Testing Critical Slowing Down as a Bifurcation Indicator in a Low-Dissipation Dynamical System

M. Marconi,[1] C. Métayer,[2] A. Acquaviva,[1] J. M. Boyer,[2] A. Gomel,[3] T. Quiniou,[2] C. Masoller,[4,*] M. Giudici,[1] and J. R. Tredicce[2,3]

[1]*Université Côte d' Azur, Institut de Physique de Nice, CNRS-UMR 7010, Sophia Antipolis, France*
[2]*Université de la Nouvelle Calédonie, ISEA, BP R4-98851 Nouméa Cedex, Nouvelle Calédonie*
[3]*Universidad de Buenos Aires, Departamento de Física, Intendente Guiraldes 2160, CABA, Buenos Aires, Argentina*
[4]*Departamento de Física, Universitat Politecnica de Catalunya, St Nebridi 22, Barcelona 08222, Spain*

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We study a two-dimensional low-dissipation nonautonomous dynamical system, with a control parameter that is swept linearly in time across a transcritical bifurcation. We investigate the relaxation time of a perturbation applied to a variable of the system and we show that critical slowing down may occur at a parameter value well above the bifurcation point. We test experimentally the occurrence of critical slowing down by applying a perturbation to the accessible control parameter and we find that this perturbation leaves the system behavior unaltered, thus providing no useful information on the occurrence of critical slowing down. The theoretical analysis reveals the reasons why these tests fail in predicting an incoming bifurcation.

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There has always been a special interest in trying to predict transitions, crisis, and catastrophic events. Today, a huge amount of research is devoted to determine good indicators applicable on time series obtained from real systems that may anticipate a change in its behavior or, in the language of dynamical systems, a bifurcation [1–3]. This is particularly relevant in disciplines like medicine, biology, atmospheric science, ecology, sociology, and economy, where these predictions may avoid a disaster or, at least, they may be useful to prepare the system to a behavioral change. For example, it has been conjectured that the advent of an epilepsy attack is the result of a phase transition [4,5], that climate on Earth is actually very close to a tipping point [6], that extremely intense pulses in lasers may result from a bifurcation of a chaotic attractor [7,8], and that evolutive specialization in ecology [9,10] is also the consequence of a bifurcation. We may say that any behavioral change in a real system is connected to the existence of a bifurcation in the corresponding dynamical system and that the prediction of these changes depends on the possibility of establishing reliable indicators alerting of the incoming bifurcation.

A well-established indicator that follows from the definition of bifurcation is known as "critical slowing down" (CSD). When the system approaches a bifurcation, its relaxation time after a perturbation grows asymptotically and this divergence is referred to as CSD [11]. CSD is often associated with an increase of the variance and of the autocorrelation of a system variable [12]. Nevertheless, it has been observed that these indicators are not always reliable for alerting on an incoming bifurcation [13,14].

On the other hand, real systems evolve toward a bifurcation because one or more parameters are changing in time. For example, the level of CO_2 in the atmosphere is an evolving parameter that may lead the Earth's climate system to a bifurcation [15].

In this Letter, we address the fundamental question whether CSD is always a good indicator of an incoming bifurcation in a system where a parameter is linearly changing in time. By definition, CSD can be identified by perturbing the dynamical system. Unfortunately, in real systems, this perturbation cannot be implemented in the variables but rather in the parameters that are accessible in the experiments. Hence, a second question that we address here is whether a perturbation of an evolving control parameter can be a reliable probe for testing the occurrence of a bifurcation. We answer to these questions by presenting a real nonautonomous system, with a control parameter that is swept linearly in time, where CSD appears only after the bifurcation has already occurred, hence when it might be too late to reverse the change in behavior. Furthermore, we show that a perturbation in the accessible control parameter is unable to provide any indication on the occurrence of CSD or the bifurcation crossing.

We begin by considering a simple two-dimensional dynamical system describing a class-B laser [8],

$$\begin{aligned} dS/dt &= -S(1-N), \\ dN/dt &= -\gamma(N-A+SN). \end{aligned} \tag{1}$$

Here S is proportional to the light intensity and N to the atomic population inversion; A is proportional to the pump

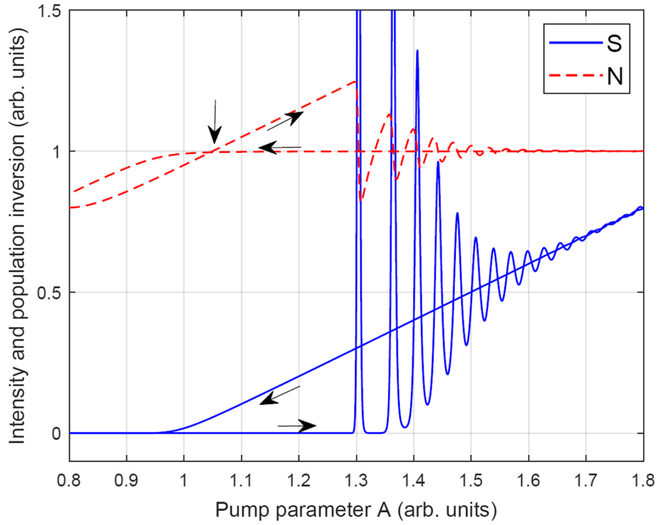

FIG. 1. Intensity S and population inversion N as a function of the pump A. The pump is swept at a velocity $b = 0.0005$; $\gamma = 0.01$. The initial conditions are $S_0 = 0.001$, $N_0 = 0.8$, $A_0 = 0.8$. The arrows indicate how S and N evolve as the pump is increased and then decreased. The vertical arrow shows that $N = 1$ is reached when $A = 1.05$.

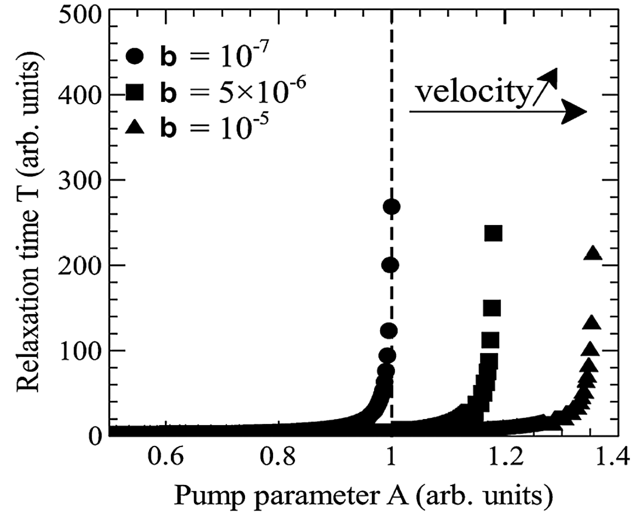

FIG. 2. Relaxation time T of the laser intensity S after this variable was perturbed by a short pulse as a function of the pump value A. Three values of the sweep velocity b are considered, while $\gamma = 2.7 \times 10^{-5}$ is kept constant. The occurrence of CSD is marked by the asymptotic growth of T. For increasing b, CSD takes place at values of A larger than the bifurcation point ($A = 1$, indicated with a dashed line).

and γ is the ratio between the decay rates of the population inversion (γ_p) and the intensity (γ_i). The time t is normalized to γ_i. This dynamical system exhibits a transcritical bifurcation at $A = 1$. For $A < 1$, the solution $(S, N) = (0, A)$ is stable. For $A > 1$, the solution $(S, N) = (A - 1, 1)$ is stable. We vary the pump parameter A with a triangular ramp of speed b, always smaller than the decay rate of the variables of the system,

$$A(t) = A_0 + bt \quad \text{for } t \leq t_0,$$
$$A(t) = A_0 + bt_0 - b(t - t_0) \quad \text{for } t_0 \leq t \leq 2t_0, \tag{2}$$

where A_0 is the initial value of the pump and t_0 is the duration of the ramp-up and ramp-down.

The evolution of the laser intensity S as a function of the pump parameter A is plotted in Fig. 1. S grows significantly at a value well beyond the bifurcation point $A = 1$. This delayed reaction of a laser when the pump is swept across the threshold was studied theoretically [16] and experimentally [17,18]. Critical slowing down was put in evidence in [18] by measuring the asymptotical growth of this delay as a function of the speed of the pump change. For a vanishing speed, this delay diverges, thus revealing the presence of CSD at the laser threshold. As shown in Fig. 1, the intensity remains close to zero on a large interval during which the pump continues to grow beyond the threshold value. Hence, the system accumulates energy, which is suddenly released, leading to a spikelike variation of the intensity. If the system is underdamped ($\gamma < 1$), as in the situation considered in Fig. 1, relaxation oscillations occur until an asymptotic solution is reached, where the intensity follows the pump. This behavior is typical of class-B lasers [19–21], such as semiconductor or solid-state lasers.

Here, we test numerically the occurrence of CSD in this system through a small perturbation ΔS in the laser intensity at different values of the pump parameter A. We measure the time taken by the perturbation to decrease to $1/e$ of its initial value. Our results are plotted in Fig. 2. We notice that, for slow ramps, the relaxation time diverges at the bifurcation point, i.e., when we approach $A = 1$, as demonstrated in [18]. However, for larger values of b, CSD does not take place at $A = 1$ but at a higher value of the pump parameter, which increases with b.

In order to explain these observations, we note that, when $S \approx 0$ (i.e., before the laser turns on), the evolution of a perturbation ϵ of the intensity is governed by

$$\frac{d\epsilon}{dt} = \epsilon(N - 1). \tag{3}$$

The relaxation time diverges when $d\epsilon/dt = 0$, so when N becomes equal to 1. Importantly, the amplitude of the perturbation has no effect whatsoever during this stage ($S \approx 0$). Because the pump parameter grows linearly in time [as indicated in Eq. (2) for $t \leq t_0$], the equation governing the evolution of N is

$$\frac{dN}{dt} = -\gamma(N - A_0 - bt), \tag{4}$$

whose solution is $N(t) = A_0 + bt - b(1 - e^{-t\gamma})/\gamma$ [22]. Then, by imposing $N = 1$, we can calculate analytically the critical pump value (A_c) at which the relaxation time diverges and CSD takes place,

$$A_c = 1 + \frac{b}{\gamma} + \frac{b}{\gamma} W[-e^{-\frac{\gamma}{b}(1-A_0)-1}], \tag{5}$$

with W being the Lambert w function. Hence, for the parameters used in Figs. 1 and 2, A_c depends mainly on the ratio b/γ. As this ratio increases, A_c grows above the value at which the bifurcation takes place ($A = 1$). The effect of A_0 on A_c is negligible provided that $A_0 < 1 - b/\gamma$. In agreement with this analytical estimation, in the simulations, using the parameters of Fig. 2, one finds $A_c = 1.004$, 1.18, and 1.37 for $b = 1 \times 10^{-7}$, 5×10^{-6}, and 1×10^{-5}, respectively, while in Fig. 1, $N = 1$ is reached when $A = 1.05$, indicated by the vertical arrow.

Therefore, we have identified a system with a time-swept parameter in which CSD takes place well beyond the bifurcation point, contradicting the common belief that CSD is an indicator of an upcoming bifurcation. Two ingredients are needed for the dynamical system to behave in this counterintuitive manner: a fast sweeping rate of the parameter and low dissipation. The system considered is a class-B laser, where γ (the ratio between the decay rates of the population inversion γ_p and the intensity γ_i) is significantly smaller than 1.

To meet this requirement, we perform experiments with a diode-pumped solid-state laser (SSL) Nd:YVO4 emitting at 1.060 μm. In this laser, γ_p is on the order of 2×10^4 s^{-1} [23], while γ_i is 5×10^9 s^{-1} (see Supplemental Material [24]), leading to $\gamma \approx 4 \times 10^{-6}$.

SSL threshold is observed for a bias current J of the diode pump $J = J_{th} = 147$ mA. The diode pump laser can be modulated by a triangular ramp applied to its bias current, hence sweeping linearly the pump intensity from a zero level ($J = 88$ mA, which corresponds to the diode pump threshold) up to 1.4 times the threshold value of the SSL ($J = 1.4 \times J_{th} = 208$ mA). The laser package is thermally stabilized in a temperature range where the SSL emits on the same single longitudinal mode in the whole swept pump range. The input and output signals (the bias current of the diode pump and the intensity of the SSL, respectively) are monitored on a digital oscilloscope. The ramp duration can be varied from 0.05 s to 0.25 ms. The speed of the fastest ramp, as defined in Eq. (2), is $b = 1.12 \times 10^{-6}$, hence $b/\gamma = 0.28$ (see Supplemental Material [24]). According to Eq. (5), this upper value of b/γ, together with the possibility of controlling experimentally b, makes this laser an ideal system to test the prediction of this equation. Unfortunately, as in the majority of real systems, it is not possible to perturb directly the laser variables (S, N) to probe the occurrence of CSD. However, real systems can be perturbed through their control parameters and we may check their influence on the variables.

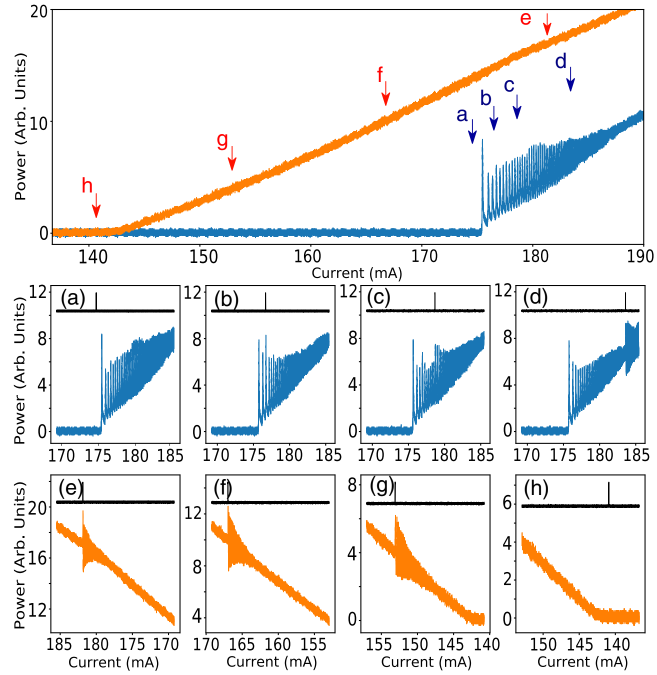

FIG. 3. Upper: laser output intensity as a function of the bias current of the diode pump that is swept in time by using a triangular ramp (see text for details). The intensity trace obtained during the positive (negative) slope of the ramp is displayed in blue (orange). (a)–(h) A short pulse (1 μs width and 40 mA height) is superimposed over the pump ramp at the positions indicated by the arrows on the upper panel. The effect of each perturbation on the laser intensity is shown in the corresponding panels.

The SSL intensity output versus the time-varying pump level is shown in the upper panel of Fig. 3 for a modulation rate of 100 Hz. We notice, in agreement with Fig. 1, that the laser intensity grows significantly only when the pump current is well above the threshold (the intensity spike occurs at $J = J_{on} = 175.5$ mA, which corresponds to $1.2J_{th}$). The first lasing peak is followed by damped relaxation oscillations whose frequency increases as the pump increases. The delayed response of the SSL in terms of the pump level has been investigated for different speed of the bias current ramp b and we have observed that it follows the b^{-1} law predicted in [16]. This delay is almost absent when ramping down the pump current and the laser switches off at $J \approx J_{th}$. The difference between the pump value at which the laser starts to emit ($J = J_{on}$) and the pump value at which it switches off ($J \approx J_{th}$) leads to the well-known dynamical hysteresis of Fig. 3, which has also been observed in [18].

For the low modulation rate used in Fig. 3, we estimate $b = 5.6 \times 10^{-8}$ and $b/\gamma = 0.014$ (see Supplemental Material [24]). Hence, according to Eq. (5), CSD is expected to occur very close to the SSL threshold ($J = J_{th}$). We test the occurrence of CSD by adding to the bias current of the diode pump a perturbation pulse that is synchronous with the current ramp. By varying the phase of the two signals, we can place the perturbation at arbitrary positions of the ramp and analyze the response in the intensity variable. We apply a pulse of 40 mA with a duration of 0.5 μs at full width at half maximum. We superimpose it to the pump ramp and, in Fig. 3, we show the most relevant positions, marked by the arrows in the upper panel.

The evolution of the perturbation depends clearly on its position on the ramp, as shown by the Figs. 3(a)–3(h). When the pump current is ramped up and the pulse is applied before the laser emits the first spike and switches on [Fig. 3(a)], the intensity is not affected and it remains at the level of the experimental noise. This is observed for any position of the perturbation in the interval $0 < J < J_{\mathrm{on}} = 1.2 J_{\mathrm{th}}$. If the perturbation is applied after the laser has switched on, the pulse may enhance the next relaxation oscillation peak [Fig. 3(b)]. Instead, when it is applied between two relaxation oscillation peaks, it will decrease the amplitude of the next relaxation oscillation. In any case, the perturbation pulse induces a new transient in the relaxation process started after the laser switches on. This can be seen in Fig. 3(c) and 3(d), where the perturbation is applied when the laser oscillations are significantly damped. One can notice that the relaxation is faster when the perturbation is applied closer to the top of the ramp, i.e., at the maximum value of the pump current. When the pump current is ramped down, the evolution of the perturbation is not interacting with another relaxation process and, therefore, it is more clearly visualized: during the ramp-down, the perturbation induces damped relaxation oscillations. As the perturbation is applied closer to the bifurcation point where the laser switches off, the damping time increases, while the relaxation oscillations frequency decreases [Figs. 3(e)–3(g)]. Finally, after the laser switches off, the perturbation does not induce any reaction on the intensity variable [Fig. 3(h)].

These experimental evidences indicate that no signature of CSD, nor of the bifurcation crossing at $J = J_{\mathrm{th}}$, can be found by perturbing the pump parameter when the system evolves from the off state to the on state. This surprising behavior is observed for any speed of the ramp, even for the highest ones, where CSD is expected to occur well beyond the threshold value of the SSL. When b is increased, the laser switches on at an increasing pump level (for example, with a ramp duration of 0.5 ms, $J_{\mathrm{on}} = 1.3 \times J_{\mathrm{th}}$) according to the law predicted in [16], and no effect on the intensity output is noticed when the pump perturbation is applied in the interval $0 < J < J_{\mathrm{on}}$. Instead, the perturbation pulse does have an effect on the intensity output when the laser is in the on state. In this case, intensity exhibits a spike followed by damped relaxation oscillations whose frequency and damping rate decrease as the perturbation is applied closer to the bifurcation point ($J = J_{\mathrm{th}}$).

In order to understand why perturbing the control parameter is not a reliable method to probe the occurrence

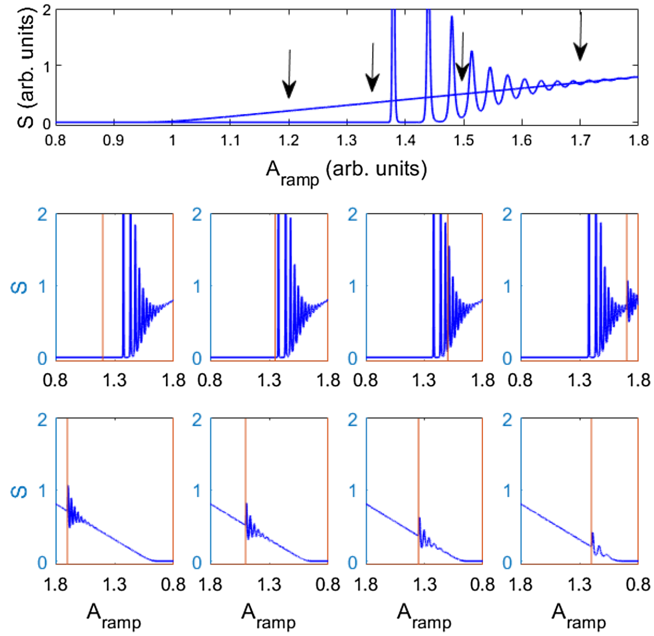

FIG. 4. Top: intensity dynamics as a function of the pump. The parameters are as in Fig. 1. The arrows and vertical lines indicate the position at which we apply a short pulse to the pump (see text for details). Middle (bottom): intensity dynamics when the pulse is applied during the upward (downward) ramp.

of CSD in our laser, we have used Eqs. (1) and (2) to analyze numerically the effect of a short pulse in the pump parameter, i.e., $A(t) = A_{\mathrm{ramp}}(t) + A_p(t)$, where $A_{\mathrm{ramp}}(t)$ is the triangular signal described by Eq. (2) and $A_p(t)$ is a short rectangular pulse. The results obtained, displayed in Fig. 4, are in very good agreement with the experimental findings. We remark that, for the parameters used in Fig. 4, $b/\gamma = 0.05$, and therefore, according to Eq. (5), CSD occurs at $A \sim 1.05$. Nevertheless, as in the experiments, no response to the pulse is observed in the laser intensity, as long as the laser is off. Simulations including noise show similar results (see Supplemental Material [24]).

The response of the system to a short perturbation in the pump can be understood by analyzing the structure of the equations. When the laser is off, the intensity S vanishes and N must, in principle, follow the pump parameter A. However, because N is a slow variable ($\gamma \ll 1$), it is unable to follow a sudden variation of A, as, for example, when A is perturbed by a short pulse. Hence, the pump pulse does not affect the value of the variable N, which will just continue to follow the pump ramp and S will remain close to zero, even if the perturbation pulse is applied when $A > A_c$, i.e., when the pump parameter is beyond the critical point where $N = 1$ and CSD occurs. In fact, no response in the S variable to the pump pulse can be observed before the laser switches on. After the first laser spike, $S > 0$ and this variable will respond directly to a perturbation pulse in the pump, thus "bypassing" the low pass filtering of the variable N. In this condition, the relaxation process following the pump perturbation is observable in S, and an effect of the pump pulse on the intensity output can be measured; however, this occurs only after the laser has turned on.

In conclusion, we have shown that, in a low-dissipation nonautonomous system with a control parameter that is swept linearly in time, CSD is not always a reliable indicator of an incoming bifurcation. By considering a two-dimensional real system featuring a transcritical bifurcation, we have demonstrated that CSD may occur beyond the bifurcation point, which makes it useless for alerting of an incoming behavioral change of the system. Moreover, we have shown that a perturbation of an evolving parameter might not be able to identify CSD: this occurs when the parameter affects directly a slow variable. In this case, a fast perturbation pulse may leave this variable unchanged and will have no effect on the system output. These results can be generalized to low-dissipation nonautonomous systems with dimension ≥ 2 that have a transcritical bifurcation, and we are currently studying their extension to other types of bifurcations.

We believe that our results have an important impact in environmental studies, in particular, in ecosystems' dynamics [25–27], because the evolution of populations is often described by coupled nonlinear rate equations, such as those considered here, and control parameters such as the amount of water or food available can vary in time.

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[*]Corresponding author.
cristina.masoller@upc.edu

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