Viscous Properties of a Degenerate One-Dimensional Fermi Gas

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We study the viscous properties of a system of weakly interacting spin-$\frac{1}{2}$ fermions in one dimension. Accounting for the effect of interactions on the quasiparticle energy spectrum, we obtain the bulk viscosity of this system at low temperatures. Our result is valid for frequencies that are small compared with the rate of fermion backscattering. For frequencies larger than this exponentially small rate, the excitations of the system become decoupled from the center of mass motion, and the fluid is described by two-fluid hydrodynamics. We calculate the three transport coefficients required to describe viscous dissipation in this regime.

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Hydrodynamics is a classical description of the mechanical and thermal properties of a fluid near equilibrium [1]. The application of hydrodynamics to low-dimensional quantum liquids has provoked a great deal of excitement and has important implications for both experiment [2–5] and theory [6–8]. Applied to the collective behavior of electrons in quantum wires or carbon nanotubes, hydrodynamics would potentially offer new insight into the transport properties of these systems [9,10]. From a theoretical perspective, reconciling the behavior of a one-dimensional (1D) quantum liquid with that of a classical dissipative fluid raises important questions.

In particular, many quantities of interest are directly related to dissipation. The dissipative dynamics of a 1D fluid is characterized by two transport coefficients, the thermal conductivity and the bulk viscosity. Ultimately, the calculation of these parameters requires input from a microscopic model. Much progress has been made in understanding 1D systems through the study of integrable models [11]. However, integrability precludes the relaxation of excitations, and thus, these models are incapable of accounting for dissipative effects in real fluids [12,13]. For the particular case of spinless quantum liquids in one dimension, however, the bulk viscosity and thermal conductivity can be evaluated [14,15] within the Tomonaga-Luttinger liquid framework [16,17].

One-dimensional systems of spin-$\frac{1}{2}$ fermions generally relax much more rapidly than their spinless counterparts. Thus, the results [14,15] for the transport coefficients of spinless systems do not apply to those with spin. Given the importance of systems of spin-$\frac{1}{2}$ fermions, a means of calculating their transport coefficients would be beneficial. Unfortunately, spin-charge separation [17–19] frustrates the application of the Tomonaga-Luttinger liquid theory for this purpose. On the other hand, the regime in which the transport coefficients are largest is, in fact, theoretically accessible. Indeed, transport coefficients are proportional to the relaxation time of the system [20] and, thus, are large for weakly interacting systems. Furthermore, as long as the relevant energy scale—in this case, temperature—is large compared with the interactions, the effect of spin-charge separation can be neglected [21]. For these reasons, in this Letter, we study the case of weakly interacting spin-$\frac{1}{2}$ fermions.

The transport coefficients of 1D systems reflect their unique relaxation properties. A 1D Fermi gas at low temperatures exhibits two disparate relaxation rates [22]. The slowest relaxation process involves backscattering of particles, in which, say, a right mover is converted to a left mover. For such a process to occur, a hole must pass through the bottom of the band. Hence, the rate for these processes takes the activated form $1/\tau \sim e^{-E_F/T}$, where E_F is the Fermi energy and T is the temperature. On the other hand, typical particle-hole excitations relax much more rapidly, with a characteristic rate $1/\tau_{\mathrm{ex}}$ that scales as a power of T.

Transport coefficients are associated with specific perturbations of the system. The thermal conductivity κ quantifies the heat current that arises from the application of a temperature gradient. Since currents in one dimension are associated with an imbalance between the right and left movers, the thermal conductivity is dominated by fermion backscattering, i.e., $\kappa \propto \tau$ [15,22–24]. The bulk viscosity ζ, on the other hand, is a measure of the amount of entropy generated by a change in the fluid density—a perturbation that affects right and left movers equally. This perturbation creates particle-hole excitations, and thus, $\zeta \propto \tau_{\mathrm{ex}}$ [14,25]. The calculation of ζ is, in general, more challenging than that of κ since particle-hole excitations exhibit a spectrum of relaxation rates, whereas κ is dominated by a single rate

[23]. The study of the bulk viscosity of a 1D gas of spin-$\frac{1}{2}$ particles is our main goal. Taken together with an understanding of thermal transport, our results give a complete hydrodynamic description of a 1D gas of spin-$\frac{1}{2}$ fermions.

The vast majority of experimentally relevant 1D systems of fermions, including quantum wires in GaAs as well as cold atomic gases, exhibits single particle energy spectra that are quadratic in momentum. It is well known that a straightforward calculation of the bulk viscosity in this case gives $\zeta = 0$ [20,26]. This presents a theoretical challenge which we overcome by properly accounting for the renormalization of the energy spectrum by the interactions.

Classical hydrodynamics describes the properties of the system at frequencies ω that are small compared with the slowest relaxation rate. In our system, the slowest relaxation process is fermion backscattering, and therefore, this condition is $\omega \ll 1/\tau$. The presence of two disparate scales of relaxation rates, $1/\tau$ and $1/\tau_{\mathrm{ex}}$, ensures that there exists a broad frequency range $1/\tau \ll \omega \ll 1/\tau_{\mathrm{ex}}$. In this regime, particle hole excitations, due to their rapid equilibration, behave as a gas moving with a well-defined velocity. In the absence of backscattering, this velocity can differ from the velocity of the center of mass. Therefore, the system is properly described by two-fluid hydrodynamics, similar to the theory developed for superfluid ^{4}He [27]. This conclusion is central to recent theoretical work on the superfluidlike behavior displayed by 1D liquids at finite frequencies [28,29]. In two-fluid hydrodynamics, bulk viscosity is described by three transport coefficients. We obtain analytic expressions for these quantities.

We start by considering the thermodynamic equilibrium state of a noninteracting 1D Fermi gas. In the absence of a magnetic field, the spins are degenerate, and the occupation numbers of the fermion states depend only on the momentum

$$n_p^{(0)} = \frac{1}{\exp\left(\frac{\epsilon_p - up - \mu}{T}\right) + 1}. \tag{1}$$

Here, $\epsilon_p = p^2/2m$ is the energy of the fermion with momentum p, while μ is the chemical potential. The appearance of the term $-up$ in Eq. (1) is dictated by the conservation of momentum in a uniform system. The physical meaning of the parameter u is the velocity of the gas.

In the presence of an infinitesimal gradient of velocity $\partial_x u$, weak interactions in the Fermi gas lead to scattering of particles, resulting in dissipation. The power W dissipated in the system is

$$W = \zeta L (\partial_x u)^2, \tag{2}$$

where ζ is the bulk viscosity and L is the system size [1]. Below, we use Eq. (2) to evaluate ζ.

To obtain the power dissipated in the Fermi gas, we employ the standard expression for the entropy $S = -2\sum_p [n_p \ln n_p + (1-n_p)\ln(1-n_p)]$ in terms of the occupation numbers n_p. Differentiation of S with respect to time gives the entropy production rate

$$\dot{S} = -2\sum_p \dot{n}_p \ln\frac{n_p}{1-n_p}. \tag{3}$$

Substitution of the unperturbed occupation numbers (1) for n_p within the logarithm in Eq. (3) results in $\dot{S} = 0$ by virtue of the laws of conservation of the number of particles, momentum, and energy. On the other hand, a small velocity gradient generates a correction to the occupation numbers $\delta n_p \propto \partial_x u$. Substituting $n_p = n_p^{(0)} + \delta n_p$ into Eq. (3) and expanding to first order in δn_p, one finds the dissipation rate $W = T\dot{S}$ in the form

$$W = -2T\sum_p \frac{\dot{n}_p \delta n_p}{n_p^{(0)}(1-n_p^{(0)})}. \tag{4}$$

Both $\dot{n}_p$ and δn_p are proportional to the perturbation $\partial_x u$. Therefore, $W \propto (\partial_x u)^2$, as expected from Eq. (2).

We will obtain $\dot{n}_p$ and δn_p by using the Boltzmann equation, which can be written as a combination of the relation

$$\dot{n}_p = \partial_t n_p + (\partial_p \epsilon_p)\partial_x n_p, \tag{5}$$

with the expression for $\dot{n}_p$ in terms of the collision integral, $\dot{n}_p = I[n_p]$. Because the correction δn_p is small, to leading order one can use unperturbed occupation numbers $n_p^{(0)}$ in the right-hand side of Eq. (5). The correction δn_p should then be found by solving $\dot{n}_p = I[n_p^{(0)} + \delta n_p]$.

A nonzero gradient of velocity $\partial_x u$ of the gas results in a time-dependent density of particles, which, in turn, gives rise to a time-dependent chemical potential $\mu(t)$ and temperature $T(t)$. Keeping this in mind, substitution of the unperturbed occupation numbers (1) into Eq. (5) yields

$$\dot{n}_p = \frac{1}{T} n_p^{(0)}(1-n_p^{(0)})\left[\frac{\partial_t T}{T}\left(\frac{p^2}{2m} - \mu\right) + \partial_t \mu + \frac{p^2}{m}\partial_x u\right]. \tag{6}$$

Here, for simplicity, we have chosen a point in space where $u = 0$. Because the collisions of particles leading to the equilibration of the system conserve particle number and energy, $\dot{n}_p$ must satisfy the conditions

$$\int \dot{n}_p dp = 0, \qquad \int \epsilon_p \dot{n}_p dp = 0. \tag{7}$$

These constraints enable one to obtain the rates of change of the temperature and chemical potential,

$$\frac{\partial_t T}{T} = \frac{\partial_t \mu}{\mu} = -2\partial_x u. \tag{8}$$

The substitution of Eq. (8) into Eq. (6) yields $\dot{n}_p = 0$.

This conclusion implies that the dissipation rate (4) vanishes, and thus, the bulk viscosity $\zeta = 0$. This can be understood as follows [14]. Consider a system with a fixed number of particles N in a box of size $L(t)$ such that $N/L = n$. From the standard continuity equation for the particle density n, we conclude that its time dependence is given by $(\partial_t n)/n = -\partial_x u$. Then the gradient of velocity can be related to the time derivative of the system size, $(\partial_t L)/L = \partial_x u$. Given that the energy levels $\epsilon_p = p^2/2m$ are multiples of $(2\pi\hbar/L)^2/2m$, we conclude that $(\partial_t \epsilon_p)/\epsilon_p = -2\partial_x u$. Equation (8) then indicates that the temperature and chemical potentials change at the same rate as ϵ_p, and the ratio $(\epsilon_p - \mu)/T$ in the Fermi-Dirac distribution Eq. (1) for $u = 0$ remains unchanged. Thus, the perturbation $\partial_x u$ does not drive the system out of equilibrium, resulting in no dissipation, and $\zeta = 0$. An analogous result for a classical ideal gas is well known [20].

The above argument requires that the particle energies scale as $\epsilon_p \propto p^2$ (or, more precisely, as any power of $|p|$). In an interacting system the fermion energies are affected by other particles, and the scaling $\epsilon_p \propto p^2$ no longer holds. To account for this effect, we consider the usual two-particle interactions described by the Hamiltonian

$$\hat{V} = \frac{1}{2L}\sum_{\substack{p,p',q\\ \sigma,\sigma'}} V(q) a^\dagger_{p+q,\sigma} a^\dagger_{p'-q,\sigma'} a_{p',\sigma'} a_{p,\sigma}. \tag{9}$$

Here, $V(q)$ is the Fourier transform of the interaction potential, and $a_{p,\sigma}$ is the annihilation operator of a fermion with momentum p and a z component of spin σ. Assuming that the interactions are weak, we will limit our treatment to first order perturbation theory in $\hat{V}$. In this approximation, the energy of the state with occupation numbers $n_{p,\sigma}$ has the form

$$E = \sum_{p,\sigma} \frac{p^2}{2m} n_{p,\sigma} + \frac{1}{2L}\sum_{\substack{p,p'\\ \sigma,\sigma'}} [V(0) - V(p-p')\delta_{\sigma,\sigma'}] n_{p,\sigma} n_{p',\sigma'}. \tag{10}$$

Since the energy of the many-body state is a functional of the occupation numbers $n_{p,\sigma}$, the quasiparticle energies can be obtained as $\epsilon_{p,\sigma} = \delta E/\delta n_{p,\sigma}$, resulting in

$$\epsilon_p = \frac{p^2}{2m} + \int \frac{dp'}{2\pi\hbar}[2V(0) - V(p-p')] n_{p'}, \tag{11}$$

where we have assumed spin degeneracy and omitted the spin indices. The energy spectrum (11) is no longer quadratic in p. For a generic interaction, this will result in nonvanishing $\dot{n}_p$, which we will evaluate to first order in $\hat{V}$.

It is worth noting that the low-energy properties of one-dimensional systems of interacting fermions are usually described within the framework of Luttinger liquid theory [17], in which the elementary excitations have bosonic statistics. On the other hand, it was shown in Ref. [21] that, for weak interactions, the curvature of the spectrum suppresses the Luttinger liquid effects, and the simple perturbative treatment of interactions is appropriate. For particles with energies of the order of T, the criterion of Ref. [21] is $p_F V(0)/\hbar \ll T$, where $p_F = \sqrt{2m\mu}$ is the Fermi momentum [30]. To account for the effects of interactions in the Boltzmann equation formalism, we notice that the first-order expressions for the energy of the system (10) and quasiparticle energy (11) are consistent with Fermi liquid theory [31]. The evaluation of the transport coefficients in this approach was performed in Refs. [26,32]. Below, we simplify and adapt the evaluation [26] of $\dot{n}_p$ induced by a small gradient of velocity to the case of one dimension and weak interactions.

To proceed, we observe that Eqs. (1)–(5) are still applicable, provided that the quasiparticle energies ϵ_p include the Fermi liquid corrections [26,32]. Evaluation of $\dot{n}_p$ should now allow for the possibility of ϵ_p depending on T and μ, which enter via the occupation numbers in Eq. (11). Then, substitution of Eq. (1) for n_p in the right-hand side of Eq. (5) yields

$$\dot{n}_p = \frac{1}{T} n_p^{(0)}(1 - n_p^{(0)}) \left[\left(\frac{\epsilon_p - \mu}{T} - \frac{\partial \epsilon_p}{\partial T} \right) \partial_t T + \left(1 - \frac{\partial \epsilon_p}{\partial \mu} \right) \partial_t \mu + p(\partial_p \epsilon_p) \partial_x u \right]. \tag{12}$$

Now, we substitute Eq. (11) for ϵ_p and obtain $\dot{n}_p$ in linear order in the interaction potential. The values of time derivatives $\partial_t T$ and $\partial_t \mu$ are fixed by the conservation laws (7). For quasiparticles with energies near the Fermi level, $|\epsilon_p - \mu| \sim T$, to leading order in temperature, we find

$$\dot{n}_p = \frac{\gamma}{4\mu T} n_p^{(0)}(1 - n_p^{(0)}) \left[v_F^2 (|p| - p_F)^2 - \frac{\pi^2 T^2}{3} \right] \partial_x u. \tag{13}$$

Here, the dimensionless parameter

$$\gamma = \frac{V(0) - V(2p_F) + 2p_F V'(2p_F) - 2p_F^2 V''(2p_F)}{2\pi\hbar v_F} \tag{14}$$

characterizes the strength of interactions, and $v_F = \sqrt{2\mu/m}$ is the Fermi velocity.

In order to obtain the dissipation rate (4), one should find a small correction δn_p to the equilibrium distribution function (1) by inverting the collision integral: $\dot{n}_p = I[n_p^{(0)} + \delta n_p]$. For small $\delta n_p \propto \partial_x u$, the latter can

be linearized. The linearized collision integral for 1D spin-$\frac{1}{2}$ fermions was studied in Ref. [30]. Remarkably, in the low-temperature regime, the correction to $n_p^{(0)}$ with momentum dependence of Eq. (13) is an eigenmode of the collision integral, with the relaxation rate

$$\frac{1}{\tau_2} = \frac{9[V(0)V(2p_F) - V(2p_F)^2 - 2p_F V(0)V'(2p_F)]^2}{64\pi^3\hbar^5 v_F^4} T. \tag{15}$$

The latter statement means that, to leading order in $T/\mu \ll 1$, the naive relaxation time approximation $\dot{n}_p = -\delta n_p/\tau_2$ is exact.

Next, we substitute $\delta n_p = -\tau_2 \dot{n}_p$ and Eq. (13) into the expression (4) for the dissipation rate and use Eq. (2) to obtain the bulk viscosity

$$\zeta = \frac{2\pi^3}{45}\frac{\gamma^2 T^4 \tau_2}{\hbar v_F \mu^2}. \tag{16}$$

This result in combination with Eqs. (14) and (15) gives a microscopic expression for the bulk viscosity of the degenerate 1D gas of spin-$\frac{1}{2}$ fermions. Given the temperature dependence of the relaxation time $\tau_2 \propto 1/T$, we conclude that $\zeta \propto T^3$.

Our result (16), derived assuming a time-independent perturbation $\partial_x u$, is applicable at frequencies $\omega \ll 1/\tau$. Now, we consider the bulk viscosity of the system at frequencies in the range $1/\tau \ll \omega \ll 1/\tau_{\rm ex}$, where the backscattering rate is exponentially small, $1/\tau \propto e^{-E_F/T}$, and the quasiparticle relaxation rate $1/\tau_{\rm ex} = 1/\tau_2 \propto T$. As discussed above, in this regime, the system is described by two-fluid hydrodynamics originally developed for superfluid ^{4}He [27] and adapted to one dimension [29]. The rate of viscous dissipation in this theory is controlled by three transport coefficients, ζ_1, ζ_2, and ζ_3

$$\frac{W}{L} = \zeta_2(\partial_x v_n)^2 + \zeta_3[\partial_x(j - \rho v_n)]^2 + 2\zeta_1[\partial_x(j - \rho v_n)](\partial_x v_n). \tag{17}$$

Here, v_n is the velocity of the normal component of the fluid, j is the mass current, and ρ is the mass density.

To obtain microscopic expressions for the bulk viscosities in Eq. (17) for the 1D Fermi gas, first, we notice that, in the two-fluid regime, one can assume $1/\tau = 0$, thereby neglecting the backscattering of fermions. Then, the numbers of the right- and left-moving fermions are conserved, and instead of μ, the occupation numbers are described by two chemical potentials $\mu_{R,L} = \mu \pm \delta\mu/2$

$$n_p^{(0)} = \frac{1}{\exp\left(\frac{\epsilon_p - up - \mu - (\delta\mu/2)\mathrm{sgn}p}{T}\right) + 1}. \tag{18}$$

For $\delta\mu \neq 0$, the center of mass velocity of the Fermi gas is different from the velocity u of the gas of elementary excitations.

Next, we relate the parameters of the distribution function (18) to v_n and j in Eq. (17). The gas of particle-hole excitations plays the role of the normal component of the fluid [28,29], and thus, $v_n = u$. Then, using Eq. (18), we express the mass current in terms of u and $\delta\mu$

$$j = \rho u + \frac{m}{\pi\hbar}\delta\mu. \tag{19}$$

The form of the first term is dictated by the Galilean invariance of the system. The second term is the mass current analog of the well-known Landauer formula $I = (e^2/\pi\hbar)V$ for the electric current $I = ej/m$ in terms of voltage $V = \delta\mu/e$. Thus, Eq. (19) yields $j - \rho v_n = (m/\pi\hbar)\delta\mu$.

To obtain the dissipation rate in the Fermi gas, we repeat the steps leading to Eq. (13) for $\dot{n}_p$, while using the unperturbed distribution $n_p^{(0)}$ in the form (18) and allowing for small gradients $\partial_x u$ and $\partial_x \delta\mu$. To linear order in the gradients, we obtain

$$\dot{n}_p = \frac{1}{4\mu T} n_p^{(0)}(1 - n_p^{(0)})\left[v_F^2(|p| - p_F)^2 - \frac{\pi^2 T^2}{3}\right] \times \left(\gamma\partial_x u - \frac{1}{2p_F}\partial_x\delta\mu\right). \tag{20}$$

Substituting Eq. (20) along with $\delta n_p = -\tau_2 \dot{n}_p$ into Eq. (4), we obtain the rate of dissipation in a 1D Fermi gas in the two-fluid regime. Replacing $u = v_n$ and $\delta\mu = (\pi\hbar/m)(j - \rho v_n)$ in the resulting expression gives Eq. (17) with

$$\zeta_1 = -\frac{\zeta}{\rho\gamma}, \qquad \zeta_2 = \zeta, \qquad \zeta_3 = \frac{\zeta}{(\rho\gamma)^2}, \tag{21}$$

where ζ is given by Eq. (16) and we have applied the low-temperature expression $\rho = 2mp_F/\pi\hbar$. The result $\zeta_2 = \zeta$ follows immediately from the fact that, in the single-fluid regime, $\delta\mu = 0$. Indeed, in this case, Eq. (19) yields $j = \rho v_n$, and Eq. (17) is identical to Eq. (2).

To assess the relative importance of ζ_1, ζ_2, and ζ_3, we compare the quantities $\rho\zeta_1$, ζ_2, and $\rho^2\zeta_3$, which all have the same dimension. In the limit of weak interactions considered here, $\gamma \ll 1$, they are very different in magnitude: $\rho^2\zeta_3 \gg \rho|\zeta_1| \gg \zeta_2$. This result is related to our earlier observation that the nonequilibrium response $\dot{n}_p$ to a small gradient $\partial_x u$ vanishes in the absence of interactions. This subtle feature of systems of particles with quadratic spectra does not apply to the response to the gradient $\partial_x\delta\mu$ in the two-fluid regime, resulting in $\rho^2\zeta_3 \gg \zeta_2$. An important application of our result (21) is to understand the attenuation of sound modes, which, in the two-fluid regime, is controlled by the parameter $\tilde{\zeta} = \zeta_2 - 2\rho\zeta_1 + \rho^2\zeta_3$ [29].

Our result (21) indicates that, for weakly interacting fermions, the first two contributions are negligible and to leading order $\tilde{\zeta} = \rho^2\zeta_3$.

We have focused on the experimentally relevant and theoretically challenging case of a quadratic single-particle spectrum. If the spectrum is not quadratic, the effect of weak interactions on the spectrum need not be considered. In this case, we expect that ζ will have a form similar to Eq. (16) without the small parameter γ. In particular, it will have the same temperature dependence as our result. Finally, the approach presented here is also applicable to the case of spinless electrons. We have verified that the results for the bulk viscosity would be consistent with those of Ref. [14] in the regime of weak interactions.

To summarize, we studied viscous dissipation in a 1D gas of spin-$\frac{1}{2}$ fermions. At the lowest frequencies $\omega \ll 1/\tau$, the gas can be described by classical hydrodynamics, and its bulk viscosity is given by our result (16). At frequencies above the backscattering rate, $1/\tau \ll \omega \ll 1/\tau_{\mathrm{ex}}$, two-fluid hydrodynamics is applicable, in which the viscous effects are described by three transport coefficients. Our analytic expressions for these coefficients are given by Eq. (21). Our results are valid in the broad temperature range $p_F V(0)/\hbar \ll T \ll E_F$.

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