Comment on "Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime"

In this Comment, we elaborate on several points raised in Ref. [1]. The authors claimed to have found a four-dimensional gravitational theory which fulfills the assumptions of the Lovelock theorem [2] though not its implications. To that end, they employed a *regularization* procedure already outlined in Ref. [3]. This procedure consists of rescaling the coupling constant of the Gauss-Bonnet (GB) term by $1/(D-4)$ and taking the $D \to 4$ *limit* after varying the Einstein-Gauss-Bonnet (EGB) action. The authors of Ref. [1] claim that the variation of the GB term is proportional to $(D-4)$, canceling the $1/(D-4)$ factor and hence yielding a nonvanishing contribution to the field equations in $D=4$. This claim does not stand a thorough analysis given that the variation of the kth order Lovelock Lagrangian can be decomposed as [4,5]

$$\frac{1}{\sqrt{|g|}}\frac{\delta S^{(k)}}{\delta g^{\mu\nu}} = (D-2k)A^{(k)}_{\mu\nu} + W^{(k)}_{\mu\nu}, \qquad (1)$$

where $W^{(k)}_{\mu\nu}$ is a tensor from which no $(D-2k)$ can be extracted and which does not vanish for a general $D > 2k$. The GB term is given by $k=2$, and the field equations proposed in Ref. [1] for arbitrary dimension are

$$G_{\mu\nu} + \frac{1}{M_P^2}\Lambda_0 g_{\mu\nu} + \frac{2\alpha}{M_P^2}\left(A^{(2)}_{\mu\nu} + \frac{W^{(2)}_{\mu\nu}}{D-4}\right) = 0, \qquad (2)$$

where the explicit form of $A^{(2)}_{\mu\nu}$ and $W^{(2)}_{\mu\nu}$ can be found in Ref. [5]. Although the $1/(D-4)$ rescaling compensates the factor multiplying the $A^{(2)}_{\mu\nu}$ term, the same procedure renders the $W^{(2)}_{\mu\nu}$ term indeterminate. This owes to the fact that, although $W^{(2)}_{\mu\nu}$ vanishes in $D=4$, it does so due to algebraic reasons [4,5] and not because it is proportional to some power of $(D-4)$. Hence, the indeterminate term $W^{(2)}_{\mu\nu}/(D-4)$ renders the field equations (2) ill-defined in four dimensions. Indeed, the first problem to address would be to make sense of the limit of a tensor field [4–8], since these objects are defined for integer values of D only.

Despite the above discussion, the field equations (2) are well defined in $D=4$ when constrained to particular geometries in which $W^{(2)}_{\mu\nu}$ vanishes in arbitrary dimensions. This is the case, for instance, for all conformally flat geometries, including the Friedmann-Lemaitre-Robertson-Walker or maximally symmetric solutions found in Ref. [1]. However, metric perturbations will be sensible to the ill-defined terms, hence rendering these solutions unphysical. Indeed, though linear perturbations around a maximally symmetric background are oblivious to these pathologies [1], they enter at second order through terms proportional to $1/(D-4)$ [5]. In this direction, other works also showed that an infinitely strongly coupled new scalar degree of freedom appears beyond linear order [9]. These results strongly suggest that Eq. (2) is generally ill-defined in four dimensions.

A possible way to circumvent the pathologies of the above field equations (2) would be to get rid of the $W^{(2)}_{\mu\nu}$ term. However, $A^{(2)}_{\mu\nu}$ is not divergenceless [5]. Hence, by virtue of the Bianchi identity under diffeomorphisms, we conclude that there is no diffeomorphism-invariant action whose variation gives Eq. (2) without the $W^{(2)}_{\mu\nu}$ term.

Regarding spherically symmetric metrics of the form

$$ds^2 = B(r)dt^2 - B^{-1}(r)dr^2 - r^2 d\Omega^2_{D-2}, \qquad (3)$$

for them to have a vanishing $W^{(2)}_{\mu\nu}$ in arbitrary dimension, there are conditions that $B(r)$ must fulfill [5]. The spherically symmetric geometries presented in Ref. [1] do not satisfy these requirements and, therefore, cannot be solutions of Eq. (2), since this is not a well-defined set of equations in this case. Indeed, as proven in Ref. [5], they are neither solutions of the truncated equations (2) without the $W^{(2)}_{\mu\nu}$ term. This is not surprising, since the authors of Ref. [1] derived these geometries by taking $D=4$ in the $D \geq 5$ solutions for the EGB theory found in Ref. [10] and then rescaling the GB coupling by $1/(D-4)$, instead of finding a solution to the rescaled field equations (2) in $D=4$. Furthermore, the authors of Ref. [1] state that the central curvature singularity of these geometries cannot be reached by an observer. Nevertheless, as shown in Ref. [5], radial freely falling observers do reach the singularity in finite proper time, contradicting this claim.

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