-point functions are

$$\langle 0|TO_{\pm1}(x)O_{\pm1}^*(0)|0\rangle = \frac{(\xi m)}{4\pi^2}\times 2\{\exp[K_0(m\sqrt{-x^2+i\epsilon})] \pm \exp[-K_0(m\sqrt{-x^2+i\epsilon})]\}\left(\frac{1}{-x^2+i\epsilon}\right)^{1/2}. \tag{20}$$

At short distances, the first exponential in the penultimate factor in (20) dominates for both + and − and (along with the last factor) produces the expected free-fermion scaling. But at long distances, while the O_{+1} operator goes smoothly to a conformal operator, the O_{-1} correlator goes to zero exponentially. One of the operators, the O_{-1}, disappears from the conformal theory as the O_1 and O_2^* pair in O_{+1} coalesce. Similar behavior was discovered in $1+1$ diagonal color models in Ref. [3] and dubbed “conformal coalescence.” Here we will see that it has dramatic consequences in the massive Schwinger model.

It is straightforward (if not particularly edifying) to write down the general result:

$$\left\langle 0\left|T\prod_{j=1}^{n}O_{g_j}(x_j)O_{h_j}^*(y_j)\right|0\right\rangle = \left(\frac{\xi m}{4\pi^2}\right)^n\left[\frac{\prod_{j<k}\left\{[-(x_j-x_k)^2+i\epsilon][-(y_j-y_k)^2+i\epsilon]\right\}}{\prod_{j,k}[-(x_j-y_k)^2+i\epsilon]}\right]^{1/2}$$
$$\sum_{\eta_j,\chi_j=0}^{1}\left(\prod_j(g_j)^{\eta_j+1}\right)\left(\prod_j(h_j)^{\chi_j+1}\right)\exp\left[\left(\sum_{j,k}(-1)^{\eta_j+\chi_k}K_0(x_j-y_k)\right)-\left(\sum_{j<k}(-1)^{\eta_j+\eta_k}K_0(x_j-x_k)+(-1)^{\chi_j+\chi_k}K_0(y_j-y_k)\right)\right]. \tag{21}$$

Again, these vanish identically if the number of O_{-1}’s plus the number of O_{-1}^*’s is odd and vanish exponentially if any of the O_{-1} or O_{-1}^* coordinates goes to infinity.

Mass terms.—While I find this model endlessly fascinating, it is still just a generalized free theory [14]. But I believe that the above analysis can help us to understand the nontrivial theory that results from adding a mass term. This has been discussed in many works, but, as I mentioned in the introduction, I want to focus on the three puzzles about the strong coupling limit, $m_f \ll m$ identified by Coleman in Ref. [1]. (i) Why are the lightest particles in the theory a degenerate isotriplet, even if one quark is 10 times heavier than the other? (ii) Why does the next-lightest particle have $I^{PG} = 0^{++}$ rather than 0^{--}? (iii) For $|\theta| = \pi$, how can an isodoublet quark and an isodoublet antiquark, carrying opposite electric charges, make an isodoublet bound state with electric charge zero?

I believe that conformal coalescence resolves the first puzzle in a very simple way. For $\theta = 0$, (17) implies that an isospin-invariant fermion mass term at low energies is

$$m_f(O_1 + O_2) + \text{H.c.} = m_f(O_{+1} + O_{+1}^*) \to \frac{m_f\sqrt{\xi m}}{\pi}(O_{1/2} + O_{1/2}^*), \quad (22)$$

where $O_{1/2}$ is a normalized dimension 1/2 conformal operator with

$$\langle 0|TO_{1/2}(x)O_{1/2}^*(0)|0\rangle = \left(\frac{1}{-x^2 + i\epsilon}\right)^{1/2}. \quad (23)$$

Note that (22) implies that the only quantity with dimensions that survives in the low-energy theory is $m_f\sqrt{m}$, and so the masses of the particles that appear as a result of the breaking of the conformal symmetry must be proportional to $(m_f^2 m)^{1/3}$, in agreement with Coleman's result. It is easy to see that if $\delta m = 0$ for arbitrary $\theta \neq \pm\pi$, the mass parameter becomes $m_f\sqrt{m}\cos(\theta/2)$, also in agreement with Coleman.

In the presence of an isospin-breaking term for $\theta = 0$, (22) goes to

$$m_f(O_1 + O_2) + \text{H.c.} + \delta m(O_1 - O_2) + \text{H.c.} = m_f(O_{+1} + O_{+1}^*) + \delta m(O_{-1} + O_{-1}^*). \quad (24)$$

All correlators involving the δm term go to zero exponentially at long distances, because they involve powers of $K_0(mx)$. Because the only mass scale in the low-energy theory is $(m_f^2 m)^{1/3}$, we expect that the isospin-breaking contribution is suppressed by powers of

$$K_0[m/(m_f^2 m)^{1/3}] \propto \exp[-(m/m_f)^{2/3}] \quad (25)$$

even if $\delta m \approx m_f$. The power of 2/3 was missing in an earlier version. I am grateful to a referee for encouraging me to make the argument more explicit.

I believe that the resolution of the second puzzle is in some sense obvious but that it is telling us something novel about the conformal theory. For $\theta = 0$, (17) implies that the unparticle stuff produced by O_{+1} and O_{+1}^* is G even. The G-odd stuff produced by O_{-1} and O_{-1}^* always involves the massive gauge boson and does not survive at long distances. Evidently, if we think of decreasing m_f/m from weak coupling, $m_f/m \gg 1$, to strong coupling, $m_f/m \ll 1$, the G-odd quark-antiquark states get stuck at masses of the order of m, while the G-even states continue to move down into the low-energy theory.

Finally, I believe that the resolution of the third puzzle is that it is a problem of logic rather than a problem of physics. The puzzle starts from the hypothesis that the low-energy theory for an isospin-invariant mass term with $\theta = \pi$ is a theory of particles. I believe that this hypothesis is false. For $\theta = \pi$, (17) implies that the isospin-invariant mass term

$$m_f(O_1 + O_2) + \text{H.c.} = -im_f(O_{-1} - O_{-1}^*) \to 0. \quad (26)$$

Thus, this term does not survive in the low-energy conformal theory, and the low-energy conformal symmetry persists even in the presence of the mass term. If this is correct, then I think that there must be a phase transition between weak and strong coupling that frustrates Coleman's attempt to understand the model in terms of the naive quark model.

Directions for future work.—While I believe that I have answered each of Coleman's questions, the answers suggest further questions. For $\theta = 0$ with nonzero δm, isospin symmetry-breaking effects are present at low energies but exponentially suppressed. In the low-energy theory, this like looks fine-tuning. Is this new mechanism for generating an exponential hierarchy of parameters useful for any of the hierarchy puzzles that afflict the standard model? Is there a more physical description of what it means for the unparticle stuff to have only even G parity? And, for $\theta = \pi$, what does the transition to the long-distance conformal theory look like? I hope to explore these questions further.

I am grateful for insights from Bea Noether, who was involved in the early stages of this work. I am also grateful for comments by Jacob Barandes, Alvaro DeRujula, David Kaplan, and Ashvin Vishwanath. This work was supported in part by National Science Foundation Grant No. PHY-1719924.

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