Bosonic Quantum Communication across Arbitrarily High Loss Channels

Ludovico Lami,[1,*] Martin B. Plenio,[1,†] Vittorio Giovannetti,[2,‡] and Alexander S. Holevo[3,§]
[1]*Institut für Theoretische Physik und IQST, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany*
[2]*NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, I-56127 Pisa, Italy*
[3]*Steklov Mathematical Institute, Gubkina 8, 119991 Moscow, Russia*

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A general attenuator $\Phi_{\lambda,\sigma}$ is a bosonic quantum channel that acts by combining the input with a fixed environment state σ in a beam splitter of transmissivity λ. If σ is a thermal state, the resulting channel is a thermal attenuator, whose quantum capacity vanishes for $\lambda \leq 1/2$. We study the quantum capacity of these objects for generic σ, proving a number of unexpected results. Most notably, we show that for any arbitrary value of $\lambda > 0$ there exists a suitable single-mode state $\sigma(\lambda)$ such that the quantum capacity of $\Phi_{\lambda,\sigma(\lambda)}$ is larger than a universal constant $c > 0$. Our result holds even when we fix an energy constraint at the input of the channel, and implies that quantum communication at a constant rate is possible even in the limit of arbitrarily low transmissivity, provided that the environment state is appropriately controlled. We also find examples of states σ such that the quantum capacity of $\Phi_{\lambda,\sigma}$ is not monotonic in λ. These findings may have implications for the study of communication lines running across integrated optical circuits, of which general attenuators provide natural models.

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Introduction.—Quantum optics will likely play a major role in the future of quantum communication [1–4]. Indeed, practically all quantum communication in the foreseeable future will rely on optical platforms. For this reason, the study of quantum channels acting on continuous variable (CV) systems, that is, finite ensembles of electromagnetic modes, is a core area of the rapidly developing field of quantum information [5–7].

In the best studied models of optical communication, one represents an optical fiber as a memoryless thermal attenuator channel. Mathematically, its action can be thought of as that of a beam splitter with a certain transmissivity $0 \leq \lambda \leq 1$, where the input state is mixed with a fixed environment state σ that is assumed to be thermal. This approximation is well justified when the signal rate is sufficiently low that memory effects are negligible, and when the optical fiber is so long that the "effective" environment state, resulting from averaging several elementary interactions that are effectively independent, due to the limited correlation length of the environment, is practically Gaussian and thermal, as follows from the quantum central limit theorem [8,9]. Indeed, an impressive amount of literature has been devoted to finding bounds on the quantum capacity of the thermal attenuator. We now have exact formulas for the zero-temperature case [10–15], and tight upper [15–18] and lower [10,19] bounds in all other cases.

However, the thermal noise approximation is challenged when memory effects become important [20], or when the communication channel is so short that the averaging process cannot possibly take place, as may happen, e.g., in miniaturized quantum optical circuits [21–24]. In both cases, it is conceivable that the environment state may be manipulated and engineered to facilitate communication. Namely, one could exploit memory effects to send pulses that alter it and precede the actual transmission, or one could design the integrated optical circuit that surrounds the communication line in order to control the noise that comes from other elements of the same circuit. Thus, we are led to investigate general attenuator channels, hereafter denoted with $\Phi_{\lambda,\sigma}$, where the environment state σ is no longer thermal. Unsurprisingly, such models have received increasing attention recently [9,25–29]. As discussed above, we will be interested in optimizing over the environment state so as to increase the capacity [30,31].

Other motivations for considering general attenuators stem, on the one hand, from the need to go beyond the Gaussian formalism to accomplish several tasks that are critical to quantum information, e.g., universal quantum computation [32,33], entanglement distillation [34–36], entanglement swapping [37,38], error correction [39], and state transformations in general resource theories [40,41]. On the other hand, general attenuators are among the simplest examples of non-Gaussian channels that are, nevertheless, Gaussian dilatable, meaning that they can be Stinespring dilated [42] by means of a symplectic unitary [27,28]. This makes them amenable to a quantitative analysis in many respects. For example, it has been shown that making the environment state non-Gaussian, e.g., by means of a photon addition, can be advantageous when transmitting quantum or private information [27]. In spite of their increased complexity compared to Gaussian channels, the entanglement-assisted capacity of a general attenuator can, nevertheless, be upper bounded thanks to the conditional entropy power inequality [25,26]. Similar bounds can be obtained for the quantum [29] and private [43] capacity as well, by making use of the solution to the minimum output entropy conjecture [44–46] combined with known extremality properties of Gaussian states [47,48]. Finally, we have mentioned that, by concatenating a large number n of general attenuators with a fixed total transmissivity, one typically obtains an effective channel that resembles a thermal attenuator. In this regime of large but finite n, the associated quantum capacity can be bounded thanks to the quantum Berry-Esseen inequality [[9] Corollary 13].

Here, we investigate the quantum capacity of general attenuators $\Phi_{\lambda,\sigma}$, uncovering some unexpected phenomena. It has been observed [9, Lemma 16] that output states of general attenuators with transmissivity $\lambda = 1/2$ have nonnegative Wigner functions [49,50]. At first sight, this may suggest that such channels are somewhat "classical" [51–53]. Indeed, we show that, for all convex combinations of symmetric states—and in particular, for all Gaussian states—$\Phi_{1/2,\sigma}$ is antidegradable, and therefore, its quantum capacity satisfies $Q(\Phi_{1/2,\sigma}) = 0$ [54]. Here, we call a state symmetric if it remains invariant under phase space inversion up to displacements. However, we also find an example of a state σ that does not belong to this class and that makes $Q(\Phi_{1/2,\sigma}) > 0$.

Next, we tackle the question of whether transmission of quantum information is possible even for very low values of the transmissivity $0 < \lambda \ll 1$. Intuitively, a beam splitter of transmissivity $\lambda \le 1/2$ should give away to the environment more than it transmits. By the no-cloning theorem, we could be led to conjecture that the quantum capacity $Q(\Phi_{\lambda,\sigma})$ vanishes for all σ as soon as $\lambda \le 1/2$. Indeed, this is exactly what happens for thermal attenuators. This intuition is further supported by the analysis of general finite-dimensional depolarizing channels $\Delta_{\lambda,\sigma}(\rho)$, defined by $\Delta_{\lambda,\sigma}(\rho) \coloneqq \lambda\rho + (1-\lambda)\sigma$, whose quantum capacity also vanishes for $\lambda \le 1/2$.

However, we establish the following surprising result: for all values of $\lambda > 0$, one can find suitable states $\sigma(\lambda)$ that make $Q(\Phi_{\lambda,\sigma(\lambda)}) \ge c$, where the constant $c > 0$ is universal (Theorem 2). This implies, but is stronger than, the fact that $\Phi_{\lambda,\sigma(\lambda)}$ can be used to distribute entanglement [55]. As a corollary, we also see that $Q(\Phi_{\lambda,\sigma})$ is, in general, not monotonic in λ for fixed σ. All this marks a striking difference with the aforementioned behavior of thermal attenuators and depolarizing channels and reveals that the phenomenology of general attenuators is richer than perhaps expected. Our proof is fully analytical, and goes by analyzing the single-copy coherent information associated with a specific transmission scheme. By a tour de force of inequalities, we show that the output state of the channel is majorized by that of the associated complementary channel. In turn, this makes it possible to lower bound the coherent information by applying a beautiful inequality recently proved by Ho and Verdú [67].

Notation.—The Hilbert space corresponding to an m-mode CV comprises all square-integrable functions $\mathbb{R}^m \to \mathbb{C}$, and is denoted by $\mathcal{H}_m \coloneqq L^2(\mathbb{R}^m)$. Quantum states are represented by density operators on $\mathcal{H}_m$, i.e., positive semidefinite trace class operators with unit trace. We will denote with $a_j, a_j^\dagger$, respectively, the annihilation and creation operators corresponding to the jth mode, and with $|0\rangle$ the vacuum state. The canonical commutation relations read $[a_j, a_k^\dagger] = \delta_{jk} I,\ [a_j, a_k] = 0$. The unitary displacement operators on $\mathcal{H}_m$ are constructed as $D(\alpha) \coloneqq e^{\sum_j (\alpha_j a_j^\dagger - \alpha_j^* a_j)}$, where $\alpha \in \mathbb{C}^m$; they satisfy $D(\alpha)D(\beta) = e^{\frac{1}{2}(\alpha^\intercal \beta^* - \alpha^\dagger \beta)} D(\alpha+\beta)$ for all $\alpha, \beta \in \mathbb{C}^m$.

For every trace class operator T on $\mathcal{H}_m$, its characteristic function $\chi_T : \mathbb{C}^m \to \mathbb{C}$ is defined by [6,68]

$$\chi_T(\alpha) \coloneqq \mathrm{Tr}[T D(\alpha)]. \tag{1}$$

The Wigner function W_T of T is the Fourier transform of χ_T [6,49,50,68]. Note that W_ρ is typically not pointwise positive for a generic quantum state ρ [51–53].

A beam splitter of transmissivity $0 \le \lambda \le 1$ acting on two systems of m modes each is represented by the unitary operator

$$U_\lambda \coloneqq e^{\arccos\sqrt{\lambda} \sum_j (a_j^\dagger b_j - a_j b_j^\dagger)}, \tag{2}$$

where a_j, b_j are the annihilation operators on the jth modes of the first and second system, respectively. Our main object of study is the general attenuator channel $\Phi_{\lambda,\sigma}$, which acts on an m-mode system B as

$$\Phi^B_{\lambda,\sigma}(\rho_B) \coloneqq \mathrm{Tr}_E\left[U_\lambda^{BE}(\rho_B \otimes \sigma_E)(U_\lambda^{BE})^\dagger\right]. \tag{3}$$

Dropping the system labels for simplicity, this can be cast in the language of characteristic functions as

$$\chi_{\Phi_{\lambda,\sigma}(\rho)}(\alpha) = \chi_\rho\left(\sqrt{\lambda}\alpha\right)\chi_\sigma\left(\sqrt{1-\lambda}\alpha\right). \tag{4}$$

A pictorial representation of the action of a general attenuator is provided in Fig. 1. The thermal attenuators

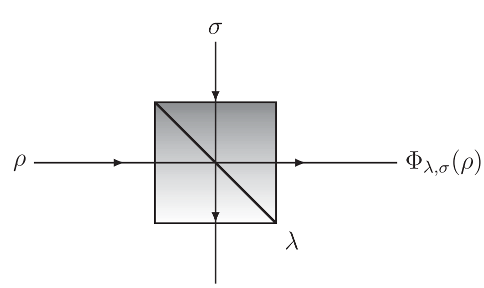

FIG. 1. A general attenuator acts by mixing the input state ρ in a beam splitter of transmissivity λ with an environment in a fixed state σ.

$\mathcal{E}_{\lambda,\nu} \coloneqq \Phi_{\lambda,\tau_\nu}$ as well as the pure loss channels $\mathcal{E}_\lambda \coloneqq \mathcal{E}_{\lambda,0} = \Phi_{\lambda,|0\rangle\langle 0|}$ are standard examples of single-mode attenuators, obtained by taking the environment to be in a thermal state $\tau_\nu \coloneqq (1/(\nu+1)) \sum_{n=0}^\infty (\nu/(\nu+1))^n |n\rangle\langle n|$, where $|n\rangle$ is the nth Fock state.

Quantum channels are useful because they can transmit quantum information. The maximum rate at which independent copies of a channel Φ acting on a system B can simulate instances of the noiseless qubit channel I_2 is called the quantum capacity of Φ, and denoted with $Q(\Phi)$. For CV systems, physical transmission of quantum data must be subjected to an energy constraint. We shall assume that the relevant Hamiltonian is the total photon number: for an m-mode system, $H_m \coloneqq \sum_{j=1}^m a_j^\dagger a_j$. The energy-constrained quantum capacity can be obtained thanks to the following modified version [14, Theorem 5] of the Lloyd-Shor-Devetak theorem [69–72]:

$$Q(\Phi, N) = \sup_k \frac{1}{k} Q_1(\Phi^{\otimes k}, kN), \tag{5}$$

$$Q_1(\Phi, N) \coloneqq \sup_{\mathrm{Tr}[\Psi_B H_B] \le N} I_{\mathrm{coh}}(A\rangle B)_{(I_A \otimes \Phi_B)(\Psi_{AB})}. \tag{6}$$

where $\Psi_{AB} \coloneqq |\Psi\rangle\langle\Psi|_{AB}$ is pure, and $I_{\mathrm{coh}}(A\rangle B)_\rho \coloneqq \mathrm{Tr}[\rho_{AB}(\log_2 \rho_{AB} - \log_2 \rho_B)]$ is the coherent information. The unconstrained quantum capacity is obtained as $Q(\Phi) \coloneqq \lim_{N\to\infty} Q(\Phi, N)$. In general, the expression in (5) is intractable. However, for the pure loss channel, the regularization is not needed, and the quantum capacity can be expressed in closed form as [10,12–15]

$$Q(\mathcal{E}_\lambda, N) = \max\{g(\lambda N) - g((1-\lambda)N), 0\}, \tag{7}$$

where $g(x) \coloneqq (x+1)\log_2(x+1) - x\log_2 x$ is the bosonic entropy. No such formula is known for the thermal attenuators, although sharp bounds are available [10,15–19].

Results.—Before expounding our findings, let us forge our intuition by looking at other channels that present some analogies with general attenuators. An obvious starting point is the thermal attenuator $\mathcal{E}_{\lambda,\nu} = \Phi_{\lambda,\tau_\nu}$. When $\lambda \le 1/2$, $\mathcal{E}_{\lambda,\nu}$ is antidegradable, meaning that tracing out B instead of E in (3) results in a channel that can simulate $\mathcal{E}_{\lambda,\nu}$ via postprocessing [11,54,73]. This implies that $Q(\mathcal{E}_{\lambda,\nu}) = 0$ for $\lambda \le 1/2$ [11 p. 3]. On a different note, we can also consider a generalized depolarizing channel in finite dimension d, acting as $\rho \mapsto \Delta_{\lambda,\sigma}(\rho) = \lambda\rho + (1-\lambda)\sigma$. As it turns out, its quantum capacity is again zero for $\lambda \le 1/2$. In fact, $\Delta_{\lambda,\sigma}$ can be obtained from an erasure channel [74] via postprocessing. Since the quantum capacity of this latter object is known [75], by data processing, we obtain that $Q(\Delta_{\lambda,\sigma}) \le \max\{(1-2\lambda)\log_2 d, 0\}$ for all σ. In particular, $Q(\Delta_{\lambda,\sigma}) = 0$ for $\lambda \le 1/2$.

Our results show that the phenomenology of general attenuators is way richer than these considerations may have suggested. We start by looking at the role of the special point $\lambda = 1/2$.

Theorem 1: Let σ be an m-mode state of the form $\sigma = \int d\mu(\alpha) D(\alpha)\sigma_0(\alpha)D(\alpha)^\dagger$, where $\alpha \in \mathbb{C}^m$, μ is a probability measure on $\mathbb{C}^m$, and the states $\sigma_0(\alpha) = V\sigma_0(\alpha)V^\dagger$ are symmetric under the phase space inversion operation $V \coloneqq (-1)^{H_m}$, with H_m being the total photon number. Then, the channel $\Phi_{1/2,\sigma}$ is antidegradable [54], and, in particular, $Q(\Phi_{1/2,\sigma}) = 0$.

Proof of Theorem 1.—Under our assumptions, it holds that $\Phi_{1/2,\sigma} = \int d\mu(\alpha) \Phi_{1/2, D(\alpha)\sigma_0(\alpha)D(\alpha)^\dagger}$. Now, since the set of antidegradable channels is convex [76, Appendix A 2], we can directly assume that μ is a Dirac measure, i.e., $\sigma = D(\alpha)\sigma_0 D(\alpha)^\dagger$ with σ_0 symmetric under phase space inversion. Acting on $\rho \otimes \sigma$ with the beam splitter unitary U_λ yields a global state with characteristic function

$$\chi_\rho(\sqrt{\lambda}\alpha - \sqrt{1-\lambda}\beta)\chi_\sigma(\sqrt{1-\lambda}\alpha + \sqrt{\lambda}\beta).$$

While the reduced state on the first system is given by (4), that on the second system has characteristic function $\chi_\rho(-\sqrt{1-\lambda}\beta)\chi_\sigma(\sqrt{\lambda}\beta)$, which coincides with that of $V\Phi_{1-\lambda, V\sigma V^\dagger}(\rho)V^\dagger$. Therefore, the weak complementary channel associated to $\Phi_{\lambda,\sigma}$ via the representation (3) can be expressed as

$$\Phi^{\mathrm{wc}}_{\lambda,\sigma} = \mathcal{V} \circ \Phi_{1-\lambda, \mathcal{V}(\sigma)},$$

where $\mathcal{V}(\cdot) \coloneqq V(\cdot)V^\dagger$.

Using the identity $VD(\alpha)V^\dagger = D(-\alpha)$, we see that, when $\sigma = D(\alpha)\sigma_0 D(\alpha)^\dagger$, we also have that $\mathcal{V}(\sigma) = \mathcal{D}_{-2\alpha}(\sigma)$, where $\mathcal{D}_z(\cdot) \coloneqq D(z)(\cdot)D(z)^\dagger$. Noting that $\Phi_{1-\lambda, \mathcal{D}_z(\sigma)} = \mathcal{D}_{\sqrt{\lambda}z} \circ \Phi_{1-\lambda,\sigma}$, we finally obtain that

$$\Phi^{\mathrm{wc}}_{\lambda,\sigma} = \mathcal{V} \circ \mathcal{D}_{-2\sqrt{\lambda}\alpha} \circ \Phi_{1-\lambda,\sigma}.$$

Thus, if $\lambda = 1/2$, the channel is equivalent to its weak complementary up to a unitary postprocessing. ■

The class of states σ to which Theorem 1 applies is invariant under symplectic unitaries and displacement operators, and it includes many states that are relevant for applications, for instance, all convex combinations of Gaussian states (e.g., classical states [77,78]) and all Fock-diagonal states. Remarkably, the above result no longer holds if we weaken the assumption on σ. To see this, for $0 \le \eta \le 1$, consider the family of single-mode states $\xi(\eta) = |\xi(\eta)\rangle\langle\xi(\eta)|$, with $|\xi(\eta)\rangle \coloneqq \sqrt{\eta}|0\rangle - \sqrt{1-\eta}|1\rangle$. A lower bound on the energy-constrained quantum capacity of the channels $\Phi_{1/2,\xi(\eta)}$ can be obtained by setting $|\Psi(\eta)_{AB}\rangle \coloneqq \sqrt{\eta(1-\eta)}|00\rangle + (1-\eta)|01\rangle + \sqrt{\eta}|10\rangle$ and by considering that [55]

$$Q(\Phi_{1/2,\xi(\eta)}, (1-\eta)^2) \ge I_{\mathrm{coh}}(A\rangle B)_{\zeta_{AB}(\eta)}, \tag{8}$$

where $\zeta_{AB}(1/2,\eta) := (I^A \otimes \Phi^B_{1/2,\xi(\eta)})(\Psi_{AB}(\eta))$, and $\Psi(\eta) := |\Psi(\eta)\rangle\langle\Psi(\eta)|$. The function on the rhs of (8) is strictly positive for all $0 < \eta < 1$ [55].

The above example shows that quantum communication can be possible on a general attenuator even for transmissivity $\lambda = 1/2$. At this point, we may wonder whether, at least for a fixed energy constraint at the input, there exists a threshold value for λ below which quantum communication becomes impossible. Our main result states that this is not the case; on the contrary, the quantum capacity can be bounded away from 0 even when λ approaches 0, if the environment state σ is chosen appropriately. Note that the bounds by Lim *et al.* [29] cannot possibly be used to draw such a conclusion [55].

Theorem 2: For all $0 < \lambda \leq 1$, there exists a single-mode (pure) state $\sigma(\lambda)$ such that

$$Q(\Phi_{\lambda,\sigma(\lambda)}) \geq Q(\Phi_{\lambda,\sigma(\lambda)}, 1/2) \geq c, \tag{9}$$

for some universal constant $c > 0$. Depending on λ, we can take $\sigma(\lambda)$ to be either the vacuum $|0\rangle$, or a superposition $\alpha|0\rangle + \beta|1\rangle$, or a Fock state $|n\rangle$ with $n \geq 2$.

Proof of Theorem 2.—Sketch of the proof. When $1/2 < \lambda \leq 1$, it suffices to set $\sigma(\lambda) = |0\rangle\langle 0|$ and leverage (7). Around $\lambda = 1/2$, positive quantum capacity follows by perturbing the lower bound in (8) thanks to the Alicki-Fannes-Winter inequality [79,80]. It remains to establish the result for $0 < \lambda \leq 1/2 - \epsilon$, where $\epsilon > 0$ is fixed. We start by making an ansatz for a state $|\Psi\rangle_{AB}$ to be plugged into (6). Let us set $|\Psi\rangle_{AB} := (1/\sqrt{2})(|01\rangle + |10\rangle)$ and $\sigma(n) := |n\rangle\langle n|$. The output state $\omega_{AB}(n,\lambda) := (I^A \otimes \Phi^B_{\lambda,\sigma(n)})(\Psi_{AB})$ can be computed, e.g., thanks to the formulas derived by Sabapathy and Winter [27, Sec. III. B]. One obtains that

$$Q(\Phi_{\lambda,\sigma(n)}, 1/2) \geq \mathcal{I}(n,\lambda) := I_{\mathrm{coh}}(A\rangle B)_{\omega_{AB}(n,\lambda)}$$
$$= H(p(n,\lambda)) - H(q(n,\lambda)),$$

where the two probability distributions $p(n,\lambda)$ and $q(n,\lambda)$ over the alphabet $\{0,\ldots,n+1\}$ are defined by

$$p_\ell(n,\lambda) := \frac{1}{2(n+1)(1-\lambda)}\binom{n+1}{\ell}(1-\lambda)^\ell\lambda^{n-\ell} \times \{(1-\lambda)(n-\ell+1) + [(n+1)(1-\lambda)-\ell]^2\},$$

$$q_\ell(n,\lambda) := \frac{1}{2(n+1)(1-\lambda)}\binom{n+1}{\ell}(1-\lambda)^\ell\lambda^{n-\ell} \times \{\lambda\ell + [(n+1)(1-\lambda)-\ell]^2\}.$$

In Fig. 2, we plotted $\mathcal{I}(n,\lambda)$ as a function of λ for increasing values of n. The lower endpoint of the range for which $\mathcal{I}(n,\lambda) \geq c$ for some fixed $c > 0$ seems to move closer and closer to 0 as n grows. However, an analytical proof of this fact is technically challenging. The crux of our argument is to show that $p(n,\lambda)$ and $q(n,\lambda)$ are in a majorization relation, that is, $p(n,\lambda) \prec q(n,\lambda)$ for all $n \geq 2$ and all $1/(n+1) \leq \lambda \leq 1/n$. Given two probability distributions r and s over the same alphabet $\{0,\ldots,N\}$, we say that r is majorized by s, and we write $r \prec s$, if $\sum_{\ell=0}^k r_\ell^\uparrow \geq \sum_{\ell=0}^k s_\ell^\uparrow$ holds for all $k = 0,\ldots,N$, where $r^\uparrow$ and $s^\uparrow$ are obtained by sorting r and s in ascending order [81]. This definition captures the intuitive notion of r being "more disordered" than s. An immediate consequence is that the entropy of r is never smaller than that of s. But more is true: a beautiful inequality recently established by Ho and Verdú [67, Theorem 3] allows us to lower bound the entropy difference as

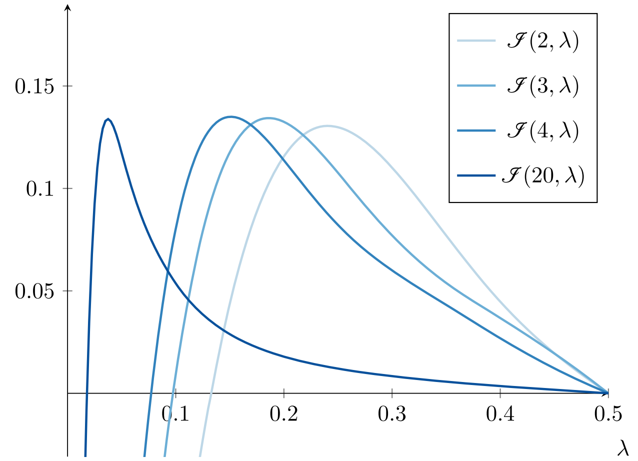

FIG. 2. The functions $\mathcal{I}(n,\lambda)$ plotted with respect to the variable λ for several values of n.

$$H(s) - H(r) \geq D(s^\uparrow \| r^\uparrow), \tag{10}$$

where $D(u\|v) := \sum_\ell u_\ell \log_2(u_\ell/v_\ell)$ is the Kullback-Leibler divergence. This latter quantity can be, in turn, lower bounded as $D(u\|v) \geq \|u-v\|_1^2/(2\ln 2)$ in terms of the total variation distance $\|u-v\|_1 := \sum_\ell |u_\ell - v_\ell|$ thanks to Pinsker's inequality [82]. We find that

$$\begin{aligned}
\mathcal{I}(n,\lambda) &= H(p(n,\lambda)) - H(q(n,\lambda)) \\
&\geq D(q^\uparrow(n,\lambda)\|p^\uparrow(n,\lambda)) \\
&\geq \frac{1}{2\ln 2}\|q^\uparrow(n,\lambda) - p^\uparrow(n,\lambda)\|_1^2 \\
&\geq \frac{2}{\ln 2}|q^\uparrow_{n+1}(n,\lambda) - p^\uparrow_{n+1}(n,\lambda)|^2 \\
&= \frac{2}{\ln 2}|p_{n-1}(n,\lambda) - q_{n+1}(n,\lambda)|^2,
\end{aligned}$$

where, in the last line, we used the fact, proven in the Supplemental Material (SM) [55], that $p_{n-1}(n,\lambda) = \max_\ell p_\ell(n,\lambda)$ and $q_{n+1}(n,\lambda) = \max_\ell q_\ell(n,\lambda)$ for all

$n \geq 2$ and $1/(n+1) \leq \lambda \leq 1/n$. It remains to lower bound $k(n, \lambda) \coloneqq |p_{n-1}(n, \lambda) - q_{n+1}(n, \lambda)|$, which is done by inspection. We find that (a) $k(2, \lambda) \geq \epsilon/4$ for all $1/3 \leq \lambda \leq 1/2 - \epsilon$; and (b) $k(n, \lambda) \geq c$ for some universal constant $c > 0$ for all $n \geq 3$ and $1/(n+1) \leq \lambda \leq 1/n$, concluding the proof. ■

Note that $Q(\Phi_{1/2,|n\rangle\langle n|}) \equiv 0$ for all n by Theorem 1, while we have just shown that $Q(\Phi_{\lambda,|n\rangle\langle n|}) > 0$ when $1/(n+1) \leq \lambda \leq 1/n$. This illustrates the rather surprising fact that $Q(\Phi_{\lambda,\sigma})$ can happen not to be monotonic in λ for a fixed σ. In the SM [55], we prove that monotonicity still holds under certain circumstances, e.g., when $\sigma = \sigma_G$ is Gaussian. Combining this with Theorem 1 also shows that $Q(\Phi_{\lambda,\sigma_G}) \equiv 0$ for all $\lambda \leq 1/2$ and all Gaussian σ_G.

From the proof, we see that, while the energy of the input of the channel in Theorem 2 is fixed, that of the environment state diverges as λ approaches 0. Intuitively, this may be due to the need for the receiver to distinguish the faint low-energy signals, which requires environmental states with highly oscillatory phase space structures and, thus, high energy. Whether this reasoning can be made rigorous is left as an open problem.

Now, we look at the optimal value of the constant c in (9). Our argument yields $c \geq 5.133 \times 10^{-6}$, while numerical investigations suggest that $c \gtrsim 0.066$. If only sufficiently small values of λ are taken into account, we can prove that $c \geq 0.0244$. To put this into perspective, elementary considerations show that $c \leq 1.377$ [55].

Conclusions.—We have studied the transmission of quantum information on general attenuator channels, which are among the simplest examples of non-Gaussian channels and may be relevant for applications. We have shown that their quantum capacity vanishes for transmissivity $1/2$ and for a wide class of environment states. At the same time, we have uncovered an unexpected phenomenon: namely, for any nonzero value of the transmissivity, there exists an environment state that makes the quantum capacity of the corresponding general attenuator larger than a universal constant. This also implies that said quantum capacity is not necessarily monotonically increasing in the transmissivity for a fixed environment state.

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[*]ludovico.lami@gmail.com
[†]martin.plenio@uni-ulm.de
[‡]vittorio.giovannetti@sns.it
[§]holevo@mi-ras.ru

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