Coherent Dynamics Enhanced by Uncorrelated Noise

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Synchronization is a widespread phenomenon observed in physical, biological, and social networks, which persists even under the influence of strong noise. Previous research on oscillators subject to common noise has shown that noise can actually facilitate synchronization, as correlations in the dynamics can be inherited from the noise itself. However, in many spatially distributed networks, such as the mammalian circadian system, the noise that different oscillators experience can be effectively uncorrelated. Here, we show that uncorrelated noise can in fact enhance synchronization when the oscillators are coupled. Strikingly, our analysis also shows that uncorrelated noise can be more effective than common noise in enhancing synchronization. We first establish these results theoretically for phase and phase-amplitude oscillators subject to either or both additive and multiplicative noise. We then confirm the predictions through experiments on coupled electrochemical oscillators. Our findings suggest that uncorrelated noise can promote rather than inhibit coherence in natural systems and that the same effect can be harnessed in engineered systems.

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Synchronization, the phenomenon in which oscillators in a population evolve in step with each other, occurs because of interactions or common driving forces among oscillators. Influences from outside an oscillator network can often be treated as noise, which is usually expected to inhibit synchronization. Indeed, small noise can result in a disproportionately large degree of asynchrony in networks of nonlocally coupled oscillators [1]. In other contexts, while network disorder can improve synchronous parallel processing performance [2], noise has also been found to limit the permissible time delays in communications for network synchronizability and hinders parallel performance [3]. Still, numerous biological systems—such as neural networks [4–6], ecological communities [7], and the cardiac and cardio-respiratory systems [8,9]—and engineered systems—such as arrays of Josephson junctions [10], lasers [11], and nanoelectromechanical devices [12]—exhibit robust synchronization even under the influence of noise.

Previous theoretical and experimental observations have demonstrated that *common noise* (in which individual oscillators experience a shared noise term) can actually induce rather than inhibit synchronization [13,14]. The understanding behind this phenomenon can be traced back to the study of coherence resonance, in which noise leads to greater temporal order in systems with irregular oscillations [15,16]; to stochastic resonance [17], which has been used to reduce the threshold to detect tactile stimuli in human sensory perception [18]; and to the effects of common driving in synchronizing chaotic or disordered systems [19,20] as well as the synchronizing effects of periodic driving with a spatially-dependent phase [21]. Synchronization induced by common noise has since been studied in a variety of oscillator networks [22–24].

Here, we establish the alternative scenario shown in Fig. 1 in which the dynamics are more synchronous in the presence of *uncorrelated noise* than in the absence of noise or even the presence of common noise. It seems intuitive that uncorrelated noise would necessarily inhibit synchronization since, unlike the common noise case, it does not have inherent order. However, uncorrelated noise is prevalent in many systems, and recent studies suggest the

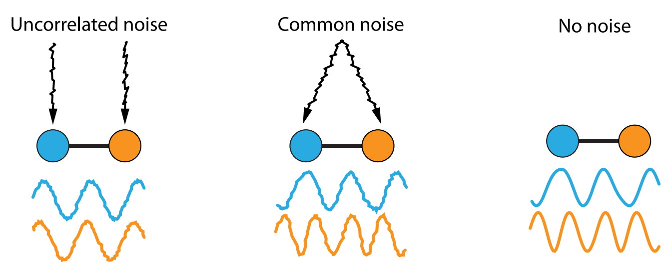

FIG. 1. Schematics of the main effect. Coupled oscillators experiencing uncorrelated noise exhibit more synchronous dynamics than those subjected to common noise or no noise. This behavior is distinct from those previously observed in oscillator models that synchronize due to large coupling or in response to specific forms of common noise.

potential for uncorrelated noise to have a positive impact on synchronization. For example, coupled neuronal networks subject to uncorrelated noise can exhibit enhanced coherence across the networks while reducing the coherence within each network [25]. Uncorrelated noise acting on a pair of oscillators has also been shown to enhance the phase coherence of one oscillator at the expense of the other [26]. Furthermore, uncorrelated noise can promote untwisted phase-locked states over twisted phase-locked states in small-world networks of Kuramoto oscillators [27] and can stabilize an otherwise unstable partially synchronized state in a globally coupled model of oscillators with biharmonic couplings [28]. However, the question of whether uncorrelated noise can enhance synchronization to a greater extent than common noise had so far remained open.

We establish our results for several forms of coupled limit-cycle oscillators governed by

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}_i(\mathbf{x}_i) + \frac{K}{N}\sum_{j=1}^{N} A_{ij}\mathbf{h}(\mathbf{x}_i, \mathbf{x}_j) + \sum_{k=1}^{n} \mathbf{g}_{ik}(\mathbf{X})\xi_{ik}, \quad (1)$$

where N is the number of oscillators, $\mathbf{x}_i$ denotes the state of oscillator i (assumed to be m dimensional), $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N)$ encodes the full state of the system, $\mathbf{f}_i$ describes the evolution of the isolated oscillators, $\mathbf{h}$ is the coupling function between two oscillators, K is the (tunable) coupling constant, and A_{ij} are the entries of the coupling matrix (assumed to be 1 if nodes i and j are coupled and 0 otherwise). We include n sources of noise determined by the state-dependent direction $\mathbf{g}_{ik}$ and the random variable ξ_{ik}, with $\langle \xi_{ik} \rangle = \langle \xi_{ik}\xi_{jl} \rangle = 0$ for all i, j, k and $l \neq k$. The ξ_{ik} term represents multiplicative noise in the case that $\mathbf{g}_{ik}$ varies with $\mathbf{X}$ and represents additive noise in the special case that $\mathbf{g}_{ik}$ is constant.

We assume that in the absence of coupling ($K = 0$) and noise ($\xi_{ik} = 0$) the isolated node dynamics approach a limit cycle $\mathbf{x}_i(t) \to \mathbf{x}_i^c(t)$, with $\mathbf{x}_i^c(t) = \mathbf{x}_i^c(t + T_i)$, where T_i is the period of oscillator i. We can always define a phase variable $\theta_i(\mathbf{x}_i)$ for oscillator i that, when restricted to the limit cycle, evolves as $\theta_i(t) = \theta_i(0) + \omega_i t$, where $\omega_i = 2\pi/T_i$ is the natural frequency. In the presence of coupling, we consider the oscillators to be more synchronized when their relative phase differences are smaller on average. Following Kuramoto [29,30], we employ the order parameter $R^2 \equiv |(1/N)\sum_j e^{\mathrm{i}\theta_j(t)}|^2$ as a measure of synchrony, where i is the imaginary unit. The time-averaged order parameter $\overline{R^2}$ is closer to 1 when oscillators are more synchronized and closer to 0 when the oscillators are less synchronized. In the results below, we say that noise enhances synchronization if $\overline{R^2}$ is larger in the presence of noise than in the absence of noise. We consider two broad forms of noise: common noise, for which $\xi_{ik} = \xi_{jk}$ for all i, j; and uncorrelated noise, for which ξ_{ik} and ξ_{jk} are independent random variables for all $i \neq j$. We are primarily interested in cases in which uncorrelated noise enhances synchronization more so than common noise.

Phase-reduced oscillators.—For weakly coupled oscillators driven by weak noise, the phase-reduction approximation can be applied to reduce the dynamics of Eq. (1) to a Kuramoto-type model with noise,

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_j A_{ij}\sin(\theta_j - \theta_i) + g_i(\theta_i)\eta_i. \quad (2)$$

The various noise terms in Eq. (1) result in a single effective noise term $g_i(\theta_i)\eta_i$ in the phase dynamics if, for instance, they are all Gaussian variables with autocorrelations of the same functional form (as shown in Sec. S1 of the Supplemental Material [31]). The effective noise η_i will be assumed to be Gaussian and white unless otherwise noted, with intensity specified by a matrix D_{ij} as $\langle \eta_i(t)\eta_j(t') \rangle = D_{ij}\delta(t - t')$, where δ is the Dirac delta function. The function $g_i(\theta_i)$, called the phase sensitivity function, arises because the effective noise acts on the phase evolution with varying intensity depending on the phase of the oscillator. In the case of common noise, $\eta_i = \eta_j$ for all i, j and all the elements of the noise intensity matrix are identical, with $D_{ij} = \sigma^2/2$ for σ denoting the noise intensity. In the case of uncorrelated noise, $\langle \eta_i\eta_j \rangle = 0$ for $i \neq j$ and the noise intensity matrix is diagonal, with $D_{ii} = \sigma^2/2$ for all diagonal elements.

We first consider the case of $N = 2$ phase oscillators with $g_i(\theta_i) = 1$, so that the multiplicative noise in Eq. (1) becomes additive in the phase approximation. By moving to a rotating frame, it is possible to take the mean natural frequency equal to zero, so that, without loss of generality, we can take $\omega_1 = \Delta\omega/2$ and $\omega_2 = -\Delta\omega/2$. In the absence of noise, the oscillators' phases will drift with respect to each other when the coupling strength K is smaller than $\Delta\omega$, as characterized by their separation angle $\phi \equiv \theta_2 - \theta_1$, while their mean angle $\Theta \equiv (\theta_1 + \theta_2)/2$ remains a constant of motion. They become phase locked as K increases above its critical value $K_c \equiv \Delta\omega$, initially with a separation angle $\phi = -\pi/2$. In the presence of Gaussian white noise with constant phase sensitivity and for any value of K, the evolution of the density of an ensemble of systems $\rho(\phi, \Theta, t)$ can be described by the Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial\phi}[(\Delta\omega + K\sin\phi)\rho] + \frac{\sigma^2}{2}\left[\frac{\partial^2\rho}{\partial\phi^2} + \frac{1}{4}\frac{\partial^2\rho}{\partial\Theta^2}\right], \quad (3)$$

where we have changed variables from θ_1 and θ_2 to ϕ and Θ. Because Eq. (3) is autonomous with respect to Θ and t, we can find steady solutions which are independent of the mean phase Θ. Direct integration in this case is possible using an integrating factor. After some simplification, the solution is

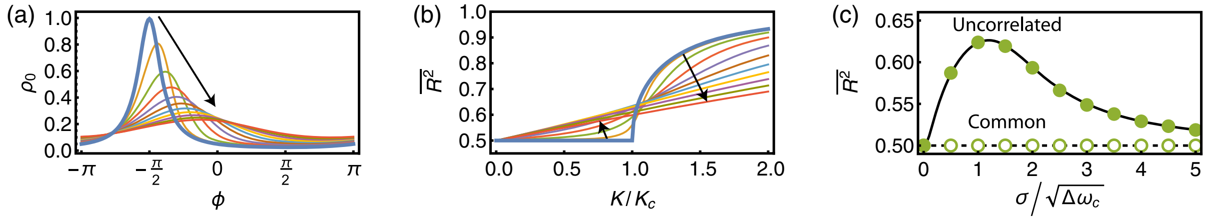

FIG. 2. Solutions of the Fokker-Planck equation (3) for two phase oscillators subject to Gaussian white noise with constant phase sensitivity $g_i(\theta) = 1$. (a) Steady ensemble density ρ_0 in Eq. (4) as a function of the phase difference ϕ for the subcritical coupling $K/K_c = 0.95$. (b) Time-averaged order parameter $\overline{R^2}$ as a function of the normalized coupling constant K/K_c. The arrows in (a) and (b) indicate the change in the solutions as σ increases from zero to $2\sqrt{\Delta\omega}$, where the zero-noise case (thick line) also corresponds to the case of common noise of any intensity. (c) Time-averaged order parameter $\overline{R^2}$ as a function of the normalized noise intensity $\sigma/\sqrt{\Delta\omega}$ at the coupling constant $K/K_c = 0.95$, where the lines show the solutions from the Fokker-Planck equation for uncorrelated noise (continuous) and common noise (dashed). The circles show the agreement with the corresponding direct numerical simulations of Eq. (2).

$$\rho_0(\phi) = A\int_0^{2\pi} d\psi \left[\frac{1}{\exp(4\pi\Delta\omega/\sigma^2)-1} + H(\phi-\psi)\right] \times \exp[2\Delta\omega(\psi-\phi)/\sigma^2 - 2K(\cos\psi - \cos\phi)/\sigma^2], \quad (4)$$

for $0 \leq \phi \leq 2\pi$, where A is a normalization constant and H is the Heaviside step function.

Figure 2(a) shows how the steady ensemble density ρ_0 varies as the noise intensity varies in the case with a subcritical coupling constant $K = 0.95K_c$. The ensemble density, which is peaked near $\phi = -\pi/2$ in the absence of noise, widens and its peak shifts toward zero as the noise intensity increases. The widening of the peak represents a loss in one form of coherence, as the oscillator phases become less correlated, but the shifting of the mean difference toward zero represents a gain in a different form of coherence, as the oscillators spend more time with similar phases. To assess the net impact on synchronization, we consider the time-averaged order parameter, which is determined from the steady-state distribution as $\overline{R^2} = \int_0^{2\pi} \cos^2(\phi/2)\rho_0(\phi)d\phi$. Figure 2(b) shows how the time-averaged order parameter varies as the noise intensity varies. The sharp, phase-locking transition at $K = K_c$ is smoothed out as the noise intensity increases. For subcritical coupling constants, the order parameter initially increases with increasing noise intensity, indicating an enhancement in synchronization in response to uncorrelated noise that does not occur in the case of common noise. This is illustrated in Fig. 2(c) for the same K as in Fig. 2(a), but a qualitatively similar effect occurs for all subcritical cases and for coupling constants just above the critical one. For large coupling constants, the system is already strongly synchronized in the absence of noise and thus synchronization is not further enhanced by noise. Time averaging of trajectories from direct numerical simulations of Eq. (2) (see Sec. S2 of the Supplemental Material [31]) agree extremely well with the solutions derived from the Fokker-Planck equation, as illustrated in Fig. 2(c).

Phase-amplitude oscillators.—We have shown that phase oscillators can exhibit enhanced synchronization under uncorrelated noise but not under common noise, assuming the noise and coupling terms are weak so that the phase-reduction approximation applies. We next consider the question of synchronization enhancement in phase-amplitude oscillators experiencing strong noise. As a prototypical example, we consider $N = 2$ coupled Stuart-Landau oscillators each with $m = 2$ degrees of freedom $\mathbf{x}_i = (x_i^{(1)}, x_i^{(2)})$, which are conveniently represented as a complex variable $z_i(t) = x_i^{(1)}(t) + \mathrm{i}x_i^{(2)}(t)$ and evolve according to

$$\frac{dz_i}{dt} = F_i(z_i) + \frac{K}{4}\sum_{j=1}^{2}(z_i z_j^* - z_j z_i^*)z_i + G_{ik}(z_1, z_2)\xi_{ik}, \quad (5)$$

where $F_i(z_i) \equiv (1 + \mathrm{i}\alpha_i)z_i - (1 - \mathrm{i}\gamma_i)|z_i|^2 z_i$ describes the intrinsic dynamics, α_i and γ_i are constants, and $*$ denotes complex conjugation. The cubic form of the coupling in Eq. (5) is selected to result in the Kuramoto-type coupling in the phase reduction, which facilitates comparisons below. In the absence of coupling and noise (when $K = 0$, $\xi_{ik} = 0$), the oscillators have a limit-cycle attractor $z_i(t) = r_i(t)e^{\mathrm{i}\theta_i(t)}$, where $r_i(t) = 1$, $\theta_i(t) = \theta_i(0) + \omega_i t$, and $\omega_i = \alpha_i + \gamma_i$.

Figure 3(a) shows the noise forces in the state space of a Stuart-Landau oscillator for three forms of noise determined by differing G_{ik}. In each case, the tangent of the noise force along the limit cycle determines the phase sensitivity function in the weak-noise regime for which the phase reduction would hold. As we proceed with our analysis of the strong-noise regime, it is instructive to compare with predictions for phase-reduced oscillators. The phase sensitivity function in the phase reduction for Eq. (5) takes the form $g_{ik}(\theta_i) = \int_0^{2\pi} d\theta_j [G_{ik}(e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2})e^{-\mathrm{i}\theta_i} - G_{ik}^*(e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2})e^{\mathrm{i}\theta_i}]/4\mathrm{i}\pi$, where $j \neq i$ [31].

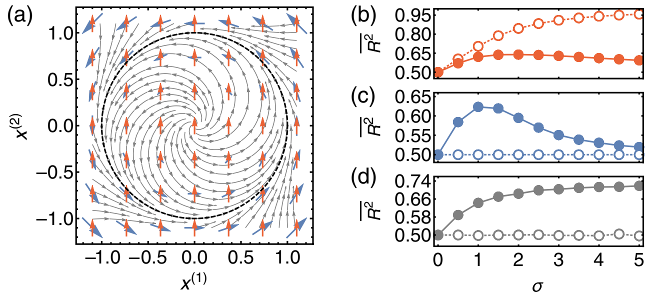

FIG. 3. Impact of noise on phase-amplitude oscillators. (a) State space of a Stuart-Landau oscillator, indicating the velocity field in the absence of noise and coupling (continuous lines), the limit-cycle attractor (dashed circle), and the noise forces (arrows). Three forms of noise are represented: Gaussian white noise with G_{i1} (vertical arrows), Gaussian white noise with G_{i2} (counterclockwise arrows), and Gamma distributed noise with G_{i3}, which acts in the direction of the uncoupled velocity field (continuous lines) when the coupling is small. (b)–(d) Time-averaged order parameter $\overline{R^2}$ as a function of the noise intensity σ for correlated (open circles) and uncorrelated (filled circles) noise corresponding to the G_{i1} (b), G_{i2} (c), and G_{i3} (d). The oscillator parameters are $\alpha_1 = 1$ and $\alpha_2 = \gamma_1 = \gamma_2 = 0$, and the coupling constant is $K = 0.95$, which is subcritical.

Taking additive noise with $G_{i1}(z_1, z_2) = \mathrm{i}$ results in multiplicative noise in the phase reduction with a trigonometric sensitivity function $g_{i1}(\theta_i) = \cos(\theta_i)$, which is expected to induce synchronization under common noise. On the other hand, taking $G_{i2}(z_1, z_2) = \mathrm{i}z_i$ results in additive noise in the phase reduction, with a constant sensitivity function $g_{i2}(\theta_i) = 1$. Noise that is modulated by the noiseless part of the dynamics, with $G_{i3}(z_1, z_2) = F_i(z_i) + (K/4)\sum_j (z_i z_j^* - z_j z_i^*) z_i$, also results in additive noise in the phase reduction. We thus expect, based on our results above for phase oscillators, that uncorrelated noise will enhance synchronization but common noise will not for G_{i2} and G_{i3} also in the unreduced system.

Figures 3(b)–3(d) assess these predictions through direct numerical simulations. For G_{i1}, common Gaussian white noise enhances synchronization significantly, as anticipated above, but uncorrelated noise also enhances synchronization to some extent. For G_{i2}, uncorrelated Gaussian white noise enhances synchronization while common noise does not, which once again agrees with the prediction above. To assess if these predictions continue to hold under non-Gaussian noise, we employed noise sampled from Gamma distribution for G_{i3}, which is dominated by brief, high-intensity bursts (see Sec. S2 of the Supplemental Material [31]). It is interesting to note that, while the enhancement for Gaussian cases are qualitatively similar to the phase approximation in Fig. 2(c), in the non-Gaussian case, the enhancement continues to grow with increasing noise intensity over the same noise range.

In summary, while the synchronization enhancement in Eq. (5) depends on specific noise features, uncorrelated noise continues to enhance synchronization beyond the phase-reduction approximation in cases where common noise does not.

Electrochemical oscillator experiments.—To test whether the effect described above can be observed in real limit-cycle systems, we performed experiments on coupled electrochemical oscillators. These oscillators, detailed in Supplemental Material, Sec. S3 [31] along with sample experimental trajectories, are described by $m = 2$ degrees of freedom, which represent the electrode potential and the concentration of the electroactive species in the vicinity of the electrode [32]. Unlike Stuart-Landau oscillators, the limit cycle in this case is not circular in the state space, and thus the phase is a complicated function that we will not attempt to describe analytically. The experimental system consists of two such electrochemical oscillators coupled together through a resistor and the shared fluid environment.

For statistical analysis, we create several realizations of the experimental system, with each realization having slightly different natural frequencies and being subject to no noise, common noise, and uncorrelated noise. The experiments are repeated for three levels of noise intensity, and the time-averaged order parameter is measured for each experimental run to assess synchronization. Figure 4 shows the statistical analysis for these experiments. We find that for low noise intensity, there is no statistically significant difference between the cases of no noise, uncorrelated noise, and common noise. For intermediate noise intensities, uncorrelated noise enhances synchronization significantly more so than common noise, confirming the effect described above. For high noise intensity, common noise exhibits a greater synchronization enhancement.

We emphasize that, in these experiments, we did not attempt to control the direction of the noise force, given that

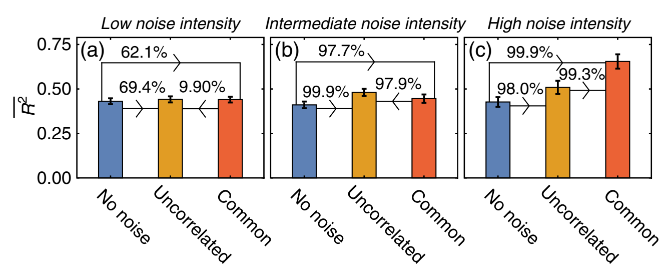

FIG. 4. Electrochemical oscillator experiments with Gaussian white noise added to the electrode potentials. (a)–(c) Bar plots of $\overline{R^2}$ at three different noise intensities, where $D = 0.025$ V for 14 realizations (a), $D = 0.05$ V for 22 realizations (b), and $D = 0.10$ V for 10 realizations (c). Error bars indicate the estimated errors in the means (i.e., the standard deviation normalized by the square root of the number of realizations), and the arrows between bar plots go from the smaller mean value to the larger mean value with percentages indicating the confidence from a paired t test.

noise can be easily applied only to the electrode potential and not to the chemical concentration, nor did we attempt to determine (or fine-tune) the phase sensitivity function, given the complexity of the limit cycle. Nevertheless, we still observe a greater degree of synchronization enhancement for the case of uncorrelated noise than for the case of common noise for intermediate noise intensity. Thus, these experiments reveal that uncorrelated noise can outperform common noise in synchronization enhancement even without careful design.

Discussion.—Our demonstration that uncorrelated noise can enhance synchronization to a greater degree than common noise reveals a new mechanism for how coherent behavior can emerge naturally in spatially-distributed noisy systems. The mechanism that generates this noise-enhanced synchronization can be interpreted as follows. On the one hand, when coupled oscillators are close to phase locking, they often spend time at relative angles that are far from zero, and their phases do not add coherently. On the other hand, uncorrelated noise allows the oscillators to escape from these large phase separations and spend more time with similar phases, even when common noise cannot do so precisely because it exerts the same effect on the phases of all oscillators. Our analysis indicates that this effect occurs prominently when the impact of the noise on coupled oscillators is independent of their phases, which means that the coherence is not inherited from a biased filtering of the noise.

These findings are counterintuitive because the noise terms acting on different oscillators exhibit permutation symmetry for common noise but not for uncorrelated noise; yet, for the systems considered here, the resulting dynamical states are more symmetric in the uncorrelated case. Such synchronization enhancement can thus be interpreted as a manifestation of asymmetry-induced symmetry [33], a recently recognized phenomenon in which some degree of asymmetry in a system actually increases the symmetry in the observed state of that system.

In this study, we observed the preferential enhancement of coherence by uncorrelated noise over common noise in a variety of coupled oscillator systems, including phase and phase-amplitude oscillators, both theoretically and experimentally. While we focused here on pairs of oscillators for clarity, we can show that this effect also occurs more generally in larger networks with a frequency gap [34], such as random networks of Janus oscillators [35,36] (see Sec. S4 of the Supplemental Material [31]) and multilayer networks relevant to the distributed mammalian neural and circadian systems [37–40] (see Sec. S5 of the Supplemental Material [31]). These results overturn the widely held assumption that uncorrelated noise necessarily tampers coherence and suggest that distributed networks that rely on synchronization, such as the network of circadian clocks distributed throughout the body, may benefit from the uncorrelated noise that they experience. In contrast with coherence induced by common noise, the enhancement due to uncorrelated noise requires nonvanishing coupling between the oscillators, thus revealing a new relationship between noise and network interactions.

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[1] Y. Kawamura, H. Nakao, and Y. Kuramoto, Phys. Rev. E **75**, 036209 (2007).
[2] G. Korniss, M. A. Novotny, H. Guclu, Z. Toroczkai, and P. A. Rikvold, Science **299**, 677 (2003).
[3] D. Hunt, G. Korniss, and B. K. Szymanski, Phys. Rev. Lett. **105**, 068701 (2010).
[4] S. L. Brown, J. Joseph, and M. Stopfer, Nat. Neurosci. **8**, 1568 (2005).
[5] S. Yamaguchi, H. Isejima, T. Matsuo, R. Okura, K. Yagita, M. Kobayashi, and H. Okamura, Science **302**, 1408 (2003).
[6] Y. Penn, M. Segal, and E. Moses, Proc. Natl. Acad. Sci. U.S.A. **113**, 3341 (2016).
[7] B. Blasius, A. Huppert, and L. Stone, Nature (London) **399**, 354 (1999).
[8] D. C. Michaels, E. P. Matyas, and J. Jalife, Circ. Res. **61**, 704 (1987).
[9] B. Kralemann, M. Frühwirth, A. Pikovsky, M. Rosenblum, T. Kenner, J. Schaefer, and M. Moser, Nat. Commun. **4**, 2418 (2013).
[10] K. Wiesenfeld, P. Colet, and S. H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996).
[11] R. A. York and R. C. Compton, IEEE Trans. Microwave Theory Tech. **39**, 1000 (1991).
[12] M. H. Matheny *et al.*, Science **363**, eaav7932 (2019).
[13] C. Zhou and J. Kurths, Phys. Rev. Lett. **88**, 230602 (2002).
[14] C. Zhou, J. Kurths, I. Z. Kiss, and J. L. Hudson, Phys. Rev. Lett. **89**, 014101 (2002).
[15] A. S. Pikovsky and J. Kurths, Phys. Rev. Lett. **78**, 775 (1997).
[16] N. Semenova, A. Zakharova, V. Anishchenko, and E. Schöll, Phys. Rev. Lett. **117**, 014102 (2016).
[17] F. Moss, L. M. Ward, and W. G. Sannita, Clin. Neurophysiol. **115**, 267 (2004).
[18] J. J. Collins, T. T. Imhoff, and P. Grigg, Nature (London) **383**, 770 (1996).
[19] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[20] Y. Braiman, J. F. Lindner, and W. L. Ditto, Nature (London) **378**, 465 (1995).
[21] S. F. Brandt, B. K. Dellen, and R. Wessel, Phys. Rev. Lett. **96**, 034104 (2006).
[22] H. Nakao, K. Arai, and Y. Kawamura, Phys. Rev. Lett. **98**, 184101 (2007).
[23] K. H. Nagai and H. Kori, Phys. Rev. E **81**, 065202(R) (2010).
[24] A. V. Pimenova, D. S. Goldobin, M. Rosenblum, and A. Pikovsky, Sci. Rep. **6**, 38518 (2016).

[25] J. H. Meng and H. Riecke, Sci. Rep. **8**, 6949 (2018).
[26] R. M. Amro, B. Lindner, and A. B. Neiman, Phys. Rev. Lett. **115**, 034101 (2015).
[27] R. K. Esfahani, F. Shahbazi, and K. A. Samani, Phys. Rev. E **86**, 036204 (2012).
[28] P. Clusella and A. Politi, Phys. Rev. E **95**, 062221 (2017).
[29] Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki, Lecture Notes in Physics Vol. 39 (Springer, Berlin, 1975), p. 420.
[30] Y. Kuramoto and H. Nakao, Phil. Trans. R. Soc. A **377**, 20190041 (2019).
[31] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.094101 for details about the phase-reduction approximation, direct numerical simulations, electrochemical oscillator experiments, networks of Janus oscillators, and layered networks of phase oscillators.
[32] M. Wickramasinghe, E. M. Mrugacz, and I. Z. Kiss, Phys. Chem. Chem. Phys. **13**, 15483 (2011).
[33] T. Nishikawa and A. E. Motter, Phys. Rev. Lett. **117**, 114101 (2016).
[34] R. M. D'Souza, J. Gómez-Gardeñes, J. Nagler, and A. Arenas, Adv. Phys. **68**, 123 (2019).
[35] Z. G. Nicolaou, D. Eroglu, and A. E. Motter, Phys. Rev. X **9**, 011017 (2019).
[36] T. Peron, D. Eroglu, F. A. Rodrigues, and Y. Moreno, Phys. Rev. Research **2**, 013255 (2020).
[37] G. B. Ermentrout, L. Glass, and B. E. Oldeman, Neural Comput. **24**, 3111 (2012).
[38] T. R. Lilley, C. Wotus, D. Taylor, J. M. Lee, and H. O. de la Iglesia, Endocrinology **153**, 732 (2012).
[39] M. H. Hastings, S. M. Maywood, and B. Brancaccio, Nat. Rev. Neurosci. **19**, 453 (2018).
[40] S. M. Reppert and D. R. Weaver, Nature (London) **418**, 935 (2002).