Comment on "Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime"

In a recent Letter [1], a general covariant four-dimensional modified gravity that propagates only a massless spin-2 graviton and bypasses Lovelock's theorem [2] was claimed to exist. Here we show that this claim is not correct. The suggested theory is a limit of the Einstein-Gauss-Bonnet theory with the field equations

$$\lim_{D\to 4}\left[\frac{1}{\kappa}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda_0 g_{\mu\nu}\right)+\frac{\alpha}{D-4}\mathcal{H}_{\mu\nu}\right]=0,$$

where the "Gauss-Bonnet (GB) tensor" (which vanishes identically in four dimensions) reads [3]

$$\mathcal{H}_{\mu\nu}=2\Big[RR_{\mu\nu}-2R_{\mu\alpha\nu\beta}R^{\alpha\beta}+R_{\mu\alpha\beta\sigma}R_{\nu}{}^{\alpha\beta\sigma}-2R_{\mu\alpha}R^{\alpha}_{\nu}$$
$$-\frac{1}{4}g_{\mu\nu}(R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}-4R_{\alpha\beta}R^{\alpha\beta}+R^2)\Big]. \tag{1}$$

For the $D\to 4$ limit to work even at the formal level, there must exist a new tensor $\mathcal{Y}_{\mu\nu}$ such that one has

$$\mathcal{H}_{\mu\nu}=(D-4)\mathcal{Y}_{\mu\nu}, \tag{2}$$

and as $D\to 4$, this new tensor should not vanish and should have a smooth limit. One can show that [4] $\mathcal{H}_{\mu\nu}$ decomposes as

$$\frac{\mathcal{H}_{\mu\nu}}{D-4}=2\frac{\mathcal{L}_{\mu\nu}}{D-4}+\frac{2(D-3)}{(D-1)(D-2)}S_{\mu\nu}, \tag{3}$$

where $\mathcal{L}_{\mu\nu}=C_{\mu\alpha\beta\gamma}C_{\nu}{}^{\alpha\beta\gamma}-\frac{1}{4}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}g_{\mu\nu}$ and

$$S_{\mu\nu}=-\frac{2(D-1)}{(D-3)}C_{\mu\rho\nu\sigma}R^{\rho\sigma}-\frac{2(D-1)}{(D-2)}R_{\mu\rho}R^{\rho}_{\nu}$$
$$+\frac{D}{(D-2)}R_{\mu\nu}R+\frac{(D-1)}{(D-2)}g_{\mu\nu}\left(R_{\rho\sigma}R^{\rho\sigma}-\frac{D+2}{4(D-1)}R^2\right).$$

The $S_{\mu\nu}$ part in Eq. (3) is smooth in the $D\to 4$ limit, but the $\mathcal{L}_{\mu\nu}$ part is undefined (0/0) and discontinuous in the sense that $\mathcal{L}_{\mu\nu}$ is *identically* zero in four dimensions and nontrivial above four dimensions. If one naively takes the limit by dropping the $\mathcal{L}_{\mu\nu}$ part in Eq. (3), then one loses the Bianchi identity since $\nabla_\mu S^{\mu\nu}\neq 0$, which is not acceptable if gravity is expected to couple to a conserved source.

The easiest way [4] to see that such a tensor $\mathcal{Y}_{\mu\nu}$ does not exist in four dimensions is to employ the first order form of the theory. The GB part of the action (without any factors)

$$I_{\mathrm{GB}}=\int_{\mathcal{M}_D}\epsilon_{a_1a_2,\ldots,a_D}R^{a_1a_2}\wedge R^{a_3a_4}\wedge e^{a_5}\wedge e^{a_6},\ldots,\wedge e^{a_D}$$

when varied with respect to the vielbein yields zero in $D=4$ dimensions, and the following $D-1$ form in $D>4$,

$$\mathcal{E}a_D=(D-4)\epsilon_{a_1a_2,\ldots,a_D}R^{a_1a_2}\wedge R^{a_3a_4}\wedge e^{a_5}\wedge e^{a_6},\ldots,\wedge e^{a_{D-1}}.$$

To relate this to $\mathcal{H}_{\mu\nu}$ (1), one can recast the last expression in spacetime indices and take its Hodge dual to get a 1-form

$$*\mathcal{E}_{\nu}=\frac{(D-4)}{4}\epsilon_{\mu_1\mu_2,\ldots,\mu_{D-1}\nu}\epsilon^{\sigma_1,\ldots,\sigma_4\mu_5,\ldots,\mu_{D-1}}{}_{\mu_D}$$
$$R^{\mu_1\mu_2}{}_{\sigma_1\sigma_2}R^{\mu_3\mu_4}{}_{\sigma_3\sigma_4}dx^{\mu_D}, \tag{4}$$

from which one defines the rank-2 tensor $\mathcal{E}_{\nu\alpha}$ as $*\mathcal{E}_{\nu}=:\mathcal{E}_{\nu\alpha}dx^{\alpha}$ whose explicit form is

$$\mathcal{E}_{\nu\alpha}=\frac{(D-4)}{4}\epsilon_{\mu_1\mu_2,\ldots,\mu_{D-1}\nu}\epsilon^{\sigma_1,\ldots,\sigma_4\mu_5,\ldots,\mu_{D-1}}{}_{\alpha}$$
$$R^{\mu_1\mu_2}{}_{\sigma_1\sigma_2}R^{\mu_3\mu_4}{}_{\sigma_3\sigma_4}. \tag{5}$$

It is clear from this expression that a $(D-4)$ factor arises only in $D>4$ dimensions; namely, even if the front factor can be canceled by multiplying with a $1/(D-4)$ as suggested in Ref. [1], the ϵ tensors in the expression explicitly show the dimensionality of the spacetime to be $D>4$. If one tries to get rid of the epsilon tensors, then one loses the front factor. In fact, expressing the epsilon factors in terms of generalized Kronecker delta tensors, one arrives at $\mathcal{E}_{\nu\alpha}=2(D-4)!\mathcal{H}_{\nu\alpha}$, in which the front factor transmutes to $(D-4)$ factorial. So the upshot is that there is no nontrivial $\mathcal{Y}_{\mu\nu}$ (2) in four dimensions as is required for the claim of Ref. [1] to work. Our result is consistent with the Lovelock's theorem which rigorously shows that in four dimensions, the only second rank symmetric, covariantly conserved tensor that is at most second order in derivatives of the metric tensor (and this is required for a massless graviton and no other degrees of freedom), besides the metric, is the Einstein tensor. Therefore, a simple rescaling of the coefficient in the EGB theory as was suggested in Ref. [1] does not yield covariant equations of a massless spin-2 theory in four dimensions. The abovementioned lack of continuity of the EGB theory at $D=4$ can also be seen from various complementary analyses [5,6] which try to obtain a well-defined limit and end up with an extra scalar degree of freedom besides the massless graviton. The resulting theory depends on how one defines the limit supporting our arguments here.

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