Singular Measures and Information Capacity of Turbulent Cascades

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How weak is the weak turbulence? Here, we analyze turbulence of weakly interacting waves using the tools of information theory. It offers a unique perspective for comparing thermal equilibrium and turbulence. The mutual information between modes is stationary and small in thermal equilibrium, yet it is shown here to grow with time for weak turbulence in a finite box. We trace this growth to the concentration of probability on the resonance surfaces, which can go all the way to a singular measure. The surprising conclusion is that no matter how small is the nonlinearity and how close to Gaussian is the statistics of any single amplitude, a stationary phase-space measure is far from Gaussian, as manifested by a large relative entropy. This is a rare piece of good news for turbulence modeling: the resolved scales carry significant information about the unresolved scales. The mutual information between large and small scales is the information capacity of turbulent cascade, setting the limit on the representation of subgrid scales in turbulence modeling.

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There are two quite different perspectives to look at the evolution of a statistical system: fluid mechanics and information theory. The first one is the continuum viewpoint, where a Hamiltonian evolution of an ensemble is treated as an incompressible flow in a phase space. Such flows generally mix leading to a uniform microcanonical equilibrium distribution. On the contrary, to deviate a system from equilibrium, one needs external forces and dissipation that break the Hamiltonian conservative evolution and lead to compressible phase-space flows, which generally produce extremely nonuniform measures [1,2]. The second perspective is the discrete viewpoint of information theory, where the evolution toward equilibrium and entropy saturation are described as the loss of all the information except integrals of motion. On the contrary, to keep a system away from equilibrium, we need to act, producing information and decreasing entropy.

Here, we make a step in synthesis of the two approaches, asking: what is the informational manifestation of nonuniform turbulent measures? Such measures are expected to have a low entropy whose limit is set by an interplay between interaction on the one hand and discreteness, coarse graining, or finite resolution on the other hand. We shall look at turbulence from the viewpoint of the mutual information (MI), which measures effective correlations between different degrees of freedom.

To keep a system away from equilibrium, environment extracts entropy thus producing information—where is this information encoded? Here, we consider turbulent systems which can be treated perturbatively as long as their statistics is close to Gaussian, such as weak wave turbulence (a similar approach can be applied to a passive scalar [3] and other systems). We show that the MI between wave modes is encoded in cumulants (not described by the traditional description in terms of occupation numbers [4]). The information production builds higher and higher correlations which concentrate sharper and sharper on the resonant surfaces, driving the distribution toward a singular measure. When nonlinearity is small, we show that the entropy decay is due to the triple moment concentrating on the three-wave resonance surface, see Fig. 1. It is unclear yet how to describe the longtime asymptotic of the entropy decay. When turbulence is driven by a random force, which provides for a phase-space diffusion and smears singularities, the entropy must saturate at a finite value, but the difference with Gaussian random-phase approximation can be large when the Reynolds number is large.

Consider a wave system defined by random amplitudes satisfying $i\dot{a}_k = \partial\mathcal{H}/\partial a_k^*$ with the Hamiltonian:

$$\mathcal{H} = \sum_{\mathbf{k}} \omega_k |a_k|^2 + \sum_{\mathbf{kpq}} \frac{1}{2}(V_{kpq} a_k^* a_p a_q + \text{c.c.})\delta^{\mathbf{k}}_{\mathbf{p+q}}.$$

Here, c.c means the complex conjugated terms, $\delta^{\mathbf{k}}_{\mathbf{p+q}}$ is the Kronecker delta, and we use the shorthand notation $a_k \equiv a(\mathbf{k})$, etc. Information theory (and processing real data of experiments and modeling) requires a discrete approach, so that we consider the number of modes N finite. Let us stress that the results of this work cannot be transferred to the continuous case. The medium is assumed scale invariant, that is both ω_k and V are homogeneous functions of degree α and m, respectively. A central problem is to describe the evolution of the phase-space

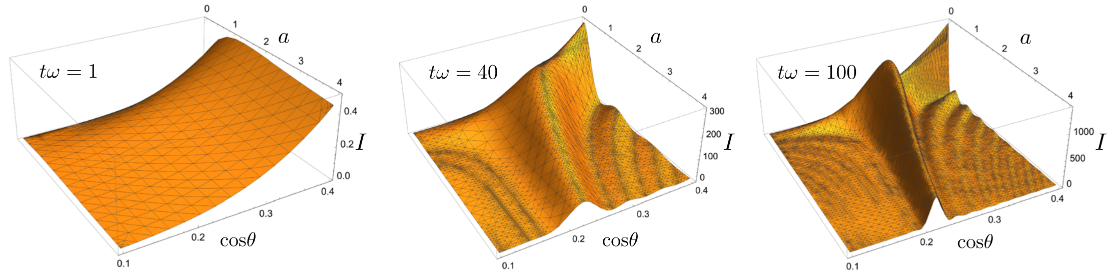

FIG. 1. Buildup of the normalized mutual information $I(\mathbf{k_1} = \mathbf{k}_2 + \mathbf{k}_3; \mathbf{k}_2, \mathbf{k}_3)$ between three interacting capillary waves in turbulence. Here, $a = k_2/k_3$, $\cos\theta = \mathbf{k_1}\cdot\mathbf{k_2}/k_1k_2$. Resonance surface is a line where MI develops a ridge at $t\omega_2 \gg 1$.

distribution $\rho(\{a_k, a_k^*\})$. We assume that the second term in the Hamiltonian is on average much smaller than the first one and that the modes are independently distributed at $t = 0$. Then the occupation numbers $n_k\delta^{\mathbf{k}}_{\mathbf{k}'} \coloneqq \langle a_k a_k^*\rangle$ satisfy a closed kinetic equation [4–10]:

$$\frac{dn_k}{dt} \equiv St_k = \sum_{\mathbf{k}_1\mathbf{k}_2} \mathrm{Im}(V_{k12}J_{k12} - 2V^*_{1k2}J^*_{1k2})\delta^{\mathbf{k}_1}_{\mathbf{k}_2+\mathbf{k}}, \quad (1)$$

$$J_{123}(t) = \frac{e^{i\omega^1_{2,3}t} - 1}{\omega^1_{2,3}} V^*_{123}(n_2n_3 - n_1n_2 - n_1n_3). \quad (2)$$

The brackets $\langle f\rangle$ indicate averaging with ρ, $\langle a_i^* a_j a_k\rangle = J_{ijk}\delta^{\mathbf{i}}_{\mathbf{j}+\mathbf{k}}$ and $\omega^1_{2,3} \equiv \omega_1 - \omega_2 - \omega_3$. Substituting (2) into (1) gives the collision integral of the kinetic equation, which is a direct analog of the Boltzmann equation for dilute gases: $St_k = \sum_{\mathbf{k}_1\mathbf{k}_2}(U_{k12} - 2U_{1k2})$ with

$$U_{ijk} = \pi|V_{ijk}|^2\delta(\omega^i_{jk})\delta^{\mathbf{i}}_{\mathbf{j}+\mathbf{k}}(n_jn_k - n_in_k - n_in_j). \quad (3)$$

The interaction time $t_{NL}(k) \simeq n_k/St_k$ is assumed much larger than the wave period, so that the nonlinearity parameter $\epsilon_k^2 = 1/\omega_k t_{NL}(k)$ is small. It is known that the Boltzmann kinetic equation is the first term of a regular cluster expansion only at thermal equilibrium, while even weak nonequilibrium leads to either temporal growth or space divergences already in the density expansion of kinetic coefficients—viscosity, diffusivity, and thermal conductivity [11–16]. The wave kinetic equation satisfactory describes stationary and nonstationary spectra of wave turbulence [4,6,8], yet the singularities hidden behind this nice picture have not been analyzed. Here, we open this Pandora's box and start such an analysis using a general approach of information theory, assuming N and ϵ finite. Subtleties related to limits $N \to \infty$ and $\epsilon \to 0$ are subject of the ongoing work [8–10,17,18].

The self-consistent weak-turbulence description of the one-mode statistics in terms of the occupation numbers n_k [4–8,17] guarantees that the statistics of any single amplitude stays close to Gaussian, i.e., $q(|a_k|)$ is Rayleigh for every $\boldsymbol{k}$. That tempts one to approximate the whole distribution using only the set of n_k, assuming that the amplitudes are independent and the phases are random:

$$q(\{a_k, a_k^*\}) = \prod_k (2\pi)^{-1} q(|a_k|), \quad (4)$$

which implies a Gaussian approximation for $\rho(\{a_k, a_k^*\})$. Here, we show that ρ is quite different. The difference between distributions can be measured by the relative entropy (Kullback-Leibler divergence), which is the price of non-optimal coding in information theory: $D(\rho|q) = \langle\ln(\rho/q)\rangle$. It was noted in [6] that random-phase approximation is not an accurate description of wave turbulence, the relative entropy quantifies that. Since q is a product, its entropy is a direct sum of the entropies of noninteracting modes: $S(q) = \sum_{\mathbf{k}} \ln(e\pi n_k) = \sum_{\mathbf{k}} S_k$. The relative entropy then coincides with the multimode mutual information $D(\rho|q) = \sum_{\mathbf{k}} S_k - S(\rho) \coloneqq I(\{a_k, a_k^*\})$. We keep in mind that the mutual information is defined for any subsystems, A and B, via their entropies: $I(A, B) = S(A) + S(B) - S(A, B)$. For example, the mutual information between two parts of the message measures how much of the future part we can predict given the part already received.

Starting with a Gaussian ρ at $t = 0$, at the times $1/\omega_k \ll t \ll t_{NL}$, the distribution ρ can be determined by the second and third cumulants using conditional entropy maximum (see Supplemental Material [19] for details):

$$\rho = \frac{1}{Z}\exp\left[-\sum_{\mathbf{k}}\alpha_k|a_k|^2 + \sum_{\mathbf{kpq}} F_{kpq}a_k^*a_pa_q + \text{c.c.}\right]. \quad (5)$$

For ρ to be normalizable, by (5) we mean a truncated series in powers of ϵ. Here, we consider terms up to second order. Then the parameters α, F of the distribution can be expressed via the moments J and n:

$$F_{123} = \frac{J^*_{123}}{2n_1n_2n_3}, \quad \frac{1}{\alpha_i} = n_i - \sum_{\mathbf{k}_1\mathbf{k}_2}\frac{|J_{i12}|^2 + 2|J_{12i}|^2}{2n_1^2n_2^2}. \quad (6)$$

We saw that the integral of the imaginary part of the third moment saturates on the short timescale $1/\omega_k$, so that continuing concentration of the third moment on the resonant surface had no influence on the kinetic equation (3) which remains universally valid. However, the relative entropy is the sum of the mutual information, $I_{\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_i,\mathbf{k}_j}$, of all resonant triads,

$$D(\rho|q) = \sum_{\mathbf{k}_i,\mathbf{k}_j} I_{\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_i,\mathbf{k}_j} = \sum_{\mathbf{k}_i\mathbf{k}_j} \frac{|J_{i+jij}|^2}{2n_i n_j n_{i+j}}, \qquad (7)$$

and is determined by the squared cumulant modulus, which depends dramatically on whether the system is in thermal equilibrium or not. The equilibrium $n_k = T/\omega_k$ is special because the last bracket in (2) is proportional to $\omega^1_{2,3}$, so the third cumulant is regular everywhere in k space and saturates after few wave periods, $J_{123}(t) \to -V^*_{123}T^2/\omega_1\omega_2\omega_3$, so that the relative entropy is smaller than the total entropy as long as nonlinearity is small:

$$D(\rho|q) = T\sum_{ij} \frac{|V_{i+j,ij}|^2}{\omega_i\omega_j\omega_{i+j}} \approx \left(\frac{E_{\text{int}}}{T}\right)^2. \qquad (8)$$

Away from equilibrium, on the contrary, with time the third cumulant (2) concentrates in a close vicinity of the resonance surface. It leads to a profound difference between statistics of a wave system in equilibrium and in turbulence. The equilibrium probability of a configuration $\{a_1, a_2, a_3\}$ is insensitive to resonances, because it is determined by the instantaneous interaction energy divided by the (uniform) temperature: $\exp[-\mathcal{H}/T]$, since $F^*_{123} = J_{123}/2n_1n_2n_3 = -V^*_{123}/2T$ in this case. For turbulence, as long as (2) is valid, the interaction energy is additionally weighted by the resonance factor $(n_1^{-1} - n_2^{-1} - n_3^{-1})/(\omega_1 - \omega_2 - \omega_3)$ as the probability is the result of a time averaging. The measure in the phase space is thus regular in equilibrium and tends to singular in turbulence.

In turbulence, the squared cumulants in the relative entropy (7) have a secular growth as long as (2) holds: $\lim_{t\to\infty}|(e^{\imath\Delta t} - 1)/\Delta|^2 = 2\pi t\delta(\Delta)$. Liouville's theorem requires that this increase of the mutual information and decrease in total entropy is exactly equal to the growth of $S(q)$ due to the change in n_k according to (3):

$$\frac{dS(q)}{dt} = \sum_{\mathbf{k}} \frac{1}{n_k}\frac{dn_k}{dt} = \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \frac{1}{2n_1n_2n_3}\frac{d}{dt}|J_{123}|^2. \qquad (9)$$

Indeed, the kinetic equation describes spreading of n_k and approach to equilibrium, accompanied by the growth of the entropy of amplitudes $S(q)$, while the Hamiltonian evolution by itself does not change the full entropy $S(\rho)$. A nonequilibrium state requires pumping and damping by the environment. If its action makes n_k and $S(q)$ stationary, then the information production is ultimately due to the entropy extraction by the environment: $dD(\rho|q)/dt = -[dS(\rho)/dt]_{\text{env}}$. We see that stationarity of the second moment does not mean stationary distribution. On the contrary, the third moment (and other cumulants) are getting more and more singular, reflecting the total entropy decrease and the growth of the relative entropy between the true distribution and the random-phase Gaussian approximation:

$$D(\rho|q) = t\sum_{\mathbf{kps}} U_{kps} \frac{n_p n_s - n_k n_p - n_k n_s}{n_k n_p n_s} > 0. \qquad (10)$$

Contribution to the relative entropy of every cumulant is proportional to its square. The fourth cumulant is $\propto V^2$, so its contribution is proportional to V^4 and can be neglected in this order. Formula (10) can be written as $D = t\sum_k t^{-1}_{NL}(k)$ and is the first term of the expansion in powers of time, valid at $t < t_{NL}$. The terms with higher powers of time will involve higher cumulants. One can estimate $t^{-1}_{NL}(k) \simeq \sum_j |V_{k+j,kj}|^2 n_j/\omega_j \simeq \omega_k\epsilon_k \propto k^{2m+d-s-\alpha}$. At $t \simeq t_{NL}(k)$, when nonlinearity at the three-wave resonant surfaces $\omega_j + \omega_k = \omega_{j+k}$ is getting of order unity, the triple moment is expected to stabilize. At that stage the entropy change already is not small, but could be comparable to the total entropy. At later time, the total entropy decrease is modified, but does not necessarily stop, contrary to what one may suggest. The reason is that the entropy extraction depends on the environment. We illustrate that for two qualitatively different ways of pumping the system.

Let us first add to the rhs of $\imath\partial a_k/\partial t = \partial\mathcal{H}/\partial a^*_k$ a random force and a damping, $f_k - \gamma_k a_k$, with $\langle f_k(0)f^*_j(t)\rangle = \delta_{kj}P_k\delta(t)$. When force and damping are in detailed balance, that is $\omega_k P_k/\gamma_k$ is independent of k, the system is brought to thermal equipartition where the entropy is maximal and stationary. If, however, the detailed balance is broken, the environment provides for the entropy change which depends on the distribution:

$$\left[\frac{dS(\rho)}{dt}\right]_{\text{env}} = \sum_{\mathbf{k}} P_k \int \prod_i \frac{da_i da^*_i}{2\imath\rho}\left|\frac{\partial\rho}{\partial a_k}\right|^2 - \sum_{\mathbf{k}} 2\gamma_k. \qquad (11)$$

Averaging now is over the force statistics. Let us show that if the steady distribution ρ exists, it must have very sharp gradients, proportional to the Reynolds number, so that the entropy $S(\rho)$ is much smaller than $S(q)$. At the initial perturbative stage, the distribution is given by (5); we substitute (2), (6) into (11) and obtain

$$\sum_{\mathbf{k}} \int \prod_i \frac{da_i da^*_i}{2\imath}\rho^{-1}\left|\frac{\partial\rho}{\partial a_k}\right|^2 = \sum_{\mathbf{k}} \alpha_k + O(J^4). \qquad (12)$$

Here, α_k is given by (6) where the last two terms are initially small. The pumping then produces much less entropy than the dissipation region absorbs (any nonequilibrium state consumes information, that is exists between a low-entropy source and a high-entropy sink). Indeed, the energy spectral density $\omega_j n_j$ is a decreasing function of j in a direct energy cascade, so $\sum_{\mathbf{k}} P_k n_k^{-1} < (\omega_j n_j)^{-1} \sum_{\mathbf{k}} \omega_k P_k$ for any $j > k_{\rm pump}$ and

$$\left[\frac{dS(\rho)}{dt}\right]_{\rm env} < \frac{1}{\omega_j n_j} \sum_{\mathbf{k}} (\omega_k P_k - 2\gamma_k \omega_k n_k) = 0 \qquad (13)$$

follows from the energy balance $\sum_k \omega_k (P_k - 2\gamma_k n_k) = 0$. For a developed turbulence with a wide inertial interval, $k_{\rm damp}/k_{\rm pump} = {\rm Re} \gg 1$, the spectrum is $n_k \propto k^{-s}$, and the ratio of the negative damping term to the positive pumping term in (11) can be estimated as $\omega_{\rm pump} n_{\rm pump}/\omega_{\rm damp} n_{\rm damp} \equiv {\rm Re}^{s-\alpha} \gg 1$. Direct energy cascade requires $s > \alpha$, and indeed the entropy absorption by the small-scale dissipation region by far exceeds the entropy production by the pumping region (it is other way around for an inverse cascade [29]). However, this is only true at the initial perturbative stage. As time proceeds, the growth of the cumulants and deviation of distribution from Gaussian decreases dD/dt by increasing the pumping contribution. For developed turbulence, the gradients $\partial\rho/\partial a_k$ in the pumping region must increase by a large factor ${\rm Re}^{(s-\alpha)/2}$ to reach the steady measure, which is thus very close to singular.

Another way of creating nonequilibrium is by adding to the Hamiltonian equations of motion the terms $\gamma_k a_k$, where positive γ_k corresponds to an instability and negative to dissipation. Averaging in this case is over the ensemble of initial conditions. The entropy rate of change $dS_{\rm env}/dt = 2\sum_{\mathbf{k}} \gamma_k \le 0$ is now independent of the distribution and negative for a steady direct cascade for the same reasons of the energy conservation $\sum_{\mathbf{k}} 2\omega_k \gamma_k n_k = 0$ and $\omega_k n_k$ being larger in the instability region. That means that the entropy decreases nonstop and the measure goes all the way to singular unless coarse graining saturates the entropy decrease. A profound difference between turbulent measures generated by additive force and instability was probably first noticed in [20].

To verify our other predictions, one needs to obtain numerically and experimentally multidimensional probability distributions. The simplest is to start from two modes. Because of translation invariance, the second moment, $\langle a_{\mathbf{k}} a_{\mathbf{p}}^* \rangle = 0$ for $\mathbf{k} \neq \mathbf{p}$, but the fourth cumulant is generally nonzero and so is the mutual information (introduced in [30] for one-dimensional models of turbulence). In thermal equilibrium and for nonresonant modes in turbulence, steady-state MI must be small for small nonlinearity: $I_{\mathbf{k},\mathbf{p}} = |J_{\mathbf{k},\mathbf{p},\mathbf{k}+\mathbf{p}}|^4/(4n_{\mathbf{k}} n_{\mathbf{p}} n_{\mathbf{k}+\mathbf{p}})^2 + |J_{\mathbf{k},\mathbf{p},\mathbf{k}-\mathbf{p}}|^4/(4n_{\mathbf{k}} n_{\mathbf{p}} n_{\mathbf{k}-\mathbf{p}})^2 \propto \epsilon^4$ for two modes. The ϵ^2 contribution requires minimum three modes: $I_{\mathbf{k},\mathbf{p},\mathbf{q}} = S(a_k) + S(a_p) + S(a_q) - S(a_k, a_p, a_q) = |J_{\mathbf{k},\mathbf{p},\mathbf{q}}|^2/2n_k n_p n_q$. On the contrary, we expect order-unity cumulants (as seen, for instance, in [31]) and substantially non-Gaussian stationary joint distribution for resonant modes in turbulence. Finding that distribution is a well-posed task for a future work.

Our consideration of the mutual information growth allows solving the old puzzle: why the direction of the formation of the turbulent spectrum $n_k \propto k^{-m-d}$ is determined by the energetic capacity? When the total energy $\sum_{\mathbf{k}} \omega_k n_k$ diverges at infinity ($m < \alpha$, infinite capacity), the formation proceeds from large to small scales, that is from pumping to dissipation [32]. In the opposite finite-capacity case, $m > \alpha$, formation of the cascade was surprisingly found to start from small and proceeds to large scales, that is opposite to the cascade direction [6,33,34]. We note that it is the growth of the density of three-mode mutual information (learning rate) in a ball of radius k, $I(k) \equiv \sum_{p \le k} I_{\mathbf{k},\mathbf{p},\mathbf{k}-\mathbf{p}}$, that must determine the direction of the evolution, since it quantifies the buildup of multimode correlations necessary for a steady nonequilibrium state. For $n_k \propto k^{-s} = k^{-m-d}$, the growth rate of $I(k)$ scales as $dI(k)/dt \propto k^{2m+d-s-\alpha} = k^{m-\alpha}$. One then can characterize the directionality of the information transfer by the sign of $m - \alpha$—when it is positive, correlations must be established first at small scales and then propagate to larger scales. That consideration puts on the firm information-theoretical ground the heuristic arguments of [33]. Since the energetic capacity is also finite for the Kolmogorov spectrum of the incompressible turbulence, a tantalizing question is whether it is also formed starting from small scales. Note that we characterized evolution by the mutual information growth rate, which is to be distinguished from the transfer entropy [35], that characterizes cause-effect relationships in a steady state. Note that though the MI between noninteracting Gaussian modes is zero, the MI between points in the physical space, $I(x_1, \ldots, x_N) = N \ln(e\pi N^{-1} \sum_{k=1}^{N} n_k) - \sum_k \ln(e\pi n_k)$, is positive whenever n_k are not all the same.

Let us briefly compare our findings with similar effects near thermal equilibrium. In the simplest case of the linear response of a dilute gas, nonequilibrium anomalies in cumulants appear as long-distance divergencies due to Dorfman-Cohen memory effects in multiple collisions [11–13,36], see nice explanation in [12] (we wonder why Peierls himself did not recognize that similar effects must be behind the wave kinetic equation, which he first derived). Another case is the two-particle single-time correlation function in two distinct space points. Outside of the radius of molecular forces, this correlation is zero in thermal equilibrium and nonzero away from it [15,16]. Nonequilibrium buildup of long spatiotemporal correlation is a counterpart to our spectral singularities.

Another analogy worth exploring is with many-body localization [37], where phase correlations prevent thermalization and keep the system in a low-entropy state. There is a vast literature devoted to cumulant anomalies away from equilibrium, see, e.g., books [1,2,12–16] and numerous references there. We believe that complementarity of information theory and singular measures will lead to a unified approach to these anomalies.

We conclude reiterating our main results: the probability distribution of weak wave turbulence is very far from Gaussian, the mutual information is substantial for resonant modes.

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