Symmetry Breaking at All Temperatures

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We explore the existence of conformal field theories that persistently break a global symmetry at finite temperature. We identify vector models in $(3-\epsilon)$ spatial dimensions that have internal symmetries broken at any temperature. We study these systems in the small ϵ regime and in the large rank limit. The latter displays a conformal manifold and a moduli space of vacua deformed at finite temperature. We touch upon a candidate in $d=2$ dimensions.

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Introduction.—Spontaneous symmetry breaking is common at low temperatures (see, for instance, [1–4]). In this Letter, we investigate the possibility of symmetry breaking that persists at all temperatures. We address the problem by studying conformal field theories (CFTs) for which demonstrating the breaking at any finite temperature is sufficient. For a more detailed analysis, we refer the reader to [5].

Let us begin by considering quantum field theories (QFTs) in $d+1$ spacetime dimensions. It is commonly expected that at high temperatures ($\beta_{\rm th}\to 0$), their symmetries are unbroken. This is because at finite temperature we do not minimize the energy but instead we minimize

$$F = E - \frac{S}{\beta_{\rm th}}$$

(where S is the entropy), and hence at high temperature the high entropy states dominate. Since such states are typically disordered, one usually expects that, for high enough temperatures, the symmetry will be restored.

There are many examples in the literature of systems that break some symmetries at intermediate temperatures. For instance, in [6], the author demonstrated that for an $O(N)\times O(N)$-symmetric scalar field theory in $(3+1)$ dimensions, it is possible to have a phase at nonzero temperatures where one of the $O(N)$ groups is broken. However, since the theory considered there is not UV complete, the above phenomenon could only be verified up to temperature scales below the UV cutoff. For more discussions on symmetry breaking at intermediate temperatures, we refer the reader to [7–17]. In this Letter, however, our focus will be on the true high temperature limit.

Using the relationship between finite temperature and a theory on a circle, one concludes that in $d=2$ only discrete symmetries can break spontaneously at finite temperature [18], and in $d=1$ no symmetries whatsoever can break at finite temperature.

The AdS/CFT correspondence links the question of symmetry restoration at high temperatures with the no-hair "theorem." According to the AdS/CFT correspondence [19–21], a conformal theory in $\mathbb{R}^{d,1}$ is dual to the Poincaré patch of AdS_{d+2}. The field theory at finite temperature corresponds to a black brane geometry in AdS_{d+2} [22]. The statement that there is symmetry breaking in the CFT translates to hair on the black brane [23–25]. To our knowledge no such hair has been exhibited for uncharged black branes that dominate the thermal ensemble. See [26–31] for some relevant references.

The notion of arbitrarily high temperature has to be sharpened. In lattice systems with finitely many degrees of freedom per site, infinite temperature corresponds to the unit density matrix

$$e^{-\beta_{\rm th}H} \to \mathbb{I}.$$

Let us now take some order parameter localized at a site. Since in the state $\mathbb{I}$ all sites decouple, the expectation values of such local operators vanish. Hence, for such lattice systems the symmetries are restored at sufficiently high temperature [32].

We would like to consider temperatures that are much larger than the inverse correlation length but much smaller than the inverse lattice spacing distance. In fact, in QFTs, the state $\mathbb{I}$ does not necessarily make sense, and the high temperature limit is potentially nontrivial.

A QFT does not necessarily require a lattice to be defined. It can be UV complete by itself. The short distance limit is then described by a CFT. The question about the behavior of the theory at very high temperatures can be then translated into a question about CFT at nonzero temperature. Since there is no inherent scale in a CFT, any nonzero temperature is equivalent to any other nonzero temperature.

Therefore we may ask this question: *Are there unitary, local, nontrivial CFTs that break global symmetry at finite temperature?*

Unitarity appears to be important since in ensembles with a chemical potential $e^{-\beta_{\mathrm{th}}H-i\mu Q}$ it is already known that one can sometimes guarantee symmetry breaking for any radius of the thermal circle [33–37]. This is due to cancellations as a result of a purely imaginary chemical potential. On the other hand, for the ensemble $e^{-\beta_{\mathrm{th}}H}$, no such example exists to our knowledge.

The main point in this Letter is the construction of conformal models in $d = 3 - \epsilon$ dimensions that break a symmetry at finite temperature (strictly speaking, CFTs in fractional dimensions are not fully unitary models [38]). We will also provide some hints for a model in $d = 2$.

Our examples are in a class of conformal vector models. We first argue that such symmetry breaking at finite temperature cannot occur in models with a single quadratic Casimir. This covers many familiar quantum magnets. In the biconical class of fixed points [39–42], which have two quadratic Casimirs, we find examples of symmetry breaking at finite temperature.

We treat the biconical models both in the limit of small ϵ and in the limit of finite ϵ and large rank. We find that the two approaches overlap and agree. These biconical CFTs have symmetry group $O(m) \times O(N-m)$. For instance (and without loss of generality) if $m < N/2$, the unbroken symmetry group is $O(m-1) \times O(N-m)$. Therefore, there is no thermal gap [43], and instead we have Nambu-Goldstone bosons living on S^{m-1}. In the equal rank case $2m = N$, no symmetry breaking occurs at finite temperature.

We find some special features when studying the large rank limit of the biconical models. We find an exactly marginal operator and a moduli space of vacua, i.e., a degenerate family of ground states, though these models have no supersymmetry. (A similar thing happens in [44–46].) Moreover, the ground state energy of the thermal effective potential does not depend on temperature. In addition, the moduli space of vacua does not disappear at finite temperature but instead is deformed. This allows us to establish symmetry breaking in $d = 3 - \epsilon$ dimensions for finite but small enough ϵ. In $d = 2$, the Nambu-Goldstone bosons on S^{m-1} are lifted by small nonperturbative effects, and hence, strictly speaking, no symmetry breakdown occurs. It is still interesting, though, that the thermal gap is exponentially small for large m.

An interesting special case is the class of models with symmetry $O(1) \times O(N-1)$. We find that within the ϵ expansion, the symmetry is broken at finite temperature to $O(N-1)$. These models are therefore possible candidates for a unitary CFT in $2+1$ dimensions with persistent symmetry breaking at finite temperature.

Quantum critical points with such behavior at finite temperature would lead to rather unfamiliar phase diagrams.

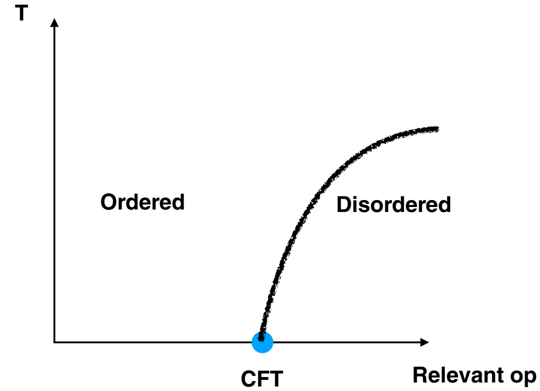

FIG. 1. A possible phase diagram in a theory where the critical point breaks a symmetry at finite temperature.

Symmetry breaking in the CFT at finite temperature implies that, had we started in the ordered zero-temperature phase, the order could persist for any temperature. Schematically, if we had just one relevant operator, one could find a phase diagram such as in Fig. 1.

Vector models.—We consider models with N real scalar fields ϕ_i, $i = 1, \ldots, N$ and potential

$$V = \frac{1}{4!}\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \tag{1}$$

in $4 - \epsilon$ spacetime dimensions. These models are interacting systems for finite positive ϵ. There are two limits in which we can carry out a perturbative study. One is when $\epsilon \ll 1$, and the other is when the number of fields N, the rank, is very large. We will study both limits, allowing us to establish a rather coherent picture for the thermal properties of such models. We will start from the limit where $\epsilon \ll 1$ is the smallest parameter.

Thermal physics in the ϵ expansion:We are interested in fixed points in the ϵ expansion [47]. Defining $\tilde{\lambda} = (\lambda/16\pi^2\epsilon)$, the fixed point equation becomes

$$\tilde{\lambda}_{ijkl} = \tilde{\lambda}_{ijmn}\tilde{\lambda}_{mnkl} + 2 \text{ permutations}. \tag{2}$$

As long as the fixed point equation, Eq. (2), is satisfied, the potential is bounded from below [42]. This follows from the fixed point equation since $\tilde{\lambda}_{ijkl}\phi_i\phi_j\phi_k\phi_l \sim \mathrm{Tr}(\tilde{\lambda}_{ijmn}\phi_i\phi_j)^2$, where the square means the square of a matrix with the indices mn.

To compute the thermal mass, we integrate out the nonzero Matsubara modes. We find that to leading order in ϵ the thermal mass squared matrix is given by

$$\mathcal{M}^2_{ij} = \frac{\beta_{\mathrm{th}}^{-2}}{24}\lambda_{ijkk} = \frac{2}{3}\pi^2\epsilon\beta_{\mathrm{th}}^{-2}\tilde{\lambda}_{ijkk}. \tag{3}$$

This is not manifestly positive for solutions of Eq. (2).

Let $G \leq O(N)$ be the symmetry group of the scalar potential Eq. (1). Suppose that G has only a single quadratic invariant. This is equivalent to requiring that the $O(N)$ fundamental representation is irreducible under G. For such models, one can prove that the thermal mass matrix is positive definite, and thus there is no symmetry breaking at finite temperature. The class of models with a single quadratic invariant includes the $O(N)$ models and the cubic, tetrahedral, bifundamental, MN, tetragonal, and Michel fixed points.

One interesting class of models not covered by the above no-go theorem are the biconical models that have $O(m) \times O(N-m)$ symmetry. These models have two quadratic invariants. Let ϕ_1 be a vector of length m and ϕ_2 be a vector of length $N-m$. We have three quartics that need to be tuned to their fixed point values α', β', γ':

$$V = \frac{\alpha'}{8}(\phi_1^2)^2 + \frac{\beta'}{8}(\phi_2^2)^2 + \frac{\gamma'}{4}\phi_1^2\phi_2^2.$$

The one-loop equations for α, β, γ (which differ from α', β', γ' by $16\pi^2\epsilon$) are

$$\alpha = \alpha^2(m+8) + \gamma^2(N-m), \tag{4}$$

$$\beta = \beta^2(N-m+8) + m\gamma^2, \tag{5}$$

$$1 = \alpha(m+2) + \beta(N-m+2) + 4\gamma. \tag{6}$$

(In the last equation, we assumed that $\gamma \neq 0$.) The solution $\alpha = \beta = \gamma = [1/(N+8)]$ is the $O(N)$-invariant fixed point. We will discard it since the no-go theorem applies to it.

The thermal mass matrix is proportional to

$$\mathcal{M}^2 \sim \begin{pmatrix} \alpha(m+2)\delta_{AB} & 0 \\ +\gamma(N-m)\delta_{AB} & \\ & \beta(N-m+2)\delta_{ab} \\ 0 & +\gamma m \delta_{ab} \end{pmatrix}. \tag{7}$$

First, consider the simpler case of equal rank, $2m = N$. We have the solution with enhanced $O(N)$ symmetry, and, in addition,

$$\alpha = \beta = \frac{m}{2(m^2+8)}, \qquad \gamma = \frac{4-m}{2(m^2+8)}. \tag{8}$$

For $m > 4$, we have $\gamma < 0$, but the potential is still increasing in all directions because $\gamma^2 < \alpha^2$. The thermal masses squared are both proportional to $\alpha(m+2) + \gamma m$. One can verify that the thermal masses are always positive. In conclusion, the equal rank biconical critical model has no symmetry breaking at finite temperature.

We now turn our attention to nonequal rank models. We will keep ϵ as the smallest parameter, but we will now take

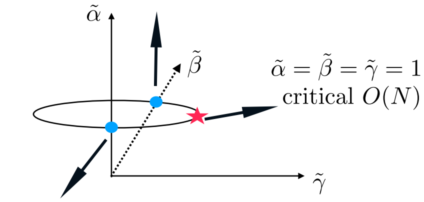

FIG. 2. A circle of fixed points in the large rank limit. The blue dots and red star surely survive the finite rank corrections, but there is another fixed point with $\gamma < 0$ that likewise survives the finite rank corrections.

large N. In the large N limit, the couplings α, β, γ all scale like $1/N$. We rescale the couplings accordingly: $\tilde{\alpha} = N\alpha, \tilde{\beta} = N\beta, \tilde{\gamma} = N\gamma$. To leading order in $1/N$,

$$\tilde{\alpha} = x\tilde{\alpha}^2 + (1-x)\tilde{\gamma}^2, \tag{9}$$

$$\tilde{\beta} = (1-x)\tilde{\beta}^2 + x\tilde{\gamma}^2, \tag{10}$$

$$1 = x\tilde{\alpha} + (1-x)\tilde{\beta}, \tag{11}$$

where we have denoted $x = m/N$.

The three beta function equations, Eqs. (9), (10), and (11), are degenerate. One gets a circle of fixed points parameterized by $\tilde{\gamma} \in \{-[1/2\sqrt{x(1-x)}], [1/2\sqrt{x(1-x)}]\}$. One expects that generic points on this conformal circle do not survive finite rank corrections (see Fig. 2).

One can check that $\tilde{\alpha}\tilde{\beta} = \tilde{\gamma}^2$ for all $\tilde{\gamma}$. Therefore, there is always a flat direction in field space at zero temperature as long as $\tilde{\gamma} < 0$. Thus, the large rank limit leads to a line of fixed points, and those with $\tilde{\gamma} < 0$ have a flat direction in field space at zero temperature. The flat direction persists even at finite temperature! Indeed, the thermal mass term in the potential is proportional to $[x\tilde{\alpha} + (1-x)\tilde{\gamma}]\phi_1^2 + [(1-x)\tilde{\beta} + x\tilde{\gamma}]\phi_2^2$. We find that it vanishes on the zero temperature flat direction as long as $\tilde{\gamma} < 0$. Therefore, the moduli space of finite temperature vacua is the hyperbola

$$\sqrt{\alpha}\phi_1^2 - \sqrt{\beta}\phi_2^2 + \frac{x\alpha + (1-x)\gamma}{12\sqrt{\alpha}} N\beta_{\rm th}^{-2} = 0, \tag{12}$$

where ϕ_1 and ϕ_2 are the thermal expectation values of the corresponding fields.

If the hyperbola is not degenerate (i.e., it does not contain the origin), regardless of the form of the small corrections due to finite rank, the vacuum would be away from the origin. In the equal rank case, the fixed point that survives the $1/N$ expansion has $\tilde{\alpha} = -\tilde{\gamma} = 1$. The hyperbola degenerates, and the origin remains as the only true vacuum. More generally, $\tilde{\gamma}$ of the fixed point that survives the expansion in $1/N$ must satisfy the radical equation

$$4x(1-x)\tilde{\gamma}_*(x)^3-20x(1-x)\tilde{\gamma}_*(x)^2+3\tilde{\gamma}_*(x)+9=0. \quad (13)$$

Thus, we obtain the biconical fixed point for $0 < x < 1$ with the following leading large N values of the couplings:

$$(\tilde{\alpha},\tilde{\beta},\tilde{\gamma}) = \left(\frac{1+\mathrm{sgn}(x-\frac{1}{2})\sqrt{1-4x(1-x)\tilde{\gamma}_*(x)^2}}{2x}, \frac{1-\mathrm{sgn}(x-\frac{1}{2})\sqrt{1-4x(1-x)\gamma_*(x)^2}}{2(1-x)}, \tilde{\gamma}_*(x)\right). \quad (14)$$

It turns out that one of the two thermal masses is negative for $x \neq 1/2$ for the fixed point (14). Therefore, the hyperbola (12) is nondegenerate. This means that one necessarily has finite temperature symmetry breaking even at finite nonequal rank.

Upon considering finite rank corrections, only one point on the hyperbola remains as the true vacuum. It is important to find which one it is since the symmetry breaking pattern depends on it.

Without loss of generality, we take $1/2 < x < 1$. One finds that the vacuum is at the vertex of the hyperbola. In terms of $\tilde{\gamma}_*$ which solves (13), we find

$$(\Phi_1^2,\Phi_2^2)=\left(0,\frac{\tilde{\gamma}_*^2(2x-2x^2)+\tilde{\gamma}_*(-2x^2+5x-3)-3x}{12\tilde{\gamma}_*[3-4x(1-x)\tilde{\gamma}_*]}\beta_{\mathrm{th}}^{-2}\right), \quad (15)$$

while for $x = 1/2$, $(\Phi_1^2, \Phi_2^2) = (0, 0)$. We conclude that for the finite nonequal rank case, we found a critical point with symmetry breaking at arbitrary nonzero temperature and the following symmetry breaking pattern:

$$O(m)\times O(N-m) \underset{\beta_{th}^{-1}>0}{\longrightarrow} \begin{cases} O(m-1)\times O(N-m) & m<\frac{N}{2} \\ O(m)\times O(N-m-1) & m>\frac{N}{2} \\ \text{no breaking} & m=\frac{N}{2} \end{cases}. \quad (16)$$

Large N analysis:Here we explore the large N limit of the biconical model with $O(m)\times O(N-m)$ symmetry and fixed m/N in d spatial dimension. While small ϵ makes the model perturbatively tractable, the large N techniques allow resummation of the perturbation series, and therefore some nonperturbative aspects of the model are elucidated in this limit. This study allows some of the results of the previous section to be extended to finite ϵ.

Large N vector modes are approximately free. Hence, the ground state approaches a Gaussian state as $N \to \infty$ [48,49], i.e., up to a normalization constant, it takes the following form in the space of fields:

$$\Psi(\phi_1,\phi_2) \propto \exp\left[-\frac{1}{2}\sum_{i=1}^{2}\int\frac{d^dk}{(2\pi)^d}\omega_i(k)|\phi_i(k)|^2\right], \quad (17)$$

where $\omega_i(k) = \sqrt{k^2+m_i^2}$. In position space, it can be written as

$$\Psi(\phi_1,\phi_2)\propto e^{-\frac{1}{4}\sum_{i=1}^{2}\int d^dx\int d^dy[\phi_i(x)-\sigma_i]D_i^{-1}(x-y)[\phi_i(y)-\sigma_i]}, \quad (18)$$

where $D_i^{-1}(x-y)$ is the Fourier transform of $2\omega_i(k)$, and two arbitrary constants σ_i parameterize the location of the Gaussian state in the space of fields.

To determine the values of m_i^2 and σ_i for the biconical model at the fixed point, we resort to this variational principle [50]:

$$\mathcal{W} = \langle\Psi|\mathcal{H}|\Psi\rangle \geq \langle 0|\mathcal{H}|0\rangle,$$
$$\mathcal{H} = \frac{1}{2}\pi_i\pi_i + \frac{1}{2}\nabla\phi_i\nabla\phi_i + \frac{g_{ij}^B}{4N}\phi_i^2\phi_j^2, \qquad 0\leq i,j\leq 2. \quad (19)$$

Here $\mathcal{W}$ is the variational functional, $|0\rangle$ is the vacuum state of the model governed by the Hamiltonian density $\mathcal{H}$, and $|\Psi\rangle$ represents a family of normalized trial states (18). The idea is to minimize the lhs with respect to the variational parameters m_i^2 and σ_i to find an approximation to the ground state energy. Evaluating $\mathcal{W}$ boils down to Gaussian integration.

In the large N limit, the renormalized couplings g_{ij} lie on a curve defined by

$$\det(g_{ij}) = 0, \qquad xg_{11} + (1-x)g_{22} = 8\pi^2\epsilon. \quad (20)$$

For each set of these couplings, the minimum of $\mathcal{W}$, which is obtained at $\mathcal{W} = 0$, lies along a flat direction in field space

$$m_i^2 = 0, \qquad \begin{pmatrix}\sigma_1^2\\ \sigma_2^2\end{pmatrix} = \begin{pmatrix}\pm\sqrt{g_{22}/g_{11}}\\ 1\end{pmatrix}\mu^{2-\epsilon}, \quad (21)$$

for $\mathrm{sign}(g_{12}) = \mp 1$, where μ is an arbitrary energy scale, and (σ_1^2, σ_2^2) is aligned along the eigenvector of g_{ij} with zero eigenvalue. Each field configuration along the flat direction can serve as a ground state of the theory.

Since $\sigma_i^2 \geq 0$, we conclude that for $g_{12} \geq 0$ there is a unique vacuum at $\mu = 0$ that respects the symmetries, whereas for $g_{12} < 0$ there is a flat direction in field space.

At the origin, scale invariance, the $O(m)$, and $O(N-m)$ symmetries are all retained. Away from the origin in field space, scale invariance is spontaneously broken. This breaking leads in turn, by (21), to the spontaneous symmetry breaking of the $O(m)$ and/or $O(N-m)$

symmetries. Hence, away from the origin, there are massless Nambu-Goldstone bosons and a dilaton.

We have therefore showed that in the strict large rank limit, there is a conformal manifold and a moduli space of vacua for $g_{12} < 0$. This is exactly as in the ϵ expansion. We will next see that the finite temperature corrections lead to a hyperbola, exactly as in the ϵ expansion.

The variational functional $\mathcal{W}$ at finite $\beta_{\rm th}$ is obtained by introducing a trial thermal state,

$$\mathcal{W} = \mathcal{F}_0 + \mathrm{Tr}[\rho_0(\mathcal{H} - \mathcal{H}_0)] \geq \mathcal{F},$$
$$\mathcal{H}_0 = \frac{1}{2}\sum_i [\pi_i^2 + (\nabla\phi_i)^2 + m_i^2(\phi_i - \sigma_i)^2], \qquad (22)$$

where $\mathcal{F}$ is the free energy density of the model and $\mathcal{F}_0$ and ρ_0 denote the free energy density and thermal density matrix associated with $\mathcal{H}_0$. In the limit $\beta_{th} \to \infty$, we recover the previous ansatz (19). For $g_{12} \geq 0$, there is a unique vacuum that respects the symmetries, and therefore we proceed to the cases with $g_{12} < 0$ where the symmetry can be broken. One finds that now $\mathcal{W} = 0$ at any point on the ridge $m_1^2 = m_2^2 = 0$,

$$\begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \sqrt{g_{22}/g_{11}} \\ 1 \end{pmatrix} \mu^{2-\epsilon} - \frac{c(\epsilon)\beta_{\rm th}^{\epsilon-2}}{12} \begin{pmatrix} x \\ 1-x \end{pmatrix}, \qquad (23)$$

where the function $c(\epsilon)$ is defined below:

$$c(\epsilon) \equiv \frac{6\Gamma(\frac{2-\epsilon}{2})\zeta(2-\epsilon)}{\pi^{\frac{4-\epsilon}{2}}}. \qquad (24)$$

Note that this function goes to 1 in $\epsilon \to 0$ limit, making (23) consistent with (12) in the small ϵ regime. Moreover, the Riemann zeta function diverges in the $\epsilon \to 1$ limit and hence restricts the validity of this analysis to $\epsilon < 1$. Such divergences of thermal expectation values of the fields are consistent with the impossibility of a symmetry-broken phase in $(2+1)$ dimensions at nonzero temperatures.

Each point on the ridge (23) corresponds to a global minimum of the free energy, and therefore it represents a thermodynamically stable phase in the large N limit. In general, the hyperbola (23) does not pass through the origin, and therefore $O(m) \times O(N-m)$ is broken at finite $\beta_{\rm th}$. The introduction of the temperature $\beta_{\rm th}$ explicitly breaks scale invariance, but a moduli space of vacua continues to exist. These results agree with the ϵ expansion.

Now comes the more difficult question regarding which of these fixed points survives at finite rank. In the ϵ expansion, we provided an explicit answer that shows that symmetry breaking indeed takes place at finite large rank. To find out the answer more generally, one needs to study $1/N$ corrections at arbitrary ϵ.

In summary, we have shown that the conformal manifold and moduli spaces of vacua exist at arbitrary d and $N = \infty$. The $1/N$ corrections needed to find out the true finite temperature vacua at finite large rank were only determined for $3-\epsilon$ dimensions with small ϵ. Therefore, we can conclude that symmetry breaking at finite temperature in the biconical models takes place in $3-\epsilon$ dimensions for finite small ϵ. It would be nice to extend this analysis to finite ϵ.

Toward a model in $2+1$ dimensions:The finite temperature symmetry breaking pattern of the biconical model is (for $n_1 < n_2$)

$$O(n_1) \times O(n_2) \to O(n_1 - 1) \times O(n_2). \qquad (25)$$

This cannot hold true all the way to $\epsilon = 1$, i.e., $2+1$ dimensions, due to the Mermin-Wagner-Hohenberg-Coleman theorem [18,51,52] (remember that we are at finite temperature). The only exception is $n_1 = 1$, in which case one can potentially have the symmetry breaking pattern (25) at finite temperature

$$\mathbb{Z}_2 \times O(N) \to O(N). \qquad (26)$$

This may in principle occur at finite temperature in $2+1$ dimensions and hence the case $n_1 = 1$ warrants some attention.

In fact, (26) does occur in the ϵ expansion. The appropriate large N limit turns out to be $\tilde{\gamma} = N\gamma, \tilde{\beta} = N\beta, \tilde{\alpha} = N\alpha$, which now leads in the large N limit to the equations

$$\tilde{\alpha} = \tilde{\gamma}^2, \qquad \tilde{\beta} = 1. \qquad (27)$$

Clearly this again parameterizes a one-dimensional conformal manifold, except that now it is unbounded and looks like a parabola. These theories describe a free field in an $O(N)$ bath where the backreaction of the free field sector on the $O(N)$ model is very small. It is crucial to find which of the fixed points on the conformal manifold correspond to fixed points that exist also for finite rank. Following the same strategy as before, one finds the following equation:

$$(\tilde{\gamma} - 1)(\tilde{\gamma} + 3) = 0.$$

Of course, $\tilde{\gamma} = 1$ is the $O(N)$ invariant fixed point, while $\tilde{\gamma} = -3$ is the new, more interesting fixed point. To leading order in the large rank expansion, the thermal masses (squared) at this new fixed point are $(2\pi^2\epsilon/3\beta_{th}^2)(-3, 1)$. Therefore we obtain this hyperbola of vacua:

$$3\Psi^2 - \vec{\phi}^2 = \frac{N}{12\beta_{th}^{2-\epsilon}}. \qquad (28)$$

One can further show that, upon including finite rank corrections, the only true vacuum that remains is the one where Ψ obtains a vacuum expectation value $\langle\Psi^2\rangle = (N/36\beta_{th}^{2-\epsilon})$ and $\langle\vec{\phi}\rangle = 0$.

Therefore, the $\mathbb{Z}_2$ symmetry at finite temperature is certainly broken at large enough finite N and small ϵ. It is possible in principle that it continues to hold true not just for small ϵ but also in $2+1$ dimensions. We hope to resolve this question in the future.

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