Superdiffusion from Emergent Classical Solitons in Quantum Spin Chains

Jacopo De Nardis,[1] Sarang Gopalakrishnan,[2] Enej Ilievski,[3] and Romain Vasseur[4]
[1]Department of Physics and Astronomy, University of Ghent, Krijgslaan 281, 9000 Gent, Belgium
[2]Department of Physics and Astronomy, CUNY College of Staten Island, Staten Island, New York 10314;
Physics Program and Initiative for the Theoretical Sciences, The Graduate Center, CUNY, New York, New York 10016, USA
[3]Faculty for Mathematics and Physics, University of Ljubljana, Jadranska ulica 19, 1000 Ljubljana, Slovenia
[4]Department of Physics, University of Massachusetts, Amherst, Massachusetts 01003, USA

(Received 6 April 2020; accepted 8 July 2020; published 12 August 2020)

Finite-temperature spin transport in the quantum Heisenberg spin chain is known to be superdiffusive, and has been conjectured to lie in the Kardar-Parisi-Zhang (KPZ) universality class. Using a kinetic theory of transport, we compute the KPZ coupling strength for the Heisenberg chain as a function of temperature, directly from microscopics; the results agree well with density-matrix renormalization group simulations. We establish a rigorous quantum-classical correspondence between the "giant quasiparticles" that govern superdiffusion and solitons in the classical continuous Landau-Lifshitz ferromagnet. We conclude that KPZ universality has the same origin in classical and quantum integrable isotropic magnets: a finite-temperature gas of low-energy classical solitons.

DOI: 10.1103/PhysRevLett.125.070601

Introduction.—The dynamics of isolated many-body systems exhibits a remarkable diversity, which we have only begun to understand in the past decade [1–3]. Dynamics in one dimension is particularly rich, as experimental and theoretical studies have shown. Although experiments often deal with systems far from equilibrium [4–13], from a theoretical perspective it is most natural to characterize dynamics in the linear regime about equilibrium states. Linear response can be probed via transport experiments [14,15] or by measuring dynamical correlations [16]. Generically, the densities of conserved quantities in lattice models undergo diffusion, as predicted by linearized hydrodynamics [17]. Integrable and many-body localized systems, however, have infinitely many local conserved charges, so simple hydrodynamic arguments fail. Transport is absent in localized systems [18–21] and in general ballistic in integrable systems [3], though anomalous transport, including both *subdiffusion* [22–24] and *superdiffusion* [3,25,26], has also been observed. The mechanisms underlying anomalous diffusion occurs remain an active open question.

The phenomenon of spin superdiffusion in the quantum Heisenberg spin chain, discovered in [25], has recently been confirmed in a number of numerical studies with tensor network simulations [27–29], and then addressed [30–35] using the framework of generalized hydrodynamics (GHD) [31,33,36–61], which extends hydrodynamics to integrable systems. The observed anomalous diffusion was initially attributed to particular properties of interacting quasiparticle excitations in the Heisenberg chain [30–32]. More recent studies, however, uncovered the presence of universal Kardar-Parisi-Zhang (KPZ) dynamics in a wide class of quantum [29] and classical [62–64] Hamiltonian systems, together with other types of superdiffusion [65,66]. These include, among others, models which are directly relevant for cold atoms experiments such as the Fermi-Hubbard chain [67,68]. At the same time, even if the KPZ equation was originally introduced to describe classical stochastic growth phenomena [69], its large dynamical universality class has recently incorporated also noisy quantum systems, as random unitary models [70] and spin chains with noise [71,72].

Stimulated by previous observations, Refs. [34,35] suggested that absence of normal diffusion originates from the long-wavelength fluctuations of local conserved charges associated with the non-Abelian continuous symmetry of the model. A common theme that emerges from all of these studies is that the excitations responsible for the observed anomalous spin diffusion in the Heisenberg chain are interacting long-wavelength spin fluctuations: either a thermal gas of "giant quasiparticles" [31,32] described by GHD equations or, alternatively, "soft gauge modes" that conventional GHD cannot capture [34,35]. These pictures have complementary advantages: the GHD approach is microscopic, but has not so far been able to reproduce the emergence of the KPZ scaling function, whereas the latter is a field-theoretical phenomenological approach which offers a plausible derivation of the KPZ equation.

In the present Letter, we elucidate the microscopic nature of spin superdiffusion by identifying the "giant quasiparticles" of the Heisenberg model with classical soliton solutions of the Landau-Lifshitz equation [73]. We achieve this systematically through an explicit semiclassical scaling

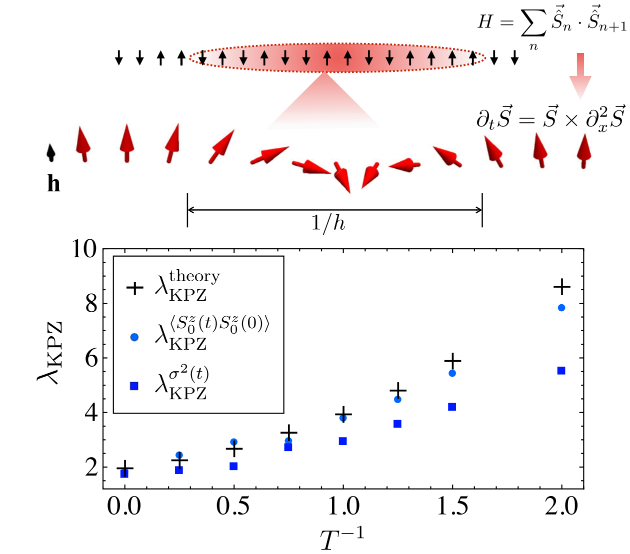

FIG. 1. Upper panel: given an external small magnetic field $h \ll 1$, the quantum-classical mapping identifies quantum eigenstates with large quantum numbers $s \propto 1/h$ of magnons with soliton waves in a classical Landau-Lifshitz magnet. Lower panel: temperature (T)-dependent coefficient of the KPZ nonlinearity predicted by the self-consistent theoretical approach, compared with numerical results from t-DMRG (see also caption of Fig. 2). At finite times, the spin profile is not precisely of the KPZ functional form as in Eq. (2); thus the value of the inferred coefficient largely depends on which quantity one uses to extract to the KPZ prediction, either the spin autocorrelation or its variance.

limit of the thermodynamic Bethe ansatz equations, thereby providing the missing link between the GHD approach and the proposal of Ref. [34]. This allows us to predict not only the correct exponent for superdiffusion but also the numerical value of the temperature-dependent coupling constant of the emergent KPZ dynamics (Fig. 1). We thus show how for integrable classical or quantum isotropic ferromagnets, the data entering the coarse-grained KPZ equation can be derived microscopically.

Model.—We consider the spin-$\frac{1}{2}$ Heisenberg Hamiltonian for a chain of size $L+1$

$$\hat{H} = J \sum_{n=-L/2}^{L/2} \vec{\hat{S}}_n \cdot \vec{\hat{S}}_{n+1}, \tag{1}$$

with $\vec{\hat{S}}_n = (S_n^x, S_n^y, S_n^z)$, and $\hat{S}_n^\alpha$ denoting spin-$\frac{1}{2}$ operators at site n. In what follows, we set $J = 1$. We focus on the spin dynamical structure factor $\langle \hat{S}_n^z(t) \hat{S}_0^z(0) \rangle$ in the thermodynamic limit $L \to \infty$, at thermal equilibrium with finite temperature $T > 0$ and *zero* magnetic field. There is now ample numerical evidence that the structure factor at late times $t \gg 1$ follows the KPZ scaling form

$$\langle \hat{S}_n^z(t) \hat{S}_0^z(0) \rangle \simeq \frac{\chi}{(\lambda_{\rm KPZ} t)^{2/3}} f_{\rm KPZ}\left(\frac{n}{(\lambda_{\rm KPZ} t)^{2/3}}\right), \tag{2}$$

where $\chi = \sum_{n=-L/2}^{L/2} \langle \hat{S}_n^z(0) \hat{S}_0^z(0) \rangle$ is the static spin susceptibility, $f_{\rm KPZ}$ is the KPZ scaling function [74,75], and $\lambda_{\rm KPZ}$ is the KPZ constant: a temperature and model-dependent coupling parameter of the emergent KPZ equation describing the hydrodynamics of the spin field. The exponent $2/3$ can be extracted from GHD via a self-consistent argument [31–33], but the scaling function and $\lambda_{\rm KPZ}$ cannot. Moreover, the framework of nonlinear fluctuating hydrodynamics [76,77], which has been used to derive KPZ equations in other contexts, does not apply straightforwardly in this situation.

Recent numerical results [62], supplemented by theoretical arguments of Refs. [34,35], have presented evidence that the same universal KPZ scaling also occurs at finite temperatures in classical integrable spin chains invariant under SO(3) rotations, whose continuum long-wavelength theory is governed by the Landau-Lifshitz (LL) equation

$$\partial_t \vec{S} = J_{\rm cl} \vec{S} \times \partial_x^2 \vec{S}, \tag{3}$$

where $\vec{S} \equiv \vec{S}(x,t)$ is a classical spin field of unit length $|\vec{S}| = 1$ on the continuum line $x \in \mathbb{R}$. In this light, it is reasonable to expect that such emergent behavior is a manifestation of a quantum-classical correspondence where certain degrees of freedom in the quantum chain are intrinsically classical in nature and behave according to (3), as proposed, e.g., in Ref. [34].

Here, we isolate the excitations relevant for KPZ dynamics. Since these turn out to be bound states of elementary magnonic excitations whose size and quantum numbers diverge as the local magnetization vanishes, we dub them "giant quasiparticles." Our picture, combined with simple kinetic arguments, yields quantitative predictions for the $\lambda_{\rm KPZ}$, and elucidates how a finite thermal density of giant quasiparticles in the spectrum of the quantum chain leads to a thermal gas of classical solitons of the Landau-Lifshitz field theory (3).

Computing the KPZ constant.—The KPZ coupling constant of the quantum Heisenberg model can be computed from the following procedure. First, we consider a thermal Gibbs state with the addition of a small magnetic field $2h$, which introduces the additional term $-2hT \sum_i \hat{S}_i^z$ to $\hat{H}$. Given that the model (1) is integrable, spin dynamics splits into two channels; a ballistic piece with spectral (Drude) weight vanishing at zero field, and a diffusive part with spin diffusion constant diverging as $D(h) = D_0/h$ in the $h \to 0$ limit [32,33]. Both transport coefficients admit closed-form expressions as sums over quasiparticles, labeled by a discrete label $s \geq 1$ (pertaining to the quantized magnetization of magnon excitations) and a continuous rapidity label $\theta \in (-\infty, \infty)$ which parametrizes their quasimomenta $p_s(\theta)$.

The spin diffusion constant assumes a spectral decomposition [32,52,78,79]

$$D = \sum_{s\geq 1} \int_{-\infty}^{\infty} d\theta D_s(\theta), \tag{4}$$

which we will use below to determine D_0.

The second step of our procedure consists of regularizing the divergence of $D(h)$ by accounting that the net magnetization observed by a quasiparticle that has traveled a distance ℓ is not precisely zero, but instead has a residual value $h(\ell)$ set by thermal magnetization fluctuations over the scale ℓ. As noted previously in Ref. [31], the motion of the giant quasiparticles that dominate spin transport is primarily diffusive, so ℓ is itself self-consistently set by $h(\ell)$. These equations relating ℓ and $h(\ell)$ now permit for a quantitative analysis of superdiffusion in terms of $D_0 = \lim_{h\to 0^+} hD$. For $h \ll 1$, this can be though of as the effective field originating from thermal fluctuations, namely $h^2 = m^2/(4\chi)^2$, with m^2 the local spin susceptibility in a interval of size ℓ, $m^2 = 4\chi/\ell$. We then infer $h = 1/\sqrt{4\chi\ell}$, and the lengthscale ℓ can be fixed self-consistently at small, finite h by $\ell^2 = 2Dt = 2D_0\sqrt{4\chi\ell}t$. This yields $\ell = (2D_0\sqrt{4\chi}t)^{2/3}$, and combining gives finally

$$D(t) = 2^{5/3} D_0^{4/3} \chi^{2/3} t^{1/3} +, \ldots. \tag{5}$$

This simple argument already suffices to predict anomalous diffusion with dynamical exponent $z = 3/2$. Remarkably, it also predicts the value of the prefactor. Even though such an approach is arguably heuristic, we wish to emphasize that a similar argument correctly predicts the exact form [31] of the diffusion constant (4) for the easy-axis *XXZ* spin chain, which has been computed by other means [32,52,78,79], so it should be taken seriously. To extract λ_{KPZ} defined in Eq. (2), we compare the variance of the spin profile σ^2 to the variance computed from the KPZ prediction (2). At any finite t there is a finite (diverging) diffusion constant $D(t) \sim t^{1/3}$, which we can define in terms of the spin variance as $\sigma^2 = \chi 2D(t)t$. This readily implies that the full temperature-dependent KPZ constant $\lambda_{\mathrm{KPZ}} \equiv \lambda_{\mathrm{KPZ}}(T)$ is given by

$$\lambda_{\mathrm{KPZ}}(T) = 4D_0(T)\sqrt{\chi(T)}/\sigma_{\mathrm{KPZ}}^{3/2}. \tag{6}$$

Here σ_{KPZ}^2 is the variance of the KPZ function $\sigma_{\mathrm{KPZ}}^2 = \int du u^2 f_{\mathrm{KPZ}}(u) \approx 0.510\,523$. Let us stress again that the above argument does not predict the KPZ scaling function, but it does fix λ_{KPZ} as a function of temperature.

Giant quasiparticles as classical soft solitons.—Our central result is the explicit form (6) for λ_{KPZ} in terms of parameter $D_0(T)$. Now we explain how to explicitly compute it. The following calculation also demystifies the nature of the "giant quasiparticles" responsible for superdiffusion: following previous work [31–33] we anticipate that these are semiclassical quasiparticles carrying large amount of spin $s \sim 1/h$, i.e., macroscopically large bound states of magnons that belong to the low-energy spectrum of the Heisenberg chain. Such states, first described in Refs. [80,81], have received a great deal of attention in the study of gauge-string dualities [82–86]. As explicitly shown in [87,88], semiclassical eigenstates manifest themselves (at the classical level) as solutions to the continuum Landau-Lifshitz model [89].

Our objective here is however not to describe individual classical spin-field configurations but rather find a classical interpretation for the giant quasiparticles immersed in a thermal background. To this end, we identify an appropriate semi-classical limit directly at the level of the thermodynamic Bethe ansatz (TBA) equations. We shall see that this will lead us directly to the classical counterpart of the GHD equations where, remarkably, the small magnetic field h will play the role of an effective Planck constant. With this in mind, we introduce a rescaled rapidity $u = \theta h$ and rescaled quasiparticle magnetization $\xi = sh$. In the limit $h \to 0^+$, we can convert the sum over s in Eq. (4) into an integral, in this way obtaining a fully classical expression for $D_0 \equiv D_0^{\mathrm{cl}}$, with

$$D_0^{\mathrm{cl}} = \int_0^{+\infty} d\xi \int_{-\infty}^{+\infty} du D(\xi, u), \tag{7}$$

and where $D(\xi, u) = \lim_{h\to 0^+}(1/h)D_{\xi/h}(u/h)$ is a finite quantity: D_0 is thus fully determined by quasiparticles with $s \to \infty$ in the limit $h \to 0^+$, with $\xi = sh$ kept fixed.

Next, we consider the scattering phase shifts between two quasiparticles with spin indices s and s' with relative rapidity θ. Here we quote the result of [90,91], $T_{s,s'}(\theta) = (1-\delta_{ss'})a_{|s-s'|}(\theta) + 2a_{|s-s'|+2}(\theta) + \cdots + 2a_{s+s'-2}(\theta) + a_{s+s'}(\theta)$, with $a_s(\theta) \equiv 1/(2\pi)\partial_\theta p_s(\theta) = 1/(2\pi)(4s/s^2 + 4\theta^2)$. Upon rescaling of parameters $\theta \to u/h$ and $s \to \xi/h$, the net phase shift of all the constituent magnons can be resummed into an integral $T_{s,s'} = \int_{h|s-s'|}^{h(s+s')}(d\zeta/2\pi)\times (4\zeta/(\zeta^2 + 4u^2))$, up to $\mathcal{O}(h)$ corrections. This readily provides an effective scattering kernel for the giant quasiparticles $T_{\xi,\xi'}^{\mathrm{giant}}(u) = \lim_{h\to 0^+} T_{\xi/h,\xi'/h}(u/h)$, reading explicitly

$$T_{\xi,\xi'}^{\mathrm{giant}}(u) = \frac{1}{\pi}\log\frac{4u^2 + (\xi+\xi')^2}{4u^2 + (\xi-\xi')^2}. \tag{8}$$

In this expression one can recognize the scattering kernel—the differential scattering phase of the two-body $\mathcal{S}$ matrix—ascribed to an elastic collision of two Landau-Lifshitz solitons [89,92] characterized by pairs of action variables (u_1, ξ) and (u_2, ξ') with $u = u_1 - u_2$. All the remaining thermodynamic state functions pertaining to the giant quasiparticles can be obtained in a similar manner by rescaling the analogous quantities in the Heisenberg chain. We will need the following standard TBA concepts: an equilibrium state is uniquely characterized by a density $\rho_s(\theta)$ of quasiparticles with quantum numbers (s, θ), the available density of states $\rho_s^{\rm tot}(\theta)$ and the associated Fermi filling fractions $n_s(\theta) \equiv \rho_s(\theta)/\rho_s^{\rm tot}(\theta)$. Finally, interactions "dress" the group velocity and magnetization (along with other local charges) carried by quasiparticles; we denote these as $v_s^{\rm eff}(\theta)$ and $m_s^{\rm dr}(\theta)$, respectively.

Writing the Fermi filling functions of the quasiparticles as $n_s(\theta) = [1 + \eta_s(\theta)]^{-1}$, we have $\eta_{\xi/h}(u/h) \to \eta(\xi, u)/h^2$, implying vanishing occupations $n_s \sim h^2\eta^{-1}(\xi, u)$ and emergent classical statistics for these modes. The rescaled ratio $\eta(\xi, u)$ is interpreted as a Boltzmann weight which obeys a two-dimensional Fredholm-type integral equation [93]

$$\log\eta(\xi,u) = 2\log h + 2\xi - \frac{h}{T}e^{\rm giant}(\xi,u) + \int_0^{+\infty}[d\xi']_h \int_{-\infty}^{+\infty} dv T^{\rm giant}_{\xi,\xi'}(u-v)[\eta(\xi',v)]^{-1}, \tag{9}$$

where we have introduced a regularized integral $\int_0^{+\infty}[d\xi]_h g(\zeta) \equiv \int_0^{+\infty} d\zeta g(\xi) - (h/2)\lim_{\xi\to 0^+} g(\xi)$ for any function $g(\xi)$ and bare energy $e^{\rm giant}(\xi, u) = 2\xi/(\xi^2 + 4u^2)$. Equation (9) can be interpreted as a semiclassical TBA equation for a finite-density soliton gas. Analogous integral equations, albeit without a regulator, have previously appeared in the context of classical thermodynamic soliton gases [98–106]. We note however that Eq. (9) only governs a particular scaling regime of classical "soft solitons" with low energy and large width.

Before we proceed with solving Eq. (9), we owe to clarify an important subtlety. Even though we are eventually only interested in the solution at $h = 0$, the limit $h \to 0^+$ can be taken only after solving (9), as h acts as a cutoff in the integral over the solitons' charge ξ. Similar integral equations can be also written for the densities of quasiparticles $\rho^{\rm tot}_{\xi/h}(u/h) \to h^2\rho^{\rm tot}(\xi, u)$, the dressed rapidity derivative of energy of the quasiparticle excitations $\varepsilon'_{\xi/h}(u/h) = h^3\varepsilon'(\xi, u)$, and the dressed magnetization $m^{\rm dr}_{\xi/h}(u/h) = h^{-1}m^{\rm dr}(\xi, u)$ [93]. The dressed magnetization diverges as h^{-1} in the $h \to 0^+$ limit, while the velocity $v^{\rm eff}_{\xi/h}(u/h) = \varepsilon'_{\xi/h}(u/h)/[2\pi\rho^{\rm tot}_{\xi/h}(u/h)]$ vanishes as $\sim h$. It is also straightforward to check that these expressions are consistent with $D(\xi, u) = \lim_{h\to 0^+}(1/h)D_{\xi/h}(u/h)$ converging to a finite function.

In the limit of infinite temperature, $T = \infty$, dependence on parameter u drops out of equation (9), which enables us to solve it exactly [101]. We find $\eta(\xi, u) = \sinh^2(\xi + h)$, in agreement with rescaling the exact analytical solution of the TBA equations at infinite temperature for the quantum chain [91]. Using this result, all other thermodynamic functions can also be obtained in a closed form, yielding $D_0(T \to \infty) = 5\pi/27$ [93]. From Eq. (6) we thus deduce that $\lambda_{\rm KPZ}(T = \infty) = 10\pi/(27\sigma_{\rm KPZ}^{3/2}) \approx 1.9265, \ldots$.

Numerical results.—Solving the semiclassical TBA Eq. (9) at finite temperature T is numerically challenging; in practice it is more convenient to solve the original quantum TBA equations and afterwards take the limit $D_0 = \lim_{h\to 0^+} hD(h)$ numerically. We compared our predictions to time-dependent density matrix renormalization group (t-DMRG) calculations, see Fig. 1 and [93] for additional numerical data, by evolving a finite-temperature state [107], with fixed maximal bond dimension equal to 800 and system size $L = 140$ and computing the dynamical structure factor (DSF) $C_n(t) = \langle \hat{S}^z_n(t)\hat{S}^z_0(0)\rangle_T$ at finite temperature T. Despite entanglement entropy growing linearly in time, we carry out computation up to times $t \sim 50J$ and estimate the maximal error by comparing values of different observables. In particular we extract the value of $\lambda_{\rm KPZ}$ at finite time by considering [given Eq. (2)]: the autocorrelation, via $\lambda^{\rm tDMRG}_{\rm KPZ}(t) = \{t^{2/3}C_0(t)/[\chi f_{\rm KPZ}(0)]\}^{-3/2}$, the variance $\sigma^2(t) = \sum_{n=-L/2}^{L/2} n^2 C_n(t)$, via $\lambda^{\rm tDMRG}_{\rm KPZ}(t) = [t^{-4/3}\sigma^2(t)/(\chi\sigma^2_{\rm KPZ})]^{3/4}$ and the mean of the absolute value $\mu(t) = \sum_{n=-L/2}^{L/2} |n|C_n(t)$ via similar relation. In the limit $t \to \infty$, all these values are expected to be equal and identify to $\lambda_{\rm KPZ}$. At the finite times accessible by the numerical simulation, we find an expected slow convergence towards the theoretically predicted value of $\lambda_{\rm KPZ}$, with corrections of order $t^{-1/3}$, consistently with other dynamical systems in the KPZ universality class [108,109] (Fig. 2).

We find good agreement with our prediction (6), especially at high temperature (Fig. 1). At lower temperatures, however, various numerical estimators for $\lambda_{\rm KPZ}$ show some discrepancy, indicating that on the accessible timescale the

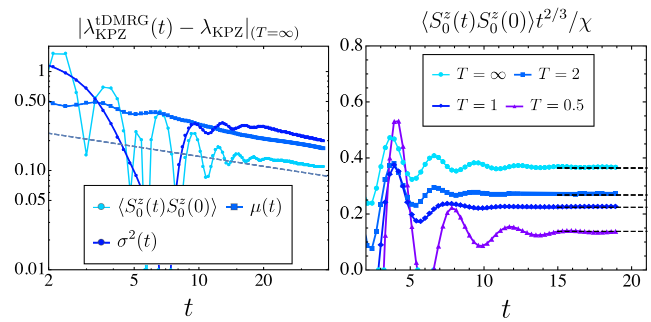

FIG. 2. Left: log-log plot of the $\lambda_{\rm KPZ}$ computed from t-DMRG numerical simulations in a Heisenberg chain at infinite temperature $T = \infty$, minus our theoretically predicted value $\lambda_{\rm KPZ}(T = \infty) = 1.9265, \ldots$, as function of the numerical simulation times t (in unit of spin coupling J). Dashed gray line represents $t^{-1/3}$. Right: spin autocorrelation as function of time multiplied by $t^{2/3}$. Convergence to KPZ scaling is reached at $t \sim O(10)$ for all considered temperatures T.

dynamical correlations have not yet relaxed sufficiently close to the asymptotic KPZ scaling form (2). We moreover observe that $\lambda_{\mathrm{KPZ}} \to \infty$ with decreasing temperature, suggesting that the classical KPZ dynamics only becomes valid on increasingly large spatiotemporal scales, whereas on shorter scales one can expect Luttinger liquid ballistic dynamics [110,111] and spinon physics [112].

Conclusion.—We have traced the microscopic origin of anomalous spin transport in the quantum Heisenberg spin-$1/2$ chain to the presence of giant quasiparticle eigenstates in its spectrum. These states admit a purely classical interpretation as a thermal gas of soft classical solitons of the isotropic Landau-Lifshitz equation. We established an explicit correspondence through the semiclassical scaling limit of the thermodynamic Bethe ansatz equations. The Fermi factors of such giant quasiparticles are vanishingly small so they become effectively classical.

Our analysis unifies the complementary pictures of KPZ superdiffusion: the generalized hydrodynamics approach of Refs. [31,32], and the effective theory of Ref. [34], which seemingly evades the conventional GHD description. In the language of GHD, one divides a system up into hydrodynamic cells of some fixed size, and constructs a thermal state within each cell. To construct such a state, one must specify both a "pseudovacuum" (i.e., a unit vector on the sphere that sets the direction of the net magnetization) and a quasiparticle distribution above this pseudovacuum. Reference [34] postulated a Landau-Lifshitz dynamics for long-wavelength spatial fluctuations of this pseudovacuum, arguing that it cannot be captured by GHD modes. However, in light of our analysis, the distinction between such "pseudovacuum fluctuations" and quasiparticles is only superficial as is depends on the cutoff: pseudovacuum fluctuations are nothing but giant quasiparticles that extend beyond the scale of a hydrodynamic cell, and can indeed be described within GHD. With that, we confirm the previous suggestion [34] that superdiffusion in the Heisenberg spin chain is due to low-energy degrees of freedom that obey an emergent Landau-Lifshitz equation; the quantum and classical systems share the same hydrodynamic description in terms of a stochastic Burgers (or equivalently KPZ) equation. Our explicit derivation provides the microscopic input for the KPZ equation, permitting to determine the temperature dependence of its coupling constant (in good agreement with numerical results); moreover, it establishes the universal nature of the low-energy solitons that cause superdiffusion. We expect the explicit mapping to a classical model to enable efficient numerical simulations that should quantitatively address important questions such as the fate of superdiffusion away from strict integrability, see [35].

Our results can be straightforwardly generalized to other integrable spin or charge models where KPZ scaling is also expected, including the spin-S integrable chains, integrable models of higher-rank symmetry [64] and Fermi-Hubbard chains [30]. A separate interesting direction for future work would be to understand the crossover from Luttinger liquid physics to KPZ dynamics at low temperature.

We are very grateful and indebted to Benjamin Doyon, Takato Yoshimura, Tomohiro Sasamoto for inspiring discussions on the semiclassical TBA equations and collaboration on the KPZ problem in the *XXX* chain; to Marko Medenjak and Brayden Ware for collaborations on closely related topics; to Utkarsh Agrawal for early collaboration on the numerical solutions to the TBA equations of the Heisenberg chain; and to Vir Bulchandani for numerous stimulating discussions. We thank the International Centre for Theoretical Sciences (ICTS) and the program "Thermalization, Many body localization and Hydrodynamics" (Code: ICTS/hydrodynamics2019/11) where this project was initiated. The MPS-based t-DMRG simulations were performed using the ITensor Library [113]. This work was supported by the National Science Foundation under NSF Grant No. DMR-1653271 (S. G.), the US Department of Energy, Office of Science, Basic Energy Sciences, under Early Career Award No. DE-SC0019168 (R. V.), the Alfred P. Sloan Foundation through a Sloan Research Fellowship (R. V.), the Research Foundation Flanders (FWO, J. D. N.), and the Slovenian Research Agency (ARRS) Program No. P1-0402 (E. I.).

[1] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).

[2] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Adv. Phys. **65**, 239 (2016).

[3] B. Bertini, F. Heidrich-Meisner, C. Karrasch, T. Prosen, R. Steinigeweg, and M. Znidaric, arXiv:2003.03334.

[4] T. Kinoshita, T. Wenger, and D. Weiss, Nature (London) **440**, 900 (2006).

[5] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, Nature (London) **449**, 324 (2007).

[6] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, Science **337**, 1318 (2012).

[7] S. Hild, T. Fukuhara, P. Schauß, J. Zeiher, M. Knap, E. Demler, I. Bloch, and C. Gross, Phys. Rev. Lett. **113**, 147205 (2014).

[8] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science **349**, 842 (2015).

[9] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner *et al.*, Nature (London) **551**, 579 (2017).

[10] S. Erne, R. Bücker, T. Gasenzer, J. Berges, and J. Schmiedmayer, Nature (London) **563**, 225 (2018).

[11] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, Phys. Rev. X **8**, 021030 (2018).

[12] J. M. Wilson, N. Malvania, Y. Le, Y. Zhang, M. Rigol, and D. S. Weiss, Science **367**, 1461 (2020).

[13] W. Kao, K.-Y. Li, K.-Y. Lin, S. Gopalakrishnan, and B. L. Lev, arXiv:2002.10475.

[14] S. Krinner, D. Stadler, D. Husmann, J.-P. Brantut, and T. Esslinger, Nature (London) **517**, 64 (2015).
[15] C. Hess, Phys. Rep. **811**, 1 (2019).
[16] M. A. Nichols, L. W. Cheuk, M. Okan, T. R. Hartke, E. Mendez, T. Senthil, E. Khatami, H. Zhang, and M. W. Zwierlein, Science **363**, 383 (2019).
[17] J. Lux, J. Müller, A. Mitra, and A. Rosch, Phys. Rev. A **89**, 053608 (2014).
[18] D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. (Amsterdam) **321**, 1126 (2006).
[19] R. Nandkishore and D. A. Huse, Annu. Rev. Condens. Matter Phys. **6**, 15 (2015).
[20] R. Vasseur and J. E. Moore, J. Stat. Mech. (2016) 064010.
[21] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Rev. Mod. Phys. **91**, 021001 (2019).
[22] Y. Bar Lev, G. Cohen, and D. R. Reichman, Phys. Rev. Lett. **114**, 100601 (2015).
[23] K. Agarwal, S. Gopalakrishnan, M. Knap, M. Müller, and E. Demler, Phys. Rev. Lett. **114**, 160401 (2015).
[24] S. Gopalakrishnan and S. Parameswaran, arXiv: 1908.10435.
[25] M. Žnidarič, Phys. Rev. Lett. **106**, 220601 (2011).
[26] V. B. Bulchandani, C. Karrasch, and J. E. Moore, arXiv:1904.09287.
[27] M. Ljubotina, M. Žnidarič, and T. Prosen, Nat. Commun. **8**, 16117 (2017).
[28] M. Ljubotina, M. Žnidarič, and T. Prosen, Phys. Rev. Lett. **122**, 210602 (2019).
[29] M. Dupont and J. E. Moore, Phys. Rev. B **101**, 121106(R) (2020).
[30] E. Ilievski, J. De Nardis, M. Medenjak, and T. Prosen, Phys. Rev. Lett. **121**, 230602 (2018).
[31] S. Gopalakrishnan and R. Vasseur, Phys. Rev. Lett. **122**, 127202 (2019).
[32] J. De Nardis, M. Medenjak, C. Karrasch, and E. Ilievski, Phys. Rev. Lett. **123**, 186601 (2019).
[33] S. Gopalakrishnan, R. Vasseur, and B. Ware, Proc. Natl. Acad. Sci. U.S.A. **116**, 16250 (2019).
[34] V. B. Bulchandani, Phys. Rev. B **101**, 041411(R) (2020).
[35] J. De Nardis, M. Medenjak, C. Karrasch, and E. Ilievski, arXiv:2001.06432.
[36] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Phys. Rev. X **6**, 041065 (2016).
[37] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Phys. Rev. Lett. **117**, 207201 (2016).
[38] B. Doyon and T. Yoshimura, SciPost Phys. **2**, 014 (2017).
[39] E. Ilievski and J. De Nardis, Phys. Rev. Lett. **119**, 020602 (2017).
[40] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Phys. Rev. Lett. **119**, 220604 (2017).
[41] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Phys. Rev. B **97**, 045407 (2018).
[42] B. Doyon and H. Spohn, SciPost Phys. **3**, 039 (2017).
[43] B. Doyon and H. Spohn, J. Stat. Mech. (2017) 073210.
[44] B. Doyon, T. Yoshimura, and J.-S. Caux, Phys. Rev. Lett. **120**, 045301 (2018).
[45] B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, Phys. Rev. Lett. **119**, 195301 (2017).
[46] X. Zotos, J. Stat. Mech. (2017) 103101.
[47] E. Ilievski and J. De Nardis, Phys. Rev. B **96**, 081118(R) (2017).
[48] M. Collura, A. De Luca, and J. Viti, Phys. Rev. B **97**, 081111(R) (2018).
[49] X. Cao, V. B. Bulchandani, and J. E. Moore, Phys. Rev. Lett. **120**, 164101 (2018).
[50] J. De Nardis, D. Bernard, and B. Doyon, Phys. Rev. Lett. **121**, 160603 (2018).
[51] S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, Phys. Rev. B **98**, 220303(R) (2018).
[52] J. D. Nardis, D. Bernard, and B. Doyon, SciPost Phys. **6**, 49 (2019).
[53] U. Agrawal, S. Gopalakrishnan, and R. Vasseur, Phys. Rev. B **99**, 174203 (2019).
[54] M. Borsi, B. Pozsgay, and L. Pristyák, Phys. Rev. X **10**, 011054 (2020).
[55] D. X. Horvath, J. High Energy Phys. 10 (2019) 020.
[56] B. Bertini, L. Piroli, and M. Kormos, Phys. Rev. B **100**, 035108 (2019).
[57] A. Bastianello, V. Alba, and J.-S. Caux, Phys. Rev. Lett. **123**, 130602 (2019).
[58] F. S. Møller and J. Schmiedmayer, SciPost Phys. **8**, 41 (2020).
[59] P. Ruggiero, P. Calabrese, B. Doyon, and J. Dubail, Phys. Rev. Lett. **124**, 140603 (2020).
[60] A. J. Friedman, S. Gopalakrishnan, and R. Vasseur, Phys. Rev. B **101**, 180302 (2020).
[61] A. Bastianello, J. D. Nardis, and A. D. Luca, arXiv:2003 .01702.
[62] A. Das, M. Kulkarni, H. Spohn, and A. Dhar, Phys. Rev. E **100**, 042116 (2019).
[63] Z. Krajnik and T. Prosen, arXiv:1909.03799.
[64] Ž. Krajnik, E. Ilievski, and T. Prosen,arXiv:2003.05957.
[65] O. Gamayun, Y. Miao, and E. Ilievski, Phys. Rev. B **99**, 140301(R) (2019).
[66] G. Misguich, N. Pavloff, and V. Pasquier, SciPost Phys. **7**, https://doi.org/10.21468/SciPostPhys.7.2.025 (2019).
[67] U. Schneider, L. Hackermuller, J. P. Ronzheimer, S. Will, S. Braun, T. Best, I. Bloch, E. Demler, S. Mandt, D. Rasch, and A. Rosch, Nat. Phys. **8**, 213 (2012).
[68] S. Scherg, T. Kohlert, J. Herbrych, J. Stolpp, P. Bordia, U. Schneider, F. Heidrich-Meisner, I. Bloch, and M. Aidelsburger, Phys. Rev. Lett. **121**, 130402 (2018).
[69] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[70] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Phys. Rev. X **7**, 031016 (2017).
[71] D. Bernard and P. L. Doussal, arXiv:1912.08458.
[72] T. Jin, A. Krajenbrink, and D. Bernard, arXiv:2001.04278 [Phys. Rev. Lett. (to be published)].
[73] M. Lakshmanan, T. W. Ruijgrok, and C. Thompson, Physica (Amsterdam) **84A**, 577 (1976).
[74] M. Prähofer and H. Spohn, J. Stat. Phys. **115**, 255 (2004).
[75] J. Quastel and H. Spohn, J. Stat. Phys. **160**, 965 (2015).
[76] A. Das, K. Damle, A. Dhar, D. A. Huse, M. Kulkarni, C. B. Mendl, and H. Spohn, J. Stat. Phys. (2019), https://doi.org/10.1007/s10955-019-02397-y.
[77] H. Spohn, J. Stat. Phys. **154**, 1191 (2014).
[78] M. Medenjak, J. D. Nardis, and T. Yoshimura, arXiv: 1911.01995.

[79] B. Doyon, arXiv:1912.01551.

[80] B. Sutherland, Phys. Rev. Lett. **74**, 816 (1995).

[81] A. Dhar and B. S. Shastry, Phys. Rev. Lett. **85**, 2813 (2000).

[82] J. A. Minahan and K. Zarembo, J. High Energy Phys. 03 (2003) 013.

[83] G. Arutyunov, S. Frolov, and M. Zamaklar, Nucl. Phys. **B778**, 1 (2007).

[84] R. Roiban, J. High Energy Phys. 07 (2007) 048.

[85] J. Minahan and O. O. Sax, Nucl. Phys. **B801**, 97 (2008).

[86] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, L. Freyhult, N. Gromov, R. A. Janik, V. Kazakov, T. Klose *et al.*, Lett. Math. Phys. **99**, 3 (2012).

[87] V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, J. High Energy Phys. 04 (2004) 024.

[88] T. Bargheer, N. Beisert, and N. Gromov, New J. Phys. **10**, 103023 (2008).

[89] L. Takhtajan, Phys. Lett. **64A**, 235 (1977).

[90] M. Takahashi, Prog. Theor. Phys. **46**, 401 (1971).

[91] M. Takahashi, Thermodynamics of One-Dimensional Solvable Models (Cambridge University Press, Cambridge, 1999).

[92] L. Faddeev, *Hamiltonian Methods in the Theory of Solitons (Classics in Mathematics)* (Springer-Verlag Berlin Heidelberg, 2007), https://www.springer.com/gp/book/9783540698432.

[93] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.070601 for extra numerical data and extra details on the thermodymamics of the solitons gas, which includes Refs. [94–97].

[94] E. Ilievski and O. Gamayun (to be published).

[95] R. Jackiw and G. Woo, Phys. Rev. D **12**, 1643 (1975).

[96] A. B. Zamolodchikov and A. B. Zamolodchikov, Ann. Phys. (N.Y.) **120**, 253 (1979).

[97] C. N. Yang and C. P. Yang, J. Math. Phys. (N.Y.) **10**, 1115 (1969).

[98] F. Mertens and H. Büttner, Phys. Lett. **84A**, 335 (1981).

[99] H. Bolterauer and M. Opper, Z. Phys. B **42**, 155 (1981).

[100] J. Timonen, R. K. Bullough, and D. J. Pilling, Phys. Rev. B **34**, 6525 (1986).

[101] K. Sasaki, Phys. Rev. B **33**, 2214 (1986).

[102] R. Bullough, Y. z. Chen, J. Timonen, V. Tognetti, and R. Vaia, Phys. Lett. **145A**, 154 (1990).

[103] N. Theodorakopoulos and N. C. Bacalis, Phys. Rev. Lett. **67**, 3018 (1991).

[104] N. Theodorakopoulos, Phys. Rev. B **52**, 9507 (1995).

[105] A. De Luca and G. Mussardo, J. Stat. Mech. (2016) 064011.

[106] A. Bastianello, B. Doyon, G. Watts, and T. Yoshimura, SciPost Phys. **4**, https://doi.org/10.21468/SciPostPhys.4.6.045 (2018).

[107] C. Karrasch, J. H. Bardarson, and J. E. Moore, Phys. Rev. Lett. **108**, 227206 (2012).

[108] K. A. Takeuchi, M. Sano, T. Sasamoto, and H. Spohn, Sci. Rep. **1**, https://doi.org/10.1038/srep00034 (2011).

[109] P. L. Ferrari and R. Frings, J. Stat. Phys. **144**, 1123 (2011).

[110] C. Karrasch, R. G. Pereira, and J. Sirker, New J. Phys. **17**, 103003 (2015).

[111] R. M. Konik and P. Fendley, Phys. Rev. B **66**, 144416 (2002).

[112] M. Mourigal, M. Enderle, A. Klöpperpieper, J.-S. Caux, A. Stunault, and H. M. Rønnow, Nat. Phys. **9**, 435 (2013).

[113] ITensor Library (version 2.1), http://itensor.org.