Standard-Model Prediction of ϵ_K with Manifest Quark-Mixing Unitarity

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The parameter ϵ_K describes CP violation in the neutral kaon system and is one of the most sensitive probes of new physics. The large uncertainties related to the charm-quark contribution to ϵ_K have so far prevented a reliable standard-model prediction. We show that Cabibbo-Kobayashi-Maskawa unitarity enforces a unique form of the $|\Delta S = 2|$ weak effective Lagrangian in which the short-distance theory uncertainty of the imaginary part is dramatically reduced. The uncertainty related to the charm-quark contribution is now at the percent level. We present the updated standard-model prediction $\epsilon_K = 2.16(6)(8)(15) \times 10^{-3}$, where the errors in parentheses correspond to QCD short-distance, long-distance, and parametric uncertainties, respectively.

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Introduction.—CP violation in the neutral kaon system, parametrized by ϵ_K, is one of the most sensitive precision probes of new physics. For decades, the large perturbative uncertainties related to the charm-quark contributions have been an impediment to fully exploiting the potential of ϵ_K. In this Letter we demonstrate how to overcome this obstacle.

The parameter ϵ_K can be defined as [1]

$$\epsilon_K \equiv e^{i\phi_\epsilon} \sin\phi_\epsilon \frac{1}{2} \arg\left(\frac{-M_{12}}{\Gamma_{12}}\right). \tag{1}$$

Here, $\phi_\epsilon = \arctan(2\Delta M_K/\Delta\Gamma_K)$, with ΔM_K and $\Delta\Gamma_K$ as the mass and lifetime difference of the weak eigenstates K_L and K_S. M_{12} and Γ_{12} are the Hermitian and anti-Hermitian parts of the Hamiltonian that determines the time evolution of the neutral kaon system. The short-distance contributions to ϵ_K are then contained in the matrix element $M_{12} = -\langle K^0|\mathcal{L}^{\Delta S=2}_{f=3}|\bar{K}^0\rangle/(2\Delta M_K)$, up to higher powers in the operator-product expansion. Both M_{12} and Γ_{12} depend on the phase convention of the Cabibbo-Kobayashi-Maskawa (CKM) matrix V. The cancellation of the phase convention in Eq. (1) is manifest if we use CKM unitarity to express the effective three-flavor $|\Delta S = 2|$ Lagrangian in terms of the minimal number of independent parameters. We therefore define the Lagrangian with manifest CKM unitarity,

$$\mathcal{L}^{\Delta S=2}_{f=3} = -\frac{G_F^2 M_W^2}{4\pi^2}\frac{1}{(\lambda_u^*)^2} Q_{S2}\{f_1\mathcal{C}_1(\mu) + iJ[f_2\mathcal{C}_2(\mu) + f_3\mathcal{C}_3(\mu)]\} + \text{H.c.} + \ldots, \tag{2}$$

in terms of the real Wilson coefficients $\mathcal{C}_i(\mu)$, $i = 1, 2, 3$, and four real, independent, rephasing-invariant parameters J, f_1, f_2, and f_3 comprising the relevant CKM matrix elements. Here, $\lambda_i \equiv V_{is}^* V_{id}$. The local four-quark operator

$$Q_{S2} = (\bar{s}_L\gamma_\mu d_L) \otimes (\bar{s}_L\gamma^\mu d_L), \tag{3}$$

defined in terms of the left-handed s- and d-quark fields, induces the $|\Delta S = 2|$ transitions. The ellipsis in Eq. (2) represents $|\Delta S = 1|$ operators that contribute to the dispersive and absorptive parts of the amplitude via nonlocal insertions, as well as operators of mass dimension higher than six [1].

The normalization factor $1/(\lambda_u^*)^2$ in Eq. (2) ensures that the resulting expression of ϵ_K in Eq. (1) is phase-convention independent if one accordingly extracts the factor $1/\lambda_u^*$ from the $|\Delta S = 1|$ Hamiltonian which contributes to Γ_{12} via a double insertion. It is evident in this decomposition that $\mathcal{C}_1$ does not contribute to ϵ_K. Moreover, the splitting into the real and imaginary part in Eq. (2) is unique. Explicitly, we have $J = \text{Im}(V_{us}V_{cb}V_{ub}^*V_{cs}^*)$ and $f_1 = |\lambda_u|^4 + \ldots$, where the ellipsis denotes real terms that are suppressed by powers of the Wolfenstein parameter λ.

By contrast, the splitting of the imaginary part among f_2 and f_3 is not unique. The choice $f_2 = 2\text{Re}(\lambda_t\lambda_u^*)$ and $f_3 = |\lambda_u|^2$ is particularly convenient in the particle data group (PDG) phase convention. It maps Eq. (2) to the Lagrangian

$$\mathcal{L}^{\Delta S=2}_{f=3} = -\frac{G_F^2 M_W^2}{4\pi^2}[\lambda_u^2 \mathscr{C}^{uu}_{S2}(\mu) + \lambda_t^2 \mathscr{C}^{tt}_{S2}(\mu) + \lambda_u\lambda_t \mathscr{C}^{ut}_{S2}(\mu)]Q_{S2} + \text{H.c.} + \ldots, \quad (4)$$

via the relations $\mathscr{C}^{uu}_{S2} \equiv \mathcal{C}_1$, $\mathscr{C}^{tt}_{S2} \equiv \mathcal{C}_2$, and $\mathscr{C}^{ut}_{S2} \equiv \mathcal{C}_3$, which are obtained by applying CKM unitarity and are valid in the PDG phase convention. This form of the effective Lagrangian, where the coefficient of $\mathscr{C}^{uu}_{S2}$ is real and thus does not contribute to ϵ_K, has been suggested in Ref. [2] as a better way to compute the matrix elements on the lattice in the four-flavor theory, and it was speculated that also the perturbative part may then converge better (see also Refs. [3,4]). Above, we showed that this minimal form is essentially dictated by CKM unitarity; we will see below that, indeed, both $\mathcal{C}_2$ and $\mathcal{C}_3$ (as opposed to $\mathcal{C}_1$) have a perfectly convergent perturbative expansion. This can be understood qualitatively by noting that $\mathcal{C}_2$ and $\mathcal{C}_3$ induce CP violation and thus require the presence of all three quark generations, while $\mathcal{C}_1$ is dominated by low-energy degrees of freedom. See also Ref. [2] for an argument at the amplitude level.

Traditionally, however, the effective Lagrangian has been given in a different form [5,6],

$$\mathcal{L}^{\Delta S=2}_{f=3} = -\frac{G_F^2 M_W^2}{4\pi^2}[\lambda_c^2 C^{cc}_{S2}(\mu) + \lambda_t^2 C^{tt}_{S2}(\mu) + \lambda_c\lambda_t C^{ct}_{S2}(\mu)]Q_{S2} + \text{H.c.} + \ldots, \quad (5)$$

which in the PDG phase conventions can be obtained from Eq. (2) via the relations $C^{cc}_{S2} \equiv \mathcal{C}_1$, $C^{ct}_{S2} \equiv 2\mathcal{C}_1 - \mathscr{C}_3$, and $C^{tt}_{S2} \equiv \mathcal{C}_1 + \mathcal{C}_2 - \mathcal{C}_3$. Here $\mathcal{C}_1$ artificially enters all three coefficients, which then all contribute to ϵ_K. This is unfortunate because the perturbative expansion of $\mathcal{C}_1$ exhibits bad convergence, as shown in Ref. [7]. Trading the short distance uncertainty in C^{cc}_{S2} for the long distance uncertainty in the theory prediction of $\mathrm{Re}(M_{12})$ cannot reduce the uncertainty—see Ref. [8], where only the uncertainty from the two-pion contribution was considered.

Clearly, Eq. (4) can be directly obtained from Eq. (2) by the replacement $\lambda_u = -\lambda_c - \lambda_t$. We will refer to Eq. (5) as "$c-t$ unitarity" and to Eq. (4) as "$u-t$ unitarity." It is customary to define the renormalization-scale-invariant (RI) Wilson coefficients $\hat{C}^{ij}_{S2} \equiv C^{ij}_{S2}(\mu)b(\mu)$, $ij = cc, ct, tt$, where the scale factor $b(\mu)$ is defined, for instance, in Refs. [6,9]. QCD corrections are then parametrized by the factors η_{tt}, η_{ct}, and η_{cc}, defined in terms of the Inami-Lim functions $S(x_i, x_j)$ (see Ref. [10]) by $\hat{C}^{tt}_{S2} = \eta_{tt}S(x_t)$, $\hat{C}^{ct}_{S2} = 2\eta_{ct}S(x_c, x_t)$, and $\hat{C}^{cc}_{S2} = \eta_{cc}S(x_c)$. Here, we defined the mass ratios $x_i \equiv m_i(m_i)^2/M_W^2$ with $m_i(m_i)$ denoting the RI $\overline{\text{MS}}$ mass. η_{tt} is known at next-to-leading-logarithmic (NLL) order in QCD, $\eta_{tt} = 0.5765(65)$ [11], while the other two are known at next-to-next-to-leading-logarithmic (NNLL) order, $\eta_{ct} = 0.496(47)$ [9] and $\eta_{cc} = 1.87(76)$ [7].

In the same way, we define the RI Wilson coefficients and the QCD correction factors for the Lagrangian in Eq. (4), namely, $\hat{\mathscr{C}}^{tt}_{S2} = \eta_{tt}\mathcal{S}_{tt}(x_c, x_t)$ and $\hat{\mathscr{C}}^{ut}_{S2} = 2\eta_{ut}\mathcal{S}_{ut}(x_c, x_t)$. Using Eqs. (4) and (5) and the unitarity relation $\lambda_c = -\lambda_u - \lambda_t$, it is readily seen that the modified Inami-Lim functions are given by $\mathcal{S}_{ut}(x_c, x_t) = S(x_c) - S(x_c, x_t)$ and $\mathcal{S}_{tt}(x_c, x_t) = S(x_t) + S(x_c) - 2S(x_c, x_t)$. The latter relation implies that η_{tt} coincides in $u-t$ and $c-t$ unitarity up to tiny corrections of order $\mathcal{O}(m_c^2/M_W^2) \sim 10^{-4}$, which we neglect, writing $\mathcal{S}_{tt}(x_c, x_t) = \mathcal{S}_{tt}(x_t)$. In what follows, we show that $\eta_{ut} = 0.402(5)$ at NNLL with an order-of-magnitude smaller uncertainty than η_{ct} and η_{cc}.

Analytic results.—In this section we will show that all ingredients for the NNLL analysis with manifest CKM unitarity of the charm contribution to ϵ_K are available in the literature. To establish the requisite relations, we display the effective five- and four-flavor Lagrangian using both the traditional c-t unitarity, giving [6,9]

$$\mathcal{L}^{\text{eff}}_{f=4,5} = -\frac{4G_F}{\sqrt{2}}\left(\sum_{k,l=u,c} V^*_{ks}V_{ld}(C_+Q^{kl}_+ + C_-Q^{kl}_-) - \lambda_t\sum_{i=3,6} C_iQ_i\right) - \frac{G_F^2M_W^2}{4\pi^2}\lambda_t^2 C_{S2}Q_{S2} - 8G_F^2\lambda_c\lambda_t\tilde{C}_7\tilde{Q}_7 + \text{H.c.}, \quad (6)$$

and u-t unitarity, giving

$$\mathcal{L}^{\text{eff}}_{f=4,5} = -\frac{4G_F}{\sqrt{2}}\left(\sum_{k,l=u,c} V^*_{ks}V_{ld}(\mathscr{C}_+Q^{kl}_+ + \mathscr{C}_-Q^{kl}_-) - \lambda_t\sum_{i=3,6} \mathscr{C}_iQ_i\right) - \frac{G_F^2M_W^2}{4\pi^2}\lambda_t^2 \mathscr{C}_{S2}Q_{S2} - 8G_F^2(\lambda_u\lambda_t + \lambda_t^2)\tilde{\mathscr{C}}_7\tilde{Q}_7 + \text{H.c.} \quad (7)$$

The Wilson coefficients in Eqs. (7) and (6) are related via

$$\mathscr{C}_i = C_i, \qquad \mathscr{C}_{S2} = C_{S2}, \qquad \tilde{\mathscr{C}}_7 = -\tilde{C}_7, \quad (8)$$

where $i = +, -, 3, \ldots, 6$. Here, $\tilde{Q}_7 \equiv m_c^2/g_s^2 Q_{S2}$, with g_s as the strong coupling constant, while the remaining operators (current-current and penguin operators) are defined in Ref. [9]. The initial conditions for all the C_i Wilson coefficients and $\tilde{C}_7$, up to next-to-next-to-leading order (NNLO), can be found in Refs. [9,11–13].

It is evident that the renormalization-group evolution of the coefficients $\mathscr{C}_i$ and C_i, as well as of $\mathscr{C}_{S2}$ and C_{S2}, is identical. We now show that also the mixing of the $\mathscr{C}_i$ into $\tilde{C}_7$ via double insertions of dimension-six operators can be obtained from results available in the literature. To this end we define the following short-hand notation for the relevant $|\Delta S = 2|$ matrix elements of double insertions of local operators O_A and O_B,

$$\langle O_A, O_B\rangle \equiv \frac{i^2}{2!}\int d^4x d^4y \langle T\{O_A(x)O_B(y)\}\rangle. \quad (9)$$

With the Lagrangian in Eq. (6) and using $(V^*_{cs}V_{ud})(V^*_{us}V_{cd}) = -\lambda_c^2 - \lambda_c\lambda_t$, the anomalous dimensions for the mixing of two $\mathscr{C}_i$s into $\tilde{C}_7$ can then be obtained from the divergent part of the amplitude

$$\begin{aligned}
&\mathcal{M}^{\Delta S=2}_{\text{double insertions}}|_{\text{div}}\\
&\propto \lambda_t^2(\langle Q_P, Q_P\rangle + \langle Q^{uu}, Q^{uu}\rangle + 2\langle Q_P, Q^{uu}\rangle)|_{\text{div}}\\
&\quad -\lambda_c\lambda_t(2\langle Q_P, Q^{cc}-Q^{uu}\rangle + \langle Q^{cc}, Q^{cc}\rangle - \langle Q^{uu}, Q^{uu}\rangle)|_{\text{div}}\\
&= \lambda_t^2(\langle Q_P, Q_P\rangle + \langle Q^{cc}, Q^{cc}\rangle + 2\langle Q_P, Q^{cc}\rangle)|_{\text{div}}\\
&\quad +\lambda_u\lambda_t(2\langle Q_P, Q^{cc}-Q^{uu}\rangle + \langle Q^{cc}, Q^{cc}\rangle - \langle Q^{uu}, Q^{uu}\rangle)|_{\text{div}}.
\end{aligned} \quad (10)$$

We introduced the short-hand notations $Q_P \equiv \sum_{i=3}^6 \mathscr{C}_i Q_i$ and $Q^{qq'} \equiv \sum_{i=+,-}\mathscr{C}_i Q_i^{qq'}$. In the first equality we utilized the observation that the divergence of the linear combination of amplitudes proportional to λ_c^2 vanishes [14],

$$(\langle Q^{cc} - Q^{uu}, Q^{cc} - Q^{uu}\rangle - 2\langle Q^{uc}, Q^{cu}\rangle)|_{\text{div}} = 0. \quad (11)$$

In the second equality we used, in addition, the unitarity relation $\lambda_c = -\lambda_u - \lambda_t$. We see that the divergent parts of the amplitudes proportional to $\lambda_c\lambda_t$ and $\lambda_u\lambda_t$ are the same up to a sign. Therefore, the corresponding anomalous dimensions can be extracted from existing literature. In the notation of Ref. [9] we have $\tilde{\gamma}^{(ut)}_{\pm,7} = -\tilde{\gamma}^{(ct)}_{\pm,7}$, where the superscripts "ut" and "ct" denote the results in $u-t$ and $c-t$ unitarity, respectively. All other contributing anomalous dimensions remain unchanged.

Note that in the second equality in Eq. (10), the amplitudes proportional to λ_t^2 involve the charm-flavored current-current operators. This is related to the appearance of an initial condition of the operator $\tilde{Q}_7$ at the weak scale proportional to λ_t^2. This charm-quark contribution to $\mathscr{C}^{tt}_{S2}$ will be neglected in this work, as discussed above. In this approximation, $\mathscr{C}^{tt}_{S2}$ is identical to C^{tt}_{S2} and can be directly taken from the literature [11].

Also the matching of the four- onto the three-flavor effective Lagrangian at μ_c changes in a simple way. Picking the coefficient of $\lambda_u\lambda_t$, the matching of the Lagrangian in Eq. (7) onto the one in Eq. (4) yields the condition

$$\begin{aligned}
&\sum_{i,j=+,-}\mathscr{C}_i(\mu_c)\mathscr{C}_j(\mu_c)(2\langle Q_i^{cc}, Q_j^{cc}\rangle\\
&\quad - 2\langle Q_i^{uc}, Q_j^{cu}\rangle - 2\langle Q_i^{uu}, Q_j^{cc}\rangle)(\mu_c)\\
&\quad + \sum_{i=3}^{6}\sum_{j=+,-}\mathscr{C}_i(\mu_c)\mathscr{C}_j(\mu_c)2\langle Q_i, Q_j^{cc} - Q_j^{uu}\rangle(\mu_c)\\
&\quad + \tilde{\mathscr{C}}_7(\mu_c)\langle\tilde{Q}_7\rangle(\mu_c) = \frac{1}{32\pi^2}\mathscr{C}^{ut}_{S2}(\mu_c)\langle Q_{S2}\rangle(\mu_c).
\end{aligned} \quad (12)$$

Alternatively, selecting the coefficient of $\lambda_c\lambda_t$, the matching of the Lagrangian in Eq. (6) onto the one in Eq. (5) yields the condition

$$\begin{aligned}
&\sum_{i,j=+,-}C_i(\mu_c)C_j(\mu_c)(2\langle Q_i^{uu}, Q_j^{uu}\rangle\\
&\quad - 2\langle Q_i^{uc}, Q_j^{cu}\rangle - 2\langle Q_i^{uu}, Q_j^{cc}\rangle)(\mu_c)\\
&\quad + \sum_{i=3}^{6}\sum_{j=+,-}C_i(\mu_c)C_j(\mu_c)2\langle Q_i, Q_j^{uu} - Q_j^{cc}\rangle(\mu_c)\\
&\quad + \tilde{C}_7(\mu_c)\langle\tilde{Q}_7\rangle(\mu_c) = \frac{1}{32\pi^2}C^{ct}_{S2}(\mu_c)\langle Q_{S2}\rangle(\mu_c).
\end{aligned} \quad (13)$$

and for the coefficient of λ_c^2 yields the condition

$$\begin{aligned}
&\sum_{i,j=+,-}C_i(\mu_c)C_j(\mu_c)((\langle Q_i^{cc} - Q_i^{uu}, Q_j^{cc} - Q_j^{uu}\rangle\\
&\quad - 2\langle Q_i^{uc}, Q_j^{cu}\rangle)(\mu_c) = \frac{1}{32\pi^2}C^{cc}_{S2}(\mu_c)\langle Q_{S2}\rangle(\mu_c).
\end{aligned} \quad (14)$$

Recalling Eq. (8), we see that $\mathscr{C}^{ut}_{S2} = 2C^{cc}_{S2} - C^{ct}_{S2}$, hence we can extract also the matching conditions from the literature.

In order to provide the explicit expressions, we parametrize the operator matrix elements as

$$\begin{aligned}
&\langle\tilde{Q}_7\rangle = r_7\langle\tilde{Q}_7\rangle^{(0)}, \qquad \langle Q_{S2}\rangle = r_{S2}\langle Q_{S2}\rangle^{(0)},\\
&\langle Q_iQ_j\rangle^{qq'}(\mu_c) = \frac{1}{32\pi^2}\frac{m_c^2(\mu_c)}{M_W^2}r^{qq'}_{ij,S2}\langle Q_{S2}\rangle^{(0)}.
\end{aligned} \quad (15)$$

Here, the superscripts $qq' = ut, ct, cc$ denote the specific flavor structures appearing in the double insertions in Eqs. (12)–(14), respectively. The matching contributions are then given in terms of the literature results by $r^{ut}_{ij,S2} = 2r^{cc}_{ij,S2} - r^{ct}_{ij,S2}$. It is interesting to note that, due to the presence of a large logarithm $\log(m_c/M_W)$ in the function $\mathcal{S}_{ut}(x_c, x_t)$, only the next-to-leading order result for η_{cc} of Ref. [15] is required. The remaining NNLO results can be found in Refs. [6,9].

Numerics.—In the previous section, we extracted all the necessary quantities to evaluate the λ_t^2 and $\lambda_u\lambda_t$ contributions to ϵ_K at NLL and NNLL accuracy, respectively. Here, we discuss the residual theory uncertainties in $u-t$ unitarity and compare them to the traditional approach of $c-t$ unitarity. To estimate the uncertainty from missing, higher-order perturbative corrections we vary the unphysical thresholds μ_t, μ_b, and μ_c in the ranges $40\text{ GeV} \le \mu_t \le 320\text{ GeV}$, $2.5\text{ GeV} \le \mu_b \le 10\text{ GeV}$, and $1\text{ GeV} \le \mu_c \le 2\text{ GeV}$. When varying one scale we keep the other two scales fixed at the values of the RI mass of the fermions, $\mu_i = m_i(m_i)$ with $i = t, b, c$. The central values for the η parameters are obtained as the average between the lowest and highest value of the three scale variations, and their scale uncertainty as half the difference of the two values. The leading, but small, parametric uncertainties of

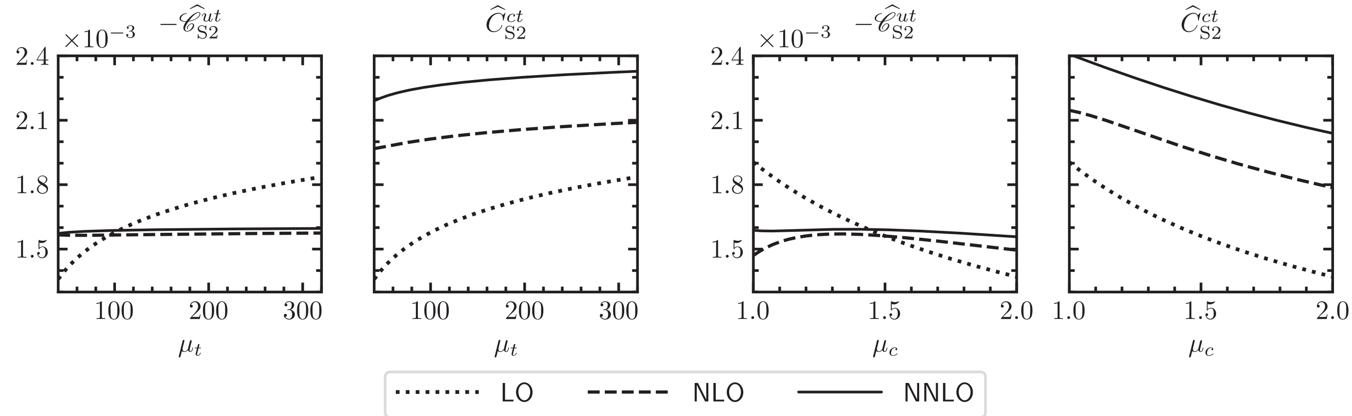

FIG. 1. Comparison of Wilson coefficients in *u*-*t* (first and third plot) and *c*-*t* unitarity (second and fourth plot). Shown is the residual renormalization-scale dependence of the RI Wilson coefficients as a proxy for their theory uncertainty. In the two plots on the left the five-flavor threshold, μ_t, is varied, while in the two on the right the three-flavor threshold, μ_c, is varied (see text for further details).

α_s and m_c are obtained by varying the parameters at their respective 1σ ranges. We find

$$\eta_{tt}^{\rm NLL} = 0.55(1 \pm 4.2\%_{\rm scales} \pm 0.1\%_{\alpha_s}),$$
$$\eta_{ut}^{\rm NNLL} = 0.402(1 \pm 1.3\%_{\rm scales} \pm 0.2\%_{\alpha_s} \pm 0.2\%_{m_c}). \quad (16)$$

Apart from the tiny correction of $\mathcal{O}(m_c^2/M_W^2) \sim 10^{-4}$, η_{tt} is not affected by the different choice of CKM unitarity. The difference in the scale uncertainty with respect to Ref. [11] is mainly due to the larger range of scale variation chosen here. By contrast, the residual scale uncertainty of η_{ut} is significantly less than the corresponding one in η_{ct} and η_{ct} in *c*-*t* unitarity. To illustrate this, we show in Fig. 1 the RI invariant Wilson coefficients $\hat{\mathscr{C}}^{ut}$ and $\hat{C}^{ct}$ as a function of the unphysical thresholds μ_t (left two panels) and μ_c (right two panels).

To obtain the standard-model prediction for ϵ_K we employ the Wolfenstein parametrization [16] of the CKM factors in Eq. (4). In the leading approximation we find $\text{Im}(\lambda_t^2) = -2\lambda^{10}A^4\bar{\eta}(1-\bar{\rho}) + \mathcal{O}(\lambda^{12})$ and $\text{Im}(\lambda_u\lambda_t) = \lambda^6A^2\bar{\eta} + \mathcal{O}(\lambda^{10})$. Numerically, the neglected terms amount to subpermil effects and can be safely neglected. Therefore, we can use the phenomenological expression (cf. Refs. [5,17,18])

$$|\epsilon_K| = \kappa_\epsilon C_\epsilon \hat{B}_K |V_{cb}|^2 \lambda^2 \bar{\eta} \times [|V_{cb}|^2(1-\bar{\rho})\eta_{tt}\mathcal{S}_{tt}(x_t) - \eta_{ut}\mathcal{S}_{ut}(x_c, x_t)], \quad (17)$$

where $C_\epsilon = (G_F^2 F_K^2 M_{K^0} M_W^2)/(6\sqrt{2}\pi^2 \Delta M_K)$. We write $\bar{\eta} = R_t \sin\beta$ and $1 - \bar{\rho} = R_t \cos\beta$, with $R_t \approx (\xi_s/\lambda)\sqrt{M_{B_s}/M_{B_d}}\sqrt{\Delta M_d/\Delta M_s}$. Here, $\xi_s = (F_{B_s}\sqrt{\hat{B}_s})/(F_{B_d}\sqrt{\hat{B}_d}) = 1.206(17)$ is a ratio of *B*-meson decay constants and bag factors that is computed on the lattice [19]. The kaon bag parameter is given by $\hat{B}_K = 0.7625(97)$ [19]. The phenomenological parameter $\kappa_\epsilon = 0.94(2)$ [18] comprises long-distance contributions not included in B_K. As input for the top-quark mass we use the $\overline{\text{MS}}$ mass $m_t(m_t) = 163.48(86)$ GeV. We obtain it by converting the pole mass $M_t = 173.1(9)$ GeV [16] to $\overline{\text{MS}}$ at three-loop accuracy using RUNDEC [20]. All remaining numerical input is taken from Ref. [16], in particular the CKM input used is $\lambda = 0.2243(5)$, $|V_{cb}| = 0.0422(8)$, and $\sin 2\beta = 0.691(17)$.

Using the η values in Eq. (16) and adding errors in quadrature we find the standard-model prediction

$$\begin{aligned} |\epsilon_K| &= (2.161 \pm 0.140_{V_{cb}} \pm 0.061_{\rm param} \pm 0.064_{\eta_{tt}} \\ &\quad \pm 0.008_{\eta_{ut}} \pm 0.027_{\hat{B}_K} \pm 0.052_{\xi_s} \pm 0.046_{\kappa_\epsilon}) \times 10^{-3}, \\ &= (2.161 \pm 0.153_{{\rm param}+V_{cb}} \\ &\quad \pm 0.076_{\rm nonpert} \pm 0.065_{\rm pert}) \times 10^{-3}, \\ &= 2.16(18) \times 10^{-3}. \end{aligned} \quad (18)$$

We see that the perturbative uncertainty (~3.0%) is now of the same order as the combined nonperturbative one (~3.5%), while the dominant uncertainties originate from the parametric, experimental uncertainties (~7.1%). Moreover, the dominant perturbative uncertainty no longer originates from η_{ct} but from the top-quark contribution, η_{tt}. Note that using the exclusive determination $|V_{cb,\rm excl}| = 0.0403(8)$ [21] and the lattice value $\kappa_\epsilon = 0.923(6)$ [22] we find $\epsilon_K = 1.81(14) \times 10^{-3}$ in tension with the experimental measurement [23].

Discussion and conclusions.—In this Letter, we showed that a manifest implementation of CKM unitarity in the effective $|\Delta S = 2|$ Hamiltonian dramatically improves the convergence behavior of the perturbative series for its imaginary part, by removing a spurious long-distance charm-quark contribution. In this way, and using only known results in the literature, we reduced the residual uncertainty of the short-distance charm-quark contribution to the weak Hamiltonian by more than an order of magnitude. The perturbative uncertainty is now dominated by the missing NNLO corrections to the top-quark contribution, as well as partially known electroweak corrections at the percent level (see Refs. [24–26]). The calculation of these corrections [27] has the potential to bring the perturbative uncertainty of ϵ_K down to the percent level, motivating a renewed effort to compute long-distance effects using lattice QCD. Our analysis reinforces the role of ϵ_K in global CKM fits as the most important test of the standard model among the kaon flavor-changing neutral-current processes.

By contrast, the real part of the $|\Delta S = 2|$ Hamiltonian is dominated by up- and charm-quark contributions, and their convergence is not improved. Hence, the calculation of these contributions is a genuine task for lattice QCD, to which a significant effort is devoted [2,28,29]. However, our results have the potential to supply useful cross checks for part of these calculations: by performing the matching to the hadronic matrix elements for ϵ_K above the charm-quark threshold we can obtain a prediction of these matrix elements that can be directly compared to a future lattice calculation. This could shed additional light onto the lattice calculation of the kaon mass difference.

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