Explicit Construction of Local Conserved Quantities in the XYZ Spin-1/2 Chain

Yuji Nozawa [*] and Kouhei Fukai
The Institute for Solid State Physics, The University of Tokyo, Kashiwa, Chiba 277-8581, Japan

(Received 6 March 2020; revised 30 March 2020; accepted 6 July 2020; published 27 August 2020)

We present a rigorous explicit expression for an extensive number of local conserved quantities in the XYZ spin-1/2 chain with general coupling constants. All the coefficients of operators in each local conserved quantity are calculated. We also confirm that our result can be applied to the case of the XXZ chain with a magnetic field in the z-axis direction.

DOI: 10.1103/PhysRevLett.125.090602

Introduction.—Understanding and describing nonequilibrium phenomena in quantum many-body systems is one of the challenging problems in physics. Particularly for the past two decades, nonequilibrium phenomena in integrable systems have been attracting more attention owing to their experimental realization with ultracold atomic gases [1–4]. An extensive number of local conserved quantities, which characterizes integrable systems, are key elements of nonequilibrium phenomena. For example, these quantities prevent systems from relaxing to a thermal state, which is described by the canonical ensemble, and it is proposed that the steady states in integrable systems are described by the generalized Gibbs ensemble [5,6], whose density matrix is constructed from an extensive number of local and quasilocal conserved quantities as well as the Hamiltonian [7,8]. The second example is the generalized hydrodynamics [9,10], which describes large scale nonequilibrium dynamics in integrable systems and is formulated from the set of continuity equations for conserved quantities. In many interacting integrable systems which are solved by the Bethe ansatz and the quantum inverse scattering methods [11,12], the existence of local conserved quantities and the mutual commutativity of them were proved from the commutativity of transfer matrices $T(\lambda)$ with different values of the spectral parameter λ: $[T(\lambda), T(\mu)] = 0$. Local conserved quantities are obtained from the expansion of $\ln T(\lambda)$ in terms of λ, which includes the Hamiltonian. Another standard method to construct local conserved quantities is to use the boost operator B [13–15]. In this method, local conserved quantities are obtained recursively from the commutation relations as $[B, Q_n] = Q_{n+1}$.

Although how to prove the existence of local conserved quantities and construct them are known, it is still difficult to obtain the explicit expression for them because the calculation is complicated in general, and one needs to find the pattern of coefficients of local conserved quantities to express general local conserved quantities. Grabowski and Mathieu investigated the problem for the XYZ spin-1/2 chain, which is a generalization of the Heisenberg spin-1/2 chain and known as an integrable spin chain [12,16–24] with the use of the boost operator. As a result, they found the explicit expression in the case of the Heisenberg chain (also called the XXX chain) [25,26]. In the more general case, so far, Q_n has been obtained only in the case of $3 \leq n \leq 6$ from the Hamiltonian $Q_2 = H$, and the explicit expression for general local conserved quantities was not found. For the study of nonequilibrium phenomena, the explicit expression for local conserved quantities is a useful tool. For example, current operators, which are fundamental ones in the study of transport phenomena [27], can be constructed from the continuity equations of the densities of local conserved quantities.

In this Letter, we present an explicit expression for all the local conserved quantities in the XYZ spin-1/2 chain with general coupling constants. We also confirm that our result can be applied to the XXZ spin-1/2 chain with a magnetic field in the z-axis direction, which is also known as a Bethe ansatz soluble model [11,28]. To obtain the expression, we have used a straightforward way with a notation called *doubling product*, which was introduced to prove the absence of local conserved quantities in the XYZ spin-1/2 chain with a magnetic field [29] and its extension. We have directly derived the conditions for the commutator of each local conserved quantity and the Hamiltonian to be zero. With the doubling-product notation, we have found the pattern of coefficients of local conserved quantities and obtained general solutions of them.

Model and k-support conserved quantities.—We consider the XYZ spin-1/2 chain without a magnetic field for periodic boundary conditions

$$H = \sum_{i=1}^{L} (J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1}), \qquad (1)$$

where X_i, Y_i, and Z_i represent the Pauli matrices σ^x, σ^y, and σ^z acting on the spin at the site i, respectively. We set all the coupling constants J_X, J_Y, and J_Z nonzero. Following Ref. [29], we define k-support conserved quantities Q_k,

$$Q_k = \sum_{l=1}^{k} \sum_{\boldsymbol{A}^l} \sum_{i=1}^{L} q_{\boldsymbol{A}^l} \boldsymbol{A}_i^l. \tag{2}$$

Here, $\boldsymbol{A}_i^l \equiv A_i^1 A_{i+1}^2 \cdots A_{i+l-1}^l$ is a sequence of l operators acting from the site i to the site $i+l-1$. Operators at both ends A^1, A^l take X, Y, or Z, and the other operators $A^2, \ldots, A^{l-1}$ take X, Y, Z, or the identity operator I. $\sum_i \boldsymbol{A}_i^l$ is called an l-support operator. Coefficients $\{q_{\boldsymbol{A}^l}\}$ are determined from the commutation relation $[Q_k, H] = 0$. For example, the Hamiltonian itself is a trivial two-support conserved quantity, and it is easily proved that all the one-support conserved quantities are $\sum_i X_i$ if $J_Y = J_Z$, $\sum_i Y_i$ if $J_Z = J_X$, and $\sum_i Z_i$ if $J_X = J_Y$. Therefore, we consider Q_k for $k \geq 2$ hereafter, and our aim is to determine the coefficients $\{q_{\boldsymbol{A}^l}\}$ of Q_k.

To describe commutation relations, we use the following notation [29]:

$$\begin{aligned} &\begin{array}{ccccc} X_i & Y_{i+1} & Z_{i+2} & & \\ & & X_{i+2} & X_{i+3} & \end{array} \\ &\equiv -\mathrm{i}[X_i Y_{i+1} Z_{i+2}, X_{i+2} X_{i+3}]/2 \\ &= X_i Y_{i+1} Y_{i+2} X_{i+3}, \end{aligned} \tag{3}$$

and we drop the subscripts hereafter for visibility. Fundamental formulas using this notation are

$$\begin{array}{ccc} XY & XY & XY \\ XX & YY & ZZ \\ = -IZ, & = ZI, & = 0, \end{array} \tag{4}$$

$$\begin{array}{ccc} XX & XX & XX \\ XX & YY & ZZ \\ = 0, & = 0, & = 0, \end{array} \tag{5}$$

$$\begin{array}{ccc} XI & XI & XI \\ XX & YY & ZZ \\ = 0, & = ZY, & = -YZ. \end{array} \tag{6}$$

Doubling-product operators and their extension.—First we consider the case that the site number L satisfies $k \leq L/2$. As shown in Ref. [29], by considering $(k+1)$-support operators in $[Q_k, H]$, k-support operators in Q_k are restricted to doubling-product operators defined as

$$\begin{aligned} &\overline{A_1 A_2 \cdots A_{k-2} A_{k-1}} \\ &= c\, A_1\, (A_1 A_2)\, (A_2 A_3) \cdots\, (A_{k-2} A_{k-1})\, A_{k-1} \\ &= \, A_1\, A_{1,2} A_{2,3} \cdots A_{k-2,k-1}\, A_{k-1}, \end{aligned} \tag{7}$$

where A_α takes one of $\{X, Y, Z\}$ and it is required that $A_\alpha \neq A_{\alpha+1}$. We define $A_{\alpha,\beta}$ by $\{A_\alpha, A_\beta, A_{\alpha,\beta}\} = \{X, Y, Z\}$ when $A_\alpha \neq A_\beta$. The coefficient $c \in \{\pm 1, \pm \mathrm{i}\}$ is determined from Eq. (7). Furthermore, after fixing a normalization factor of Q_k, nonzero coefficients of k-support operators are uniquely given by

$$\begin{aligned} &q_{\overline{A_1 A_2 \cdots A_{k-2} A_{k-1}}} \\ &\quad = s(A_1 A_2 \cdots A_{k-2} A_{k-1}) J_{A_1} J_{A_2} \cdots J_{A_{k-2}} J_{A_{k-1}}, \end{aligned} \tag{8}$$

where $s(XY) = s(YZ) = s(ZX) = -s(YX) = -s(ZY) = -s(XZ) \equiv 1$, and $s(A_1 A_2 \cdots A_{k-2} A_{k-1}) \equiv s(A_1 A_2) s(A_2 A_3) \cdots s(A_{k-2} A_{k-1})$. Therefore, for $2 \leq k \leq L/2$, Q_k is unique up to differences of smaller support conserved quantities $Q_{k'<k}$. Note that $Q_k + Q_{k'<k}$ is also a k-support conserved quantity.

To express $k'(<k)$-support operators in Q_k, it is useful to extend the definition of doubling-product operators. Let us allow the case that neighboring symbols in doubling-product operators are the same $A_\alpha = A_{\alpha+1}$. Then, in the definition Eq. (7), $A_{\alpha,\alpha+1}$ is replaced by I if $A_\alpha = A_{\alpha+1}$. When the condition $A_\alpha = A_{\alpha+1}$ satisfies at m places in an l-support operator, we call it an (l, m) operator. m is called the number of *holes* and used to study the structure of conserved quantities [25,26]. Under this definition, all the k-support operators in Q_k are $(k, 0)$ operators.

We can express (l, m) operators as

$$\begin{aligned} &\overline{\underbrace{A_1 \cdots A_1}_{1+m_1} \underbrace{A_2 \cdots A_2}_{1+m_2} \cdots\cdots \underbrace{A_{l-m-1} \cdots A_{l-m-1}}_{1+m_{l-m-1}}} \\ &\equiv \overline{A_1^{1+m_1} A_2^{1+m_2} \cdots A_{l-m-1}^{1+m_{l-m-1}}}, \end{aligned} \tag{9}$$

where $A_\alpha \neq A_{\alpha+1}$ and $m_j \geq 0$ is an integer which satisfies $\sum_{j=1}^{l-m-1} m_j = m$. For example, $\overline{X^2 Z^2} = \overline{XXZZ} = XIYIZ$ and $\overline{X^3 Z} = \overline{XXXZ} = XIIYZ$ are both (5,2) operators. When we consider commutation relations of (l, m) operators, we use the following notation:

$$\begin{aligned} &\begin{array}{c} \overline{XYZ^2} \\ \overline{Z} \end{array} \equiv \begin{array}{c} XZXIZ \\ ZZ \end{array} \\ &= -XIYIZ = -\overline{X^2 Z^2}, \end{aligned} \tag{10}$$

where $\overline{XYZ^2}$ is a (5,1) operator, and $\overline{X^2 Z^2}$ is a (5,2) operator.

Structure of Q_k.—Let us consider the commutation relation of an (l, m) operator in Q_k $\overline{A_1^{1+m_1} A_2^{1+m_2} \cdots A_{l-m-1}^{1+m_{l-m-1}}}$ and H. Candidates of operators in the commutator are $(l \pm 1)$ and l support. First, $(l-1)$-support operators are constructed by removing A_1 or A_{l-m-1}. As for A_1, this operator is

$$\begin{array}{l} \overline{A_1 A_1^{m_1} A_2^{1+m_2} \cdots A_{l-m-1}^{1+m_{l-m-1}}} \\ \overline{A_1}. \end{array} \tag{11}$$

Note that this term is nonzero only if $m_1 = 0$ and $A_1 \neq A_2$. Therefore, the number of holes is conserved, and it is an $(l-1, m)$ operator. The same holds for A_{l-m-1}. Second, $(l+1)$-support operators are constructed by adding $A_0 (\neq A_1)$ on the left side of A_1:

$$\frac{\overline{A_1^{1+m_1} A_2^{1+m_2} \cdots A_{l-m-1}^{1+m_{l-m-1}}}}{\overline{A_0}}, \tag{12}$$

or $A_{l-m} (\neq A_{l-m-1})$ on the right side of A_{l-m-1}. Therefore, these operators are $(l+1, m)$ operators. The third case of l-support operators is a bit more complicated. For example, they are given as

$$\frac{\overline{A_1^{1+m_1} \cdots A_p \cdots A_{l-m-1}^{1+m_{l-m-1}}}}{\overline{B_p}}. \tag{13}$$

If $A_p = B_p$ for $1 < p < l-1$, these operators cannot be expressed as (l, m) operators. However, these terms are cancelled and do not contribute to the commutator. In the case of $A_p \neq B_p$, from Eqs. (4)–(6), only $(l, m \pm 1)$ operators are obtained (see Supplemental Material [30] for the details).

Consequently, operators in Q_k are classified as (l, m) operators as shown in Fig. 1. Here, we fix the degrees of freedom to add $Q_{k'<k}$. For example, coefficients of $(k-2n-1, 0)$ operators $(n = 0, 1, \ldots)$ are set to zero. In Fig. 1, circles represent (l, m) operators in Q_k, and crosses shown by arrows represent operators generated by the commutation relations of (l, m) operators in Q_k and H. From this structure, Q_k is represented as

$$Q_k = \sum_{\substack{0 \le n+m \le \lfloor \frac{k}{2} \rfloor - 1, \\ n,m \ge 0}} \sum_{\substack{\bar{A}: \\ (k-2n-m,m)\text{operators}}} q_{\bar{A}}^{k-2n-m,m} \bar{A}. \tag{14}$$

Here, the sum of $\bar{A} \equiv \overline{A_1^{1+m_1} A_2^{1+m_2} \cdots A_{k-2n-2m-1}^{1+m_{k-2n-2m-1}}}$ runs over all $(k-2n-m, m)$ operators that satisfy $n \ge 0$, $m \ge 0$, and $k - 2n - 2m \ge 2$, which corresponds to circles in Fig. 1.

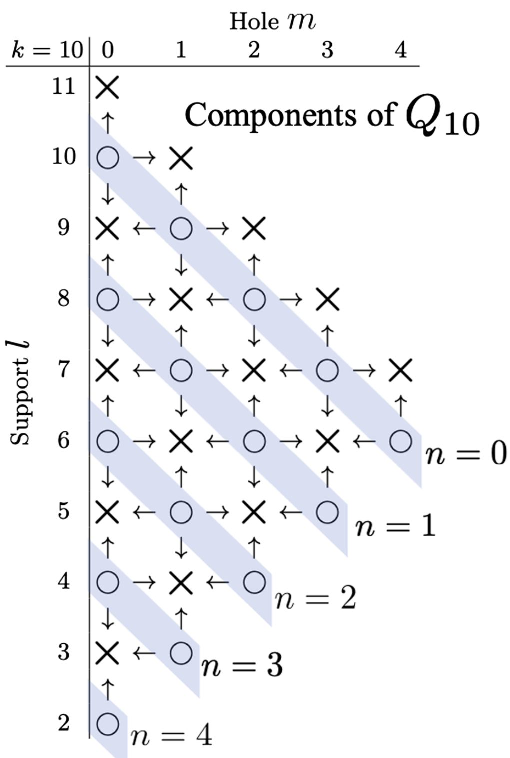

FIG. 1. Structure of a k-support conserved quantity Q_k for $k = 10$. Circles represent (l, m) operators in Q_k, where $l = k - 2n - m$. Crosses represent operators generated by the commutation relation of H and operators represented as circles, which are to be cancelled.

Recursive way.—One of the main results of this Letter is that we have found a simple recursive way to determine all the coefficients $\{q_{\bar{A}}^{k-2n-m,m}\}$ in Eq. (14) [30]. By introducing some functions, we describe the way.

First, $q_{\bar{A}}^{k-2n-m,m}$ is expressed using the function R as

$$q^{w,m}_{\overline{A_1^{1+m_1} A_2^{1+m_2} \cdots A_t^{1+m_t}}} = s(A_1 A_2 \cdots A_t)(J_X J_Y J_Z)^m \times \left(\prod_{j=1}^{t} J_{A_j}^{1-m_j} \right) R^{w,m}(A_1 A_2 \cdots A_t), \tag{15}$$

where we introduce the notation $w \equiv k - 2n - m$, $t \equiv k - 2n - 2m - 1$ for simplicity. $A_1 A_2 \cdots A_t$ is a character string of length $t \ge 1$, and $A_1, A_2, \ldots, A_t$ take one of $\{X, Y, Z\}$, respectively. s is the function we introduced in Eq. (8). For example, $q^{5,2}_{\overline{X^2 Z^2}} = (J_X J_Y J_Z)^2 R^{5,2}(XZ)$ and $q^{5,2}_{\overline{X^3 Z}} = (J_X J_Y J_Z)^2 (J_Z/J_X) R^{5,2}(XZ)$. We remark that R does not depend on where holes are because it does not depend on $m_1, m_2, \ldots, m_t$.

Second, R is represented as a linear combination of the function S:

$$R^{w,m}(A_1 A_2 \cdots A_t) = \sum_{\tilde{n}=0}^{n} g^{w,m}_{n-\tilde{n}} S_{\tilde{n}}(A_1 A_2 \cdots A_t), \tag{16}$$

where S is defined as $S_0(A_1 A_2 \cdots A_t) \equiv 1$ and

$$S_{\tilde{n}}(A_1 A_2 \cdots A_t) \equiv \sum_{1 \le j1 \le j2 \le \cdots \le j\tilde{n} \le t} J^2_{A_{j1}} J^2_{A_{j2}} \cdots J^2_{A_{j\tilde{n}}} \tag{17}$$

for $\tilde{n} \ge 1$, and $g^{w,m}_{n-\tilde{n}} (0 \le \tilde{n} \le n)$ does not depend on $A_1 A_2 \cdots A_t$. By definition, $R^{w,m}$ is a symmetric polynomial in $J^2_{A_1}, J^2_{A_2}, \ldots, J^2_{A_t}$.

Finally, $g^{w,m}_{n-\tilde{n}}$ is determined as follows. From Eq. (8), which corresponds to the case of $n = 0$, $g_0^{k-m,m} = 1$ because $R^{k-m,m} = 1$. Figure 2 shows how to determine the other $g^{w,m}_{n-\tilde{n}}$'s recursively. Suppose that $R^{w+1,m-1} = \sum_{\tilde{n}=0}^{n} g^{w+1,m-1}_{n-\tilde{n}} S_{\tilde{n}}$ (the upper left circle) and $R^{w+1,m+1} = \sum_{\tilde{n}=0}^{n-1} g^{w+1,m+1}_{n-1-\tilde{n}} S_{\tilde{n}}$ (the upper right circle) are determined. Then $R^{w,m}$ (the lower center circle) is determined as $\sum_{\tilde{n}=0}^{n} g^{w+1,m-1}_{n-\tilde{n}} a_{\tilde{n}} + \sum_{\tilde{n}=0}^{n-1} g^{w+1,m+1}_{n-1-\tilde{n}} S_{\tilde{n}+1}$. Here, in the case of $m = 0$, the term with respect to $R^{w+1,m-1}$ is regarded as zero. a_n is defined as

$$a_n \equiv \frac{J_X^2(J_Y^{2(n+2)} - J_Z^{2(n+2)}) + J_Y^2(J_Z^{2(n+2)} - J_X^{2(n+2)}) + J_Z^2(J_X^{2(n+2)} - J_Y^{2(n+2)})}{(J_X^2 - J_Y^2)(J_Y^2 - J_Z^2)(J_Z^2 - J_X^2)}. \tag{18}$$

By following this recursive way, all the other $g_{n-\tilde{n}}^{w,m}$'s are determined. a_n is characterized as the coefficient of u^2 in the remainder of the division of a monomial u^{n+2} by $(u - J_X^2)(u - J_Y^2)(u - J_Z^2)$, and plays an important role in our proof that this recursive way is correct [30]. We note that even if $J_X = J_Y$, a_n does not diverge by the characterization of a_n. a_n is also obtained from the recurrence relation $a_{n+3} = (J_X^2 + J_Y^2 + J_Z^2)a_{n+2} - (J_X^2 J_Y^2 + J_Y^2 J_X^2 + J_Z^2 J_X^2)$ $a_{n+1} + J_X^2 J_Y^2 J_Z^2 a_n$, $a_{-2} = a_{-1} = 0$, and $a_0 = 1$.

In the case of the XXX chain ($J_X = J_Y = J_Z = 1$), the pattern of $R^{w,m}$ becomes more simple [30]. It is satisfied that $R^{w,m} = R^{w+1,m-1} + R^{w+1,m+1}$ for $m \geq 1$, which reproduces the known structure called a Catalan tree in Refs. [25,26].

Explicit expression for Q_k.—By solving the recursive way discussed above explicitly, we have obtained the explicit expression for Q_k [30]. The solution for $g_{n-\tilde{n}}^{w,m} \equiv g_{n-\tilde{n}}^{k-2n-m,m}$ is k independent and given by

$$g_{n-\tilde{n}}^{w,m} = f(n - \tilde{n}, m + \tilde{n}), \tag{19}$$

where f is defined as $f(0, m) \equiv 1$ and

$$f(n, m) \equiv \frac{m}{n+m} \sum_{p=1}^{n} \binom{n+m}{p} \sum_{\substack{j1,j2,\ldots,jp \geq 1 \\ j1+j2+\cdots+jp=n}} a_{j1} a_{j2} \cdots a_{jp} \tag{20}$$

for $n \geq 1$.

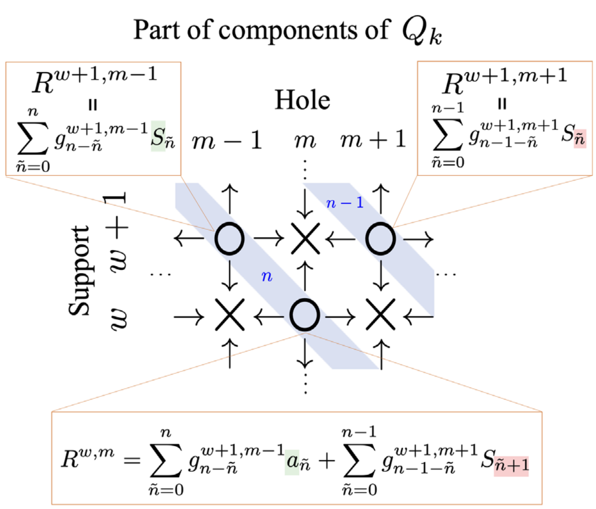

FIG. 2. Recursive way to obtain the function $R^{w,m}$ in Eq. (15). When $R^{w+1,m-1} = \sum_{\tilde{n}=0}^{n} g_{n-\tilde{n}}^{w+1,m-1} S_{\tilde{n}}$ and $R^{w+1,m+1} = \sum_{\tilde{n}=0}^{n-1} g_{n-1-\tilde{n}}^{w+1,m+1} S_{\tilde{n}}$ are determined, then $R^{w,m}$ is determined as the sum of two terms; the first term is obtained by the replacement $S_{\tilde{n}}$ to $a_{\tilde{n}}$ in $R^{w+1,m-1}$, and the second term is obtained by the replacement $S_{\tilde{n}}$ to $S_{\tilde{n}+1}$ in $R^{w+1,m+1}$.

For $k \leq 6$, the explicit expression for Q_k was calculated in Ref. [26]. Here, as an example, we present the coefficients of 0-hole operators in Q_8. They are given as $q^{8,0}(\overline{A_1 A_2 \cdots A_7}) = s(A_1 A_2 \cdots A_7) \prod_{j=1}^{7} J_{A_j}$, $q^{6,0}(\overline{A_1 A_2 \cdots A_5}) = s(A_1 A_2 \cdots A_5)(\prod_{j=1}^{5} J_{A_j}) \sum_{j=1}^{5} J_{A_j}^2$, $q^{4,0}(\overline{A_1 A_2 A_3}) = s(A_1 A_2 A_3)(\prod_{j=1}^{3} J_{A_j})[J_{A_1}^4 + J_{A_2}^4 + J_{A_3}^4 + J_{A_1}^2 J_{A_2}^2 + J_{A_2}^2 J_{A_3}^2 + J_{A_3}^2 J_{A_1}^2 + (J_X^2 + J_Y^2 + J_Z^2) \sum_{j=1}^{3} J_{A_j}^2]$, and $q^{2,0}(\overline{A_1}) = J_{A_1}[J_{A_1}^6 + 2(J_X^2 + J_Y^2 + J_Z^2)J_{A_1}^4 + (2J_X^4 + 2J_Y^4 + 2J_Z^4 + 3J_X^2 J_Y^2 + 3J_Y^2 J_Z^2 + 3J_Z^2 J_X^2)J_{A_1}^2]$.

Let us compare some properties of $Q_{2k'}$ and $Q_{2k'+1}$, where k' is a positive integer. We first consider the time-reversal symmetry. $Q_{2k'(+1)}$ consists of the sum of $(2k'(+1) - 2n - m, m)$ operators as shown in Eq. (14). Since $(2k'(+1) - 2n - m, m)$ operators act on an even (odd) number of sites as the Pauli matrices, these operators are (anti-)symmetric under time reversal. Therefore, $Q_{2k'(+1)}$ is (anti-)symmetric under time reversal. We next consider the similarity between $Q_{2k'}$ and $Q_{2k'+1}$. Since $\lfloor 2k'/2 \rfloor = \lfloor (2k'+1)/2 \rfloor = k'$, n and m included in the summation of $Q_{2k'}$ and $Q_{2k'+1}$ in Eq. (14) take the same values. For this reason, the coefficients of the operators in $Q_{2k'}$ and $Q_{2k'+1}$ have a similar structure.

We note that even if one or two coupling constants are zero, Q_k we obtained can be used. However, Q_k may be a conserved quantity multiplied by coupling constants set to zero, and it needs to be divided by the coupling constants in this case. Another difference from the case that all the coupling constants are nonzero is that k-support conserved quantities are not unique even for $2 \leq k \leq L/2$. In fact, it is known that there is another family of local conserved quantities in addition to Q_k's [26,31–34]. For example, in the case of $J_Z = 0$, $\sum_i (X_i Y_{i+1} - Y_i X_{i+1})$ is another two-support conserved quantity in addition to the Hamiltonian $Q_2 = H$ itself.

Commutativity with a magnetic field in the case of the XXZ chain.—In the case of $J_X = J_Y$, we confirm that $[Q_k, \sum_i Z_i] = 0$, i.e., Q_k is also conserved in the XXZ spin-1/2 chain with a magnetic field in the z-axis direction. Since it is known that the transfer matrix of the XXZ spin-1/2 chain commutes with the magnetic field $[T(\lambda), \sum_i Z_i] = 0$ [11], local conserved quantities obtained from the expansion of $\ln T(\lambda)$ in terms of λ also commute with the magnetic field. In addition, the uniqueness of k-support conserved quantities for $2 \leq k \leq L/2$ also holds in the presence of the magnetic field because commutation relations of the magnetic field and k-support operators generate no $(k+1)$-support operators. Therefore, Q_k is a linear combination of local conserved quantities obtained from the transfer matrix method, and $[Q_k, \sum_i Z_i] = 0$ is satisfied. We note that one can prove this commutativity explicitly by using Eqs. (14)–(20) [30].

Case of $L/2 < k \leq L$.—In the case of $L/2 < k \leq L$, a different point from the case of $2 \leq k \leq L/2$ is that commutators of different support operators can be cancelled. In this case, the conditions we impose in the above discussion for $2 \leq k \leq L/2$ become not necessary but sufficient for $[Q_k, H] = 0$. Therefore, the Q_k we obtain is also conserved for $L/2 < k \leq L$, although it is not necessarily the unique k-support conserved quantity.

Summary and outlook.—We have presented the rigorous explicit expression for k-support conserved quantities in the XYZ spin-1/2 chain $\{Q_k\}$ for $1 \leq k \leq L$. The doubling product is a useful notation to find and express them. By using this notation, we have derived a recursive way to obtain the coefficients of Q_k directly and have found the solution. We have also confirmed that Q_k for $k \geq 2$ is conserved even in the case of the XXZ model with a magnetic field in the z-axis direction. The XXX chain was the exceptional case that the expression is known [25,26], and our result has expanded the scope to general coupling constants. In particular, it enables us to analyze the coupling constants' dependence of the local conserved quantities. Once the expression Eqs. (14)–(20) is obtained, one can easily handle the local conserved quantities both analytically and numerically. Therefore, our result may serve as a new tool for the study of nonequilibrium phenomena in interacting integrable spin chains.

We thank H. Tsunetsugu and H. Katsura for fruitful discussion and comments on the manuscript. K. F. acknowledges support by the Forefront Physics and Mathematics Program to Drive Transformation (FoPM), WINGS Program, the University of Tokyo.

*y.nozawa@issp.u-tokyo.ac.jp

[1] T. Kinoshita, T. Wenger, and D. S. Weiss, A quantum Newton's cradle, Nature (London) **440**, 900 (2006).

[2] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. **80**, 885 (2008).

[3] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Colloquium: Nonequilibrium dynamics of closed interacting quantum systems, Rev. Mod. Phys. **83**, 863 (2011).

[4] T. Langen, T. Gasenzer, and J. Schmiedmayer, Prethermalization and universal dynamics in near-integrable quantum systems, J. Stat. Mech. (2016) 064009.

[5] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Relaxation in a Completely Integrable Many-Body Quantum System: An Ab Initio Study of the Dynamics of the Highly Excited States of 1D Lattice Hard-Core Bosons, Phys. Rev. Lett. **98**, 050405 (2007).

[6] F. H. L. Essler and M. Fagotti, Quench dynamics and relaxation in isolated integrable quantum spin chains, J. Stat. Mech. (**2016**) 064002.

[7] E. Ilievski, M. Medenjak, and T. Prosen, Quasilocal Conserved Operators in the Isotropic Heisenberg Spin-1/2 Chain, Phys. Rev. Lett. **115**, 120601 (2015).

[8] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, Complete Generalized Gibbs Ensembles in an Interacting Theory, Phys. Rev. Lett. **115**, 157201 (2015).

[9] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Emergent Hydrodynamics in Integrable Quantum Systems Out of Equilibrium, Phys. Rev. X **6**, 041065 (2016).

[10] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Transport in Out-of-Equilibrium XXZ Chains: Exact Profiles of Charges and Currents, Phys. Rev. Lett. **117**, 207201 (2016).

[11] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, England, 1993).

[12] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Dover Publications, New York, 2007).

[13] M. G. Tetel'man, Lorentz group for two-dimensional integrable lattice systems, Sov. Phys. JETP **55**, 306 (1982).

[14] K. Sogo and M. Wadati, Boost operator and its application to quantum Gelfand-Levitan equation for Heisenberg-Ising chain with spin one-half, Prog. Theor. Phys. **69**, 431 (1983).

[15] H. B. Thacker, Corner transfer matrices and Lorentz invariance on a lattice, Physica (Amsterdam) **18D**, 348 (1986).

[16] B. M. McCoy and T. T. Wu, Hydrogen-bonded crystals and the anisotropic heisenberg chain, Il Nuovo Cimento B (1965–1970) **56**, 311 (1968).

[17] B. Sutherland, Two-dimensional hydrogen bonded crystals without the ice rule, J. Math. Phys. (N.Y.) **11**, 3183 (1970).

[18] R. J. Baxter, Eight-Vertex Model in Lattice Statistics, Phys. Rev. Lett. **26**, 832 (1971).

[19] R. J. Baxter, One-Dimensional Anisotropic Heisenberg Chain, Phys. Rev. Lett. **26**, 834 (1971).

[20] R. J. Baxter, Partition function of the eight-vertex lattice model, Ann. Phys. (N.Y.) **70**, 193 (1972).

[21] R. J. Baxter, One-dimensional anisotropic Heisenberg chain, Ann. Phys. (N.Y.) **70**, 323 (1972).

[22] R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Some fundamental eigenvectors, Ann. Phys. (N.Y.) **76**, 1 (1973).

[23] R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. II. Equivalence to a generalized ice-type lattice model, Ann. Phys. (N.Y.) **76**, 25 (1973).

[24] R. J. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. III. Eigenvectors of the transfer matrix and hamiltonian, Ann. Phys. (N.Y.) **76**, 48 (1973).

[25] M. P. Grabowski and P. Mathieu, Quantum integrals of motion for the Heisenberg spin chain, Mod. Phys. Lett. A **09**, 2197 (1994).

[26] M. P. Grabowski and P. Mathieu, Structure of the conservation laws in quantum integrable spin chains with short range interactions, Ann. Phys. (N.Y.) **243**, 299 (1995).

[27] X. Zotos, F. Naef, and P. Prelovsek, Transport and conservation laws, Phys. Rev. B **55**, 11029 (1997).

[28] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, 1999).
[29] N. Shiraishi, Proof of the absence of local conserved quantities in the XYZ chain with a magnetic field, Europhys. Lett. **128**, 17002 (2019).
[30] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.090602 for the derivation of the explicit expression for local conserved quantities and more detailed discussions of the cases of the *XXZ* chain and the *XXX* chain.
[31] M. Grady, Infinite set of conserved charges in the Ising model, Phys. Rev. D **25**, 1103 (1982).
[32] E. V. Gusev, Limit states and potentials of planar Heisenberg dynamics with transverse magnetic field, Theor. Math. Phys. **53**, 1018 (1982).
[33] H. Itoyama and H. B. Thacker, Integrability and virasoro symmetry of the noncritical baxter/ising model, Nucl. Phys. **B320**, 541 (1989).
[34] H. Araki, Master symmetries of the XY model, Commun. Math. Phys. **132**, 155 (1990).