

Chern-Simons Origin of Superstring Integrability

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We derive the $\text{AdS}_5 \times \text{S}^5$ Green-Schwarz superstring from four-dimensional Beltrami-Chern-Simons theory reduced on a manifold with singular boundary conditions. In this construction, the Lax connection and spectral parameter of the integrable superstring have a simple geometric origin in four dimensions as gauge connection and reduction coordinate. κ symmetry arises as a certain class of singular gauge transformations, while the world-sheet metric comes from complex-structure-changing Beltrami differentials. Our approach offers the possibility of investigating integrable holography using traditional field theory methods.

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Introduction.—Integrability is an invaluable exact tool for AdS/CFT holography [1] which has provided significant evidence in favor of many of the dualities conjectured by Maldacena [2,3]. Signs of integrability were noticed early on in both gauge and string theory. At weak 't Hooft coupling the dilatation operator corresponds to an integrable spin-chain Hamiltonian [4–6], while at strong coupling superstring equations of motion are equivalent to the flatness of an auxiliary Lax connection [7,8]. Yet the origin of the integrable structure underlying these theories remains obscure.

In this Letter, we will attempt to demystify the appearance of integrability in holographic superstring theory by showing how Metsaev-Tseytlin (MT) κ -symmetric string actions [9,10] can be obtained from four-dimensional Chern-Simons (CS_4) gauge theory with suitably chosen boundary conditions. Recently, a new approach to integrability based on CS_4 theory has been proposed in [11–13]. In this approach, reducing CS_4 theory to two dimensions in the presence of defects [14] gives rise to many integrable two-dimensional field theories such as the Gross-Neveu and Wess-Zumino-Witten models, as well as the pure-spinor sigma model on $\text{AdS}_5 \times \text{S}^5$ [15].

We begin by obtaining the MT sigma model from a reduction of CS_4 theory generalizing [14]. As in that work, the mysterious Lax connection and spectral parameter of the MT sigma model have a prosaic interpretation in terms of the gauge potential and direction of reduction in the CS_4 theory [16]. The absence of fermion kinetic terms makes

the MT sigma model pathological [17]. Nonetheless, coupling it to a world-sheet metric leads to a consistent *string theory* [18] with target-space supersymmetry [19] because the action has κ symmetry [20].

We couple CS_4 theory to complex-structure-changing Beltrami differentials. In the presence of singular boundary conditions, these couplings cannot be removed by field redefinitions and reducing this Beltrami-Chern-Simons (BCS) theory to two dimensions gives the MT *string theory*. We show that BCS theory has κ symmetry, with κ transformations implemented by certain singular *gauge transformations* combined with an action on the Beltrami differentials which has compact support near the singularities. The resulting theory is essentially a conventional gauge theory of Chern-Simons type coupled to extra matter. On general grounds its observables will be Wilson lines whose interactions encode the R matrix of AdS/CFT. These can be computed using Feynman graphs with propagators and vertices derived in a conventional way from the action, taking into account the slightly unusual boundary conditions. As a result, our construction opens up the possibility of investigating integrability in string theory and holography through conventional and rigorous quantum field theory methods, albeit with singular boundary conditions.

MT sigma model from CS_4 .—Consider the CS_4 action on $\mathcal{V} = \Sigma \times C$

$$S_{\text{CS}_4} = \frac{1}{h} \int_{\mathcal{V}} \omega \wedge L_{\text{CS}}(A), \quad (1)$$

where L_{CS} is the CS Lagrangian (23), $\Sigma = \mathbb{R}^2$, $C = \mathbb{C}\mathbb{P}^1$ and the holomorphic one-form ω has n second-order poles, and $2n - 2$ first-order zeros

$$\omega = \frac{\prod(z - q_k) \prod(z - \tilde{q}_k)}{\prod(z - p_i)^2} dz. \quad (2)$$

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The gauge group of main interest in this Letter is $G = \mathfrak{psu}(2, 2|4)$ whose dual Coxeter number is zero. As a result, the theory is framing-anomaly free and h is not quantized. As reviewed in [18], $\mathfrak{psu}(2, 2|4)$ has a \mathbb{Z}_4 automorphism and we denote its i^m eigenspace by $\mathfrak{psu}(2, 2|4)^{(m)}$ ($m = 0, \dots, 3$) and recall that $\mathfrak{psu}(2, 2|4)^{(0)} = \mathfrak{so}(4, 1) \times \mathfrak{so}(5)$. As in [14], for a well-defined action we require $A_w, A_{\bar{w}}$ to have first-order zeros at $z = p_i$ and poles at

$$A_w|_{z=q_k} \sim \frac{1}{z - q_k}, \quad A_{\bar{w}}|_{z=\tilde{q}_k} \sim \frac{1}{z - \tilde{q}_k}. \quad (3)$$

Solving the equations of motion and boundary conditions gives $A = \hat{A} + A'$, where $\hat{A} = \hat{\sigma}^{-1} d\hat{\sigma}$, $A' = \hat{\sigma}^{-1} \mathcal{L} \hat{\sigma}$, with $\mathcal{L}_{\bar{z}} = 0$ and

$$\mathcal{L}_w = \frac{\prod(z - p_i)}{\prod(z - q_k)} \sum_j \frac{1}{p_j - z} \frac{\prod(p_j - q_k)}{\prod_{i \neq j} (p_j - p_i)} \partial_w \sigma_j \sigma_j^{-1}, \quad (4)$$

$$\mathcal{L}_{\bar{w}} = \frac{\prod(z - p_i)}{\prod(z - \tilde{q}_k)} \sum_j \frac{1}{p_j - z} \frac{\prod(p_j - \tilde{q}_k)}{\prod_{i \neq j} (p_j - p_i)} \partial_{\bar{w}} \sigma_j \sigma_j^{-1}. \quad (5)$$

The notation used above follows [14]. In particular, up to gauge transformations, $A_{\bar{z}}$ defines a map $\sigma: \mathbb{R}^2 \rightarrow G^n$, which extends to $\hat{\sigma}: \mathcal{V} \rightarrow G^n$, where near $z = p_i$, $\hat{\sigma} \sim \sigma_i$, with $\sigma_i: \mathbb{R}^2 \rightarrow G$. In [14] constant gauge transformations were used to set $\sigma_n = \text{Id}$. Inserting A back into S_{CS_4} localizes the action on the boundary and gives a large family of integrable sigma models, with left-acting G^n symmetry and Lax connection \mathcal{L} [14], which were also found in [21] without reference to CS_4 theory.

We now set $p_j = \exp(2\pi i j/n)$ and, in contradistinction to [14], take the limit

$$\begin{aligned} q_1 = \dots = q_{n-m} = \tilde{q}_{n-m+1} = \dots = \tilde{q}_{n-1} &\rightarrow 0, \\ \tilde{q}_1 = \dots = \tilde{q}_{n-m} = q_{n-m+1} = \dots = q_{n-1} &\rightarrow \infty, \end{aligned} \quad (6)$$

with $1 < m < n$. Upon rescaling $h \rightarrow h(-\tilde{q}_1)^{n-1}$, the $m = 2$ action is

$$\begin{aligned} S^{n,m=2} &= \frac{k}{4\pi n} \int_{\Sigma} \sum_i J_{i,w} J_{i,\bar{w}} - \sum_{i,j} \alpha_{ij} J_{i,\bar{w}} J_{j,w} \\ &\quad - \frac{k}{8\pi n^2} \int_{\Sigma} \sum_{i \neq j} \frac{p_i^3 + p_j^3}{p_i p_j (p_i - p_j)} J_{i,\bar{w}} J_{j,w} \\ &\quad + \frac{k}{12\pi n^2} \int_{\Sigma \times \mathbb{R}^+} f^{abc} X_{i,a} X_{i,b} X_{i,c}, \end{aligned} \quad (7)$$

where $J_i \equiv d\sigma_i \sigma_i^{-1}$, $X_i \equiv \sigma_i^{-1} d\sigma_i$, f^{abc} are the gauge-group structure constants, \mathbb{R}^+ is the z -plane radial direction, $\alpha_{ij} \equiv (1 + p_i/p_j + p_j/p_i)/n$ and $k = 8\pi i/h$ [22].

These models have a \mathbb{Z}_n symmetry ρ , which permutes the n copies of G , together with acting by the \mathbb{Z}_n

automorphism on each copy, and multiplies z by an n th root of unity. This fixes a subgroup $\rho(G^{(0)}) = G^{(0)} \subset G$, which for $G = \mathfrak{psu}(2, 2|4)$ is $G^{(0)} = \mathfrak{so}(4, 1) \times \mathfrak{so}(5)$. Gauging the \mathbb{Z}_n action leads to novel integrable models on generalized symmetric spaces $G/G^{(0)}$ whose equations of motion are equivalent to the flatness condition of the Lax connection

$$\begin{aligned} \mathcal{L}_w^{n,m}(z) &= \sum_{k=0}^m z^k J_w^{(k)} + \sum_{k=m+1}^{n-1} z^{k-n} J_w^{(k)}, \\ \mathcal{L}_{\bar{w}}^{n,m}(z) &= \sum_{k=0}^{n-m-1} z^k J_{\bar{w}}^{(k)} + \sum_{k=n-m}^{n-1} z^{k-n} J_{\bar{w}}^{(k)}, \end{aligned} \quad (8)$$

where $J_a \equiv J_{1,a}$, $J_{i,a} = \rho^{i-1}(J_a)$, and $J^{(k)}$ is the k th \mathbb{Z}_n eigenspace. In particular, for $m = 2$, $n = 4$ we obtain the MT σ model [9,10,23] with target-space $\mathfrak{psu}(2, 2|4)/[\mathfrak{so}(4, 1) \times \mathfrak{so}(5)]$

$$S^{n=4,m=2} = \frac{k}{4\pi} \int_{\Sigma} J_w^{(2)} J_{\bar{w}}^{(2)} - J_w^{(1)} J_{\bar{w}}^{(3)} + J_w^{(3)} J_{\bar{w}}^{(1)}. \quad (9)$$

Metric and Virasoro constraints.—The $S^{n,m}$ models, unlike their pure-spinor counterparts [14], do not have second-order kinetic terms for fermions, since the boundary conditions imposed in CS_4 to obtain them are not elliptic. This apparent disadvantage is in fact a boon: for judicious choices of G , $S^{4,2}$ has κ symmetry when coupled to the world-sheet metric.

In our derivation of $S^{4,2}$ from CS_4 theory it is not immediately clear how a world-sheet metric might arise. After all, CS_4 theory does not depend on the four-dimensional metric. More precisely, this is only true for manifolds without boundary or for everywhere regular field configurations. On the other hand, allowing singularities in the gauge field like those in Eq. (3) means that the CS_4 theory may no longer be invariant under general-coordinate transformations on the boundary or in regions near such singularities. If we separate the singular part of the gauge connection

$$A_w \equiv A_w^{(p)} + A_w^{(\text{reg})}, \quad A_{\bar{w}} \equiv A_{\bar{w}}^{(p)} + A_{\bar{w}}^{(\text{reg})}, \quad (10)$$

S_{CS_4} is not general-coordinate invariant

$$\delta_v S_{\text{CS}_4} \sim - \oint_{z=0,\infty} \omega_{\wedge} (A_w^{(p)} dv^w + A_{\bar{w}}^{(p)} dv^{\bar{w}})_{\wedge} A. \quad (11)$$

We used Green's theorem to rewrite $\delta_v S_{\text{CS}_4}$ as a contour integral around $z = 0, \infty$, where A_w and $A_{\bar{w}}$ have poles. For $S^{4,2}$ this integral reduces to

$$- \int_{\Sigma} (A_w^{(2)} A_w^{(2)} dw_{\wedge} dv^w + A_{\bar{w}}^{(2)} A_{\bar{w}}^{(2)} d\bar{w}_{\wedge} dv^{\bar{w}}). \quad (12)$$

Since this is nonzero, it appears that with our boundary conditions the CS_4 theory is no longer invariant under general-coordinate transformations.

The lack of diffeomorphism invariance (12) suggests we need to introduce a new field to restore it. We do this by varying the complex structure with the new field corresponding to the Beltrami differential β . Under such a variation, the de Rham differential d , which is the CS_4 theory's BRST operator, changes to

$$d \rightarrow d + d\bar{w}\mathcal{L}_{\beta_w\partial_w} + d\bar{z}\mathcal{L}_{\beta_z\partial_w}, \quad (13)$$

where \mathcal{L} is the Lie derivative. Since the Cartan homotopy formula gives the Lie derivative as

$$\mathcal{L}_V = [d, \iota_V], \quad (14)$$

where ι_V is the inner multiplication (contraction) by V , we are adding a BRST-exact term to the action, which should have no effect in the bulk. The $\beta_{\bar{w}}$ -dependent part of the action is

$$S_{\beta_{\bar{w}}} \equiv \frac{2}{h} \int_{\mathcal{V}} \omega_z \beta_{\bar{w}} A_w (\partial_z A_w - \partial_w A_z), \quad (15)$$

with a similar term for S_{β_z} . The combined Beltrami-Chern-Simons action

$$S_{\text{BCS}} \equiv S_{CS_4} + S_{\beta_{\bar{w}}} + S_{\beta_z} \quad (16)$$

is invariant, up to a w derivative, under a new gauge invariance

$$A \rightarrow A + \mathcal{L}_V A, \quad \beta \rightarrow \beta + \mathcal{L}_V \beta, \quad (17)$$

with gauge parameter $V \equiv v\partial_w$ for an arbitrary function v . This invariance can be used to gauge away one of the components of β , for example $\beta_z \rightarrow 0$. Working in this gauge, redefining $A_{\bar{w}}$

$$A_{\bar{w}} \rightarrow A_{\bar{w}} - \beta_{\bar{w}} A_w, \quad (18)$$

one recovers the original CS_4 action

$$S_{CS_4} + S_{\beta_{\bar{w}}} \rightarrow S_{CS_4}. \quad (19)$$

This is to be expected, since on a manifold with no boundary the CS_4 theory is metric independent and we are adding a BRST-exact Beltrami term to it. This should leave the theory unmodified, up to field redefinitions. However, in the presence of a boundary the field redefinition (18) might not be compatible with the boundary conditions. Indeed, $A_{\bar{w}}$ and A_w have poles of different order at $z = 0, \infty$, while A_z is regular. Since β should be regular on the boundary, this means that we cannot eliminate the Beltrami couplings on the boundary using field

redefinitions (18). With boundary conditions (6) and (3), the β -dependent part of the action reduces to a boundary contribution at $z = 0$

$$S_{\beta} = \frac{\delta_{z=0}}{h} \int_{\Sigma} 2\beta_{\bar{w}} A_w^{(2)} A_w^{(2)}. \quad (20)$$

This coupling restores diffeomorphism invariance at $z = 0$ and varying the action with respect to $\beta_{\bar{w}}$ leads to the Virasoro constraint

$$A_w^{(2)} A_w^{(2)} = 0. \quad (21)$$

We can introduce a similar modification to Eq. (13) along $\partial_{\bar{w}}$ and show that it is trivial up to a field redefinition away from $z = \infty$, leading to a boundary action

$$S_{\beta_w} = \frac{\delta_{z=\infty}}{h} \int_{\Sigma} 2\tilde{\beta}_w A_{\bar{w}}^{(2)} A_{\bar{w}}^{(2)}, \quad (22)$$

which restores diffeomorphism invariance at $z = \infty$.

In the Polyakov action, the world-sheet metric g is taken up to Weyl transformations. If we analytically continue this action to allow g to be a metric with complex coefficients, then the data of g , up to Weyl transformations, is equivalent to the data of a holomorphic Beltrami differential β and an antiholomorphic Beltrami differential $\bar{\beta}$ [24]. The reality condition corresponding to asking that g be a metric with real coefficients is that we ask β and $\bar{\beta}$ to be complex conjugate. For a discussion of such a factorization of the Polyakov action see [25].

Similarly, in the Polyakov action, the gauge symmetries (after gauging away Weyl transformations) are world-sheet diffeomorphisms. Infinitesimally these are sections of the tangent bundle $T\Sigma$. If we analytically continue the Polyakov action, allowing the infinitesimal world-sheet diffeomorphisms to be complex, we find the gauge transformations we used for the Beltrami differential fields. Indeed, $T\Sigma \otimes_{\mathbb{R}} \mathbb{C}$ decomposes as $T^{1,0}\Sigma$, which gives the gauge transformations for β , and $T^{0,1}\Sigma$, giving the gauge transformations for $\bar{\beta}$.

Gauge invariance in BCS theory.—Before proceeding to discuss κ symmetry in BCS theory, we briefly review how to modify gauge variations in the presence of a Beltrami deformation β_w in order for S_{BCS} to be gauge invariant. In components the CS Lagrangian is

$$\frac{1}{2} L_{\text{CS}} = A_{\bar{w}} (\partial_z A_w - \partial_w A_z) + A_z \partial_w A_{\bar{w}} - A_{\bar{w}} [A_w, A_z], \quad (23)$$

where as in the rest of the paper the trace is implicit. The $\beta_w \equiv \beta$ part of the Beltrami-deformed action is

$$S_{\beta} = -\frac{1}{h} \int_{\mathcal{V}} \omega_z (\partial_z \beta_{\bar{w}} A_w^2 + 2\beta_{\bar{w}} A_w \partial_w A_z). \quad (24)$$

S_{CS_4} is invariant under gauge transformations

$$\delta_\chi A_\mu = \partial_\mu \chi + [A_\mu, \chi], \quad (25)$$

but S_β is not

$$\delta_\chi S_\beta = 2 \int_{\mathcal{V}} \omega \beta_{\bar{w}} (\partial_{\bar{z}} A_w - \partial_w A_{\bar{z}} - [A_w, A_{\bar{z}}]) \partial_w \chi. \quad (26)$$

To cancel this we modify the gauge variation of $A_{\bar{w}}$ in accordance with Eq. (13) to

$$\delta_{\chi, \beta} A_{\bar{w}} = \partial_{\bar{w}} \chi + [A_{\bar{w}}, \chi] - \beta_{\bar{w}} \partial_w \chi, \quad (27)$$

while leaving the gauge variation of the other components of A unchanged. Since S_β does not depend on $A_{\bar{w}}$ its variation (26) is unchanged, while the gauge variation of S_{CS_4} becomes

$$\delta_{\chi, \beta} S_{\text{CS}_4} = -\delta_\chi S_\beta, \quad (28)$$

making S_{BCS} gauge invariant.

κ symmetry.—We now show that the action S_{BCS} is invariant under certain singular G -gauge transformations, which reduce to κ symmetry in the σ model. To this end, consider gauge variations

$$\delta_\xi A = d\xi + [A, \xi] \quad (29)$$

which have a simple pole near $z = 0$

$$\xi \sim \frac{1}{z} \xi^{(3)} + \dots \quad (30)$$

Since the generator of the \mathbb{Z}_4 automorphism multiplies z by $-i$, we immediately see that the singular gauge variation $\xi^{(3)}$ must be in the i^3 eigenspace of \mathbb{Z}_4 and hence is fermionic, as expected of a κ variation. The variation of S_{CS_4} is

$$\delta_\xi S_{\text{CS}_4} = \frac{1}{h} \int_{\mathcal{V}} \omega (A_{\bar{w}} d_{\bar{z}} [A_w, \xi] - A_w d_{\bar{z}} [A_{\bar{w}}, \xi]). \quad (31)$$

Near $z = 0$ the gauge fields have an expansion

$$A_{\bar{w}} \sim \frac{A_{\bar{w}}^{(3)}}{z} + \dots, \quad A_w \sim \frac{A_w^{(2)}}{z^2} + \frac{A_w^{(3)}}{z} + \dots \quad (32)$$

Inserting these into (31) we get [26]

$$\delta_\xi S_{\text{CS}_4} = \frac{\delta_{z=0}}{h} \int_{\Sigma} [A_{\bar{w}}^{(3)}, A_w^{(2)}] \xi^{(3)}. \quad (33)$$

Analogously, for gauge variations with a simple pole at $z = \infty$, we have $\xi \sim z \xi^{(1)} + \dots$, and the CS_4 action changes by

$$\delta_\xi S_{\text{CS}_4} = \frac{\delta_{z=\infty}}{h} \int_{\Sigma} [A_w^{(1)}, A_{\bar{w}}^{(2)}] \xi^{(1)}. \quad (34)$$

κ transformations can be obtained from the singular gauge transformations by requiring [18]

$$\begin{aligned} \bar{\xi}^{(1)} &\equiv A_{\bar{w}}^{(2)} \kappa_w^{(1)} + \kappa_w^{(1)} A_{\bar{w}}^{(2)}, \\ \xi^{(3)} &\equiv A_w^{(2)} \kappa_{\bar{w}}^{(3)} + \kappa_{\bar{w}}^{(3)} A_w^{(2)}, \end{aligned} \quad (35)$$

where the κ are the independent (local) parameters of κ transformations. Notice that the above expression involves selecting a particular (matrix) representation for the gauge group, and using matrix multiplication in that representation. In judiciously chosen cases, there are certain famous Fierz identities [18,27,28] [see Eq. (1.80) of [18]] that can be used to reexpress the κ variation of S_{CS_4} as

$$\begin{aligned} \delta_\kappa S_{\text{CS}_4} &= -\frac{\delta_{z=0}}{2h} \int_{\Sigma} \text{tr}(A_w^{(2)} A_w^{(2)}) \text{tr}(\Upsilon[\kappa_{\bar{w}}^{(3)}, A_{\bar{w}}^{(3)}]) \\ &\quad - \frac{\delta_{z=\infty}}{2h} \int_{\Sigma} \text{tr}(A_{\bar{w}}^{(2)} A_{\bar{w}}^{(2)}) \text{tr}(\Upsilon[\kappa_w^{(1)}, A_w^{(1)}]). \end{aligned} \quad (36)$$

Above, Υ is a suitable constant matrix which for $\mathfrak{psu}(2, 2|4)$ is $\text{diag}(1_4, -1_4)$. The lack of gauge invariance under (30) can be compensated by varying the Beltrami operators under κ transformations. Working in the gauge $\beta_{\bar{z}} = 0$, we demand

$$\delta_\kappa \beta_{\bar{w}} = \frac{\delta_{|z| \leq \varepsilon}}{2} \text{tr}(\Upsilon[\kappa_{\bar{w}}^{(3)}, A_{\bar{w}}^{(3)}]), \quad (37)$$

where $\delta_{|z| \leq \varepsilon}$ has support in an ε neighborhood of $z = 0$ only. Now, $S_{\beta_{\bar{w}}}$ is no longer gauge invariant near $z = 0$. Expanding as in Eq. (32), we find

$$\begin{aligned} \delta_\kappa S_\beta &= \frac{1}{2h} \int_{\mathcal{V}_4} \frac{\partial_{\bar{z}} \delta_{|z| \leq \varepsilon}}{z} \text{tr}(\Upsilon[\kappa_{\bar{w}}^{(3)}, A_{\bar{w}}^{(3)}]) \text{tr}(A_w^{(2)} A_w^{(2)}) \\ &= \frac{1}{2h} \oint_{z=0} \frac{dz}{z} \int_{\Sigma} \text{tr}(\Upsilon[\kappa_{\bar{w}}^{(3)}, A_{\bar{w}}^{(3)}]) \text{tr}(A_w^{(2)} A_w^{(2)}) \end{aligned} \quad (38)$$

using the identity $\partial_{\bar{z}} \delta_{|z| \leq \varepsilon} = \delta_{|z| = \varepsilon}$. This cancels the gauge noninvariance of S_{CS_4} at $z = 0$ in Eq. (36). The $z = \infty$ term can be analogously canceled by a $\tilde{\beta}_{\bar{w}}$ variation. The separate cancelations at $z = 0, \infty$, which on the world sheet correspond to self-dual- and anti-self-dual-vector representations or after Wick rotation holomorphic and antiholomorphic ones, provide a novel separation of the two sectors in four dimensions.

Conclusions.—In this Letter, we have introduced BCS theory and showed that, upon imposing suitable singular boundary conditions, it reduces to the MT superstring. The Beltrami fields and boundary gauge connection of BCS theory map to the world-sheet metric and sigma-model

fields, respectively. The Lax connection and spectral parameter appear somewhat mysteriously in string theory, but from the four-dimensional BCS point of view they are simply the gauge connection and holomorphic coordinate of the reduction. In BCS theory κ symmetry corresponds to a certain class of singular gauge transformations and κ invariance holds for gauge groups for which a suitable hypercharge matrix Υ exists. This includes $G = \mathfrak{psu}(2, 2|4)$ and its plane-wave [29] and flat-space [10] limits. We will investigate the properties of these backgrounds from the four-dimensional point of view more fully in a forthcoming paper [30].

There are other well-known integrable superstring backgrounds with a Lax connection [31–37]. In these cases, the MT coset action often needs to be supplemented by extra fermionic degrees of freedom [38] to obtain an action equivalent to the conventional κ -symmetric superstring actions [39,40] and it would be interesting to see how to extend these to BCS theory. Some of these backgrounds have target-space moduli and understanding how these appear in BCS theory could provide new insights into moduli spaces.

It would be interesting to perform a Batalin-Vilkovisky quantization of BCS theory. This introduces a tower of extra fields including conventional $b - c$ ghosts of string theory. If the \mathbb{R}^2 with coordinates w, \bar{w} is replaced by a Riemann surface Σ , then the formalism we have described includes the integral over the moduli of the world-sheet Σ . Indeed, the Beltrami differential β on Σ has zero modes which live in the Dolbeault cohomology group $H^1(\Sigma, T\Sigma)$, which is of (complex) dimension $3g - 3$ for $g > 1$ [41]. The antiholomorphic Beltrami differential has zero modes which live in the complex conjugate of this space. Together, the manifold of zero modes is the product of the moduli space with its complex conjugate: $\mathcal{M}_g \times \bar{\mathcal{M}}_g$. As we are doing an analytically continued path integral, we need to choose an integration contour. It is natural to choose the contour to be the locus where the holomorphic and antiholomorphic Beltrami differential are complex conjugate, leading to an integral over one copy of the moduli space \mathcal{M}_g . We will return to a detailed discussion of this and its relation to the Polyakov path integral over Riemann surfaces [42] in a future paper [30].

We hope our construction can shed light on the relationship between the pure-spinor and Green-Schwarz formulations of string theory. Quantizing BCS theory should also offer new insights into quantum integrability of holographic string backgrounds and connect with the quantum spectral curve approach [43,44]. In particular, we expect that compactifying the string world sheet $\Sigma = \mathbb{R} \times S^1$ will lead to diagrams involving the photon propagator “wrapping” the S^1 direction and the \mathbb{Q} functions should appear as solutions of the Baxter equation involving the transfer matrix of BCS theory.

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