Editors' Suggestion

Constructing Quantum Spin Liquids Using Combinatorial Gauge Symmetry

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We introduce the notion of combinatorial gauge symmetry—a local transformation that includes single spin rotations plus permutations of spins (or swaps of their quantum states)—that preserve the commutation and anticommutation relations among the spins. We show that Hamiltonians with simple two-body interactions contain this symmetry if the coupling matrix is a Hadamard matrix, with the combinatorial gauge symmetry being associated with the automorphism of these matrices with respect to monomial transformations. Armed with this symmetry, we address the physical problem of how to build quantum spin liquids with physically accessible interactions. In addition to its intrinsic physical significance, the problem is also tied to that of how to build topological qubits.

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Quantum liquids of spins are systems where no magnetic symmetry-breaking order should be detectable down to zero temperature [1], and instead topological order exits [2]. On the theoretical side, there are a number of model Hamiltonians where quantum spin liquid states exist [3,4]. Gauge symmetries are common in these models, whether discrete or continuous, intrinsic or emergent. Many of these gauge models, such as the $\mathbb{Z}_2$ toric code [3] and fracton models such as the X cube [5,6], are defined using multispin interactions. Here, we show that *exact* local $\mathbb{Z}_2$ gauge symmetries in these models can arise from solely two-spin interactions. That one can generate effective multispin interactions in some low energy limit of a two-spin Hamiltonian is not unexpected; what is novel is that the symmetries we discuss are *exact*. We articulate a notion of combinatorial gauge symmetry that underlies why it is possible to construct local two-spin Hamiltonians with an exact $\mathbb{Z}_2$ gauge symmetry.

Algebra-preserving transformations and monomial matrices.—We start with a set of N spin-1/2 degrees of freedom, such as the familiar spin models on a lattice with N sites. The spin operators are Pauli matrices σ_i^α, where $\alpha = x, y, z$ and $i = 1, \ldots, N$. Spins on different sites commute, while those on the same site satisfy the usual angular momentum algebra. Let us ask a simple question: which transformations of these $3N$ operators can preserve all commutation and anticommutation relations? For N bosons or fermions, this is a trivial question to answer; the allowed set of single-particle transformations belong to the unitary group $\mathrm{U}(N)$ because either the commutation or anticommutation relations need to be satisfied. But for spins, the question is harder; one cannot simply mix spatial components of different spins and retain both the intra- and intersite algebra.

The Hilbert space for N spins is 2^N dimensional and the allowed operators in this space are $2^N \times 2^N$ unitary matrices, corresponding to the group $\mathrm{SU}(2^N)$. A generic transformation on the spin operators $\sigma_i^a \to U\sigma_i^a U^\dagger$ preserves the algebra, but also acts simultaneously on many spins: it mixes the $3N$ single-spin operators σ_i^a with the other (multispin) $2^{2N} - 1 - 3N$ generators of $\mathrm{SU}(2^N)$. Therefore, if one is to remain with only single-spin terms, one must work with a much smaller subgroup of $\mathrm{SU}(2^N)$. The simplest solution is trivial: only rotate spins individually by restricting the allowed transformations to $\mathrm{SU}(2) \otimes \mathrm{SU}(2) \otimes \cdots \otimes \mathrm{SU}(2)$ or N copies of $\mathrm{SU}(2)$. A more interesting and nontrivial solution is to also allow permutations of spins. (If one wishes to connect to quantum gates, these transformations correspond to the combination of one-qubit rotations and the use of two-qubit SWAP gates.)

Any $\mathrm{SU}(2)$ transformation on spin i can be represented by a matrix in the rotation group $g_i \in \mathrm{SO}(3)$ that acts on the spatial components of the vector $\vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z)^\top$. This representation makes it convenient to combine permutations and single-spin transformations into monomial matrices. Monomial matrices are generalizations of permutation matrices such that the nonzero elements in each row and column are group elements, not simply equal to 1. Here is an $N = 4$ example:

$$\begin{pmatrix} \vec{\sigma}_1 \\ \vec{\sigma}_2 \\ \vec{\sigma}_3 \\ \vec{\sigma}_4 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & g_1 & 0 \\ 0 & 0 & 0 & g_2 \\ g_3 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{\sigma}_1 \\ \vec{\sigma}_2 \\ \vec{\sigma}_3 \\ \vec{\sigma}_4 \end{pmatrix}. \quad (1)$$

It is clear from this form that monomial matrices are orthogonal and that the product of any two monomial matrices is another monomial matrix. The example above can be written as a product of the diagonal matrix $\mathrm{Diag}(g_1, g_2, g_3, g_4)$ and a 4×4 permutation matrix.

For arbitrary N, the group of monomial matrices is a semidirect product of the group generated by the diagonal matrices $\mathrm{Diag}(g_1, \ldots, g_N)$ and the group of permutations (symmetric group) S_N. In mathematical literature, this particular form of a semidirect product is sometimes referred to as a wreath product.

To summarize the above: we are pointing out that many-body spin states admit a group of nontrivial transformations on the $3N$ spin components that preserve all spin algebras. When formulated in this way, the combination of local and permutation symmetry will allow us to construct exact lattice gauge theories using only two-body interactions.

Combinatorial gauge symmetry.—One particular subgroup of monomial transformations, such as in Eq. (1), is for SO(3) rotations by angle π around a given axis, which we take to be $\hat{x}$. This is equivalent to flipping the z component of spin. We shall use this special case to construct a microscopic model with local $\mathbb{Z}_2$ symmetry. We term our methodology *combinatorial gauge symmetry* for its relation to monomials and permutations.

Consider the lattice depicted in Fig. 1, where 4 "matter" spins μ are placed on each lattice site, and "gauge" spins σ are placed on the links. A single site (star) is isolated in Fig. 1(a), and contains the 4 matter spins and 4 gauge spins sitting on the links. The gauge spins are shared by neighboring stars, as depicted in Fig. 1(b). Each matter spin couples only to its neighboring gauge spins but not to one another (or other lattice sites). Gauge spins do not couple to each other. We encode all two-spin (ZZ) couplings between μ^z_a and σ^z_i by a 4×4 matrix W_{ai}.

The quantum fluctuations will come from two transverse fields $\tilde{\Gamma}$ and Γ acting on the gauge spins and matter spins, respectively. For generality, we allow Γ and $\tilde{\Gamma}$ to have different magnitudes.

Thus the full lattice Hamiltonian is given by

$$H = -\sum_s \Big[J \sum_{\substack{a\in s\\ i\in s}} W_{ai}\sigma^z_i\mu^z_a + \Gamma\sum_{a\in s}\mu^x_a\Big] - \tilde{\Gamma}\sum_i \sigma^x_i, \quad (2)$$

where the s are stars on the lattice.

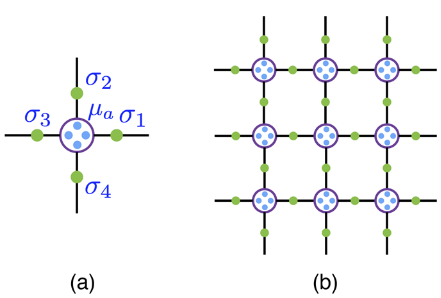

FIG. 1. (a) A single site (star) of the $\mathbb{Z}_2$ gauge theory, with 4 matter spins μ_a on the site, and 4 gauge spins σ_i on the links. (b) The full lattice.

We shall select the interaction matrix W so as to satisfy the monomial transformations as in Eq. (1) that act on the z components of the gauge and matter spins as follows:

$$\sigma^z_i \to \sum_{j=1}^{4} R_{ij}\sigma^z_j$$
$$\mu^z_a \to \sum_{b=1}^{4} \mu^z_b (L^{-1})_{ba}. \quad (3)$$

These are monomial transformations that preserve the spin commutation and anticommutation relations, as discussed above. The L ("left") and R ("right") matrices act like gauge transformations on the z components of the gauge and matter spins. These monomial matrices have elements ± 1. (Henceforth all monomial matrices will be of this kind.)

The requirement that the Hamiltonian Eq. (2) be invariant with respect to transformations Eq. (3) is equivalent to the requirement that the W matrices be invariant under the automorphism transformation $L^{-1}WR = W$, where L and R are 4×4 monomial matrices [7]. [The transverse fields are also invariant under the transformation Eq. (3), and we shall return to this point below].

Hadamard matrices [7] satisfy these conditions. These matrices have elements ± 1, and all its columns (or rows) are orthogonal vectors, i.e., $W^\top W \propto \mathbb{1}$. (They maximize the determinant of the information matrix $W^\top W$.) We pick an intuitive form of W, where the coupling between σ^z_i and μ^z_a is antiferromagnetic when $i = a$ and ferromagnetic otherwise:

$$W = \begin{pmatrix} -1 & +1 & +1 & +1 \\ +1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ +1 & +1 & +1 & -1 \end{pmatrix}. \quad (4)$$

All other choices of W are equivalent by symmetry and will not affect the spectrum. Specifically, any two Hadamard matrices W and W' are equivalent if there exist monomial matrices S_1, S_2 such that $W' = S_1^{-1} W S_2$.

Our model further restricts R to be diagonal because any off-diagonal permutation of gauge spins would deform the lattice. For example, with our choice of W in Eq. (4), the following pair satisfies the conditions above:

$$L = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
$$R = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}. \quad (5)$$

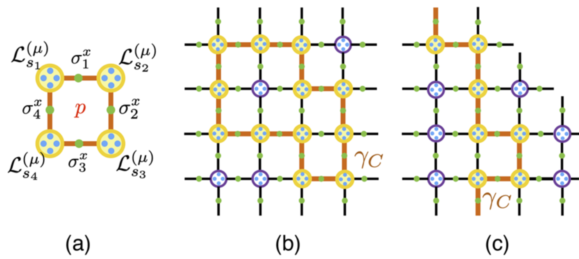

FIG. 2. (a) Operator generating the local $\mathbb{Z}_2$ gauge transformation on an elementary plaquette, G_p in Eq. (6). (b) A closed loop operator along a path γ_C. (c) An open string operator along a path in a system with boundaries.

Once we choose an R, we determine L uniquely by solving the automorphism condition: $L = WRW^{-1}$. (The number of -1's in the diagonal R matrix must be even so that the corresponding L is a monomial matrices.) Note that flipping gauge spins, even without permuting them, requires a simultaneous permutation of matter spins.

The automorphism pair (L, R) directly leads to the local $\mathbb{Z}_2$ gauge symmetry of the full lattice Hamiltonian (2). Consider an elementary plaquette p depicted in Fig. 2(a) and define the local gauge transformation

$$G_p = \prod_{s\in p}\mathcal{L}_s^{(\mu)}\prod_{s\in p}\mathcal{R}_s^{(\sigma)}, \tag{6}$$

where $\mathcal{L}_s^{(\mu)}$ denotes the operator that permutes and flips the matter spins at each corner site s of the plaquette as in Eq. (5): $\mathcal{L}_s^{(\mu)}\mu_a^z(\mathcal{L}_s^{(\mu)})^{-1} = \sum_b \mu_a^z(L^{-1})_{ba}$, as in the transformation in Eq. (3), and similarly for $\mathcal{R}_s^{(\sigma)}$ on the gauge spins: $\mathcal{R}_s^{(\sigma)}\sigma_i^z(\mathcal{R}_s^{(\sigma)})^{-1} = \sum_j R_{ij}\sigma_j^z$. $\mathcal{L}_s^{(\mu)}$ is uniquely determined by the local operator $\mathcal{R}_s^{(\sigma)}$ that flips the two gauge spins on links emanating from the site s—just as L is determined by R. Since here we restrict R to be diagonal, corresponding to only flipping the gauge spins without permuting them, the spin flip $\sigma^z \to -\sigma^z$ is simply generated by σ^x. Therefore we have $\prod_{s\in p}\mathcal{R}_s^{(\sigma)} = \prod_{i\in p}\sigma_i^x$, where i runs over all gauge spins in a plaquette. Any two L matrices commute and therefore the plaquette operators do as well, $[G_p, G_{p'}] = 0$.

The importance of G_p is that it is a local symmetry of the full lattice Hamiltonian (2): $[H, G_p] = 0$, for all p. Invariance of the Ising interaction term follows from the automorphism above, while invariance of the transverse field terms Γ and $\tilde{\Gamma}$ follows from two observations. First, all spin flips by the operator pair $(\mathcal{L}_s^{(\mu)}, \mathcal{R}_s^{(\sigma)})$ can be viewed as 180° rotations around the x axis, which commute with σ^x and μ^x. Second, the transverse fields are uniform and therefore independent of permutations. Therefore, the Hamiltonian (2) is a gauge theory with a local $\mathbb{Z}_2$ gauge symmetry that is generated by G_p. This symmetry relies on the locking of the permutations contained in the operators $\mathcal{L}_s^{(\mu)}$ to the $\mathbb{Z}_2$ transformations in the $\mathcal{R}_s^{(\sigma)}$, which is another reason that we refer to it as combinatorial gauge symmetry.

One can further construct loop or closed string symmetry operators on the lattice, as shown in Fig. 2(b). For systems with boundaries, one can also associate a symmetry operation to open strings, as depicted in Fig. 2(c). The loop (or string) operator along a path is composed of both the gauge spin flips $\prod_\ell \sigma_\ell^x$, where ℓ are the links along the path, as well as the corresponding operations on matter spins $\prod_s \mathcal{L}_s^{(\mu)}$ applied to each star along the path. In the case of closed paths, the loop operator is equivalent to a product of all plaquette operators G_p enclosed by the loop.

Special case: $\mathbb{Z}_2$ gauge theory.—The Hamiltonian Eq. (2) obeys a local $\mathbb{Z}_2$ gauge symmetry for *all* values of the parameters J, Γ, and $\tilde{\Gamma}$. Here we shall obtain, as a particular limit, an effective Hamiltonian with a 4-spin interaction on a star, which lands directly onto the more familiar $\mathbb{Z}_2$ gauge theory on the square lattice [8,9], in the following manner.

Isolate a single star with its 4 spins μ on the site and 4 gauge spins σ on the links, as depicted in Fig. 1(a). Let us freeze for the moment a given configuration of the gauge spins σ_i^z, $i = 1, 2, 3, 4$ in the z basis. The Hamiltonian (2) for each matter spin μ_a on a star can be viewed as that of a single spin in a magnetic field, whose eigenvalues are functions of σ_i^z:

$$E_a^{(\pm)}(\sigma_1^z, \sigma_2^z, \sigma_3^z, \sigma_4^z) = \pm\left[J^2\left(\sum_{i=1}^4 W_{ai}\sigma_i^z\right)^2 + \Gamma^2\right]^{1/2}. \tag{7}$$

The expression in Eq. (7) can be written, for any value of Γ and J, as

$$E_a^{(\pm)} = \pm C_0 \pm C_2\sum_{i\neq j}^4 W_{ai}W_{aj}\sigma_i^z\sigma_j^z \pm C_4 W_{a1}W_{a2}W_{a3}W_{a4}\sigma_1^z\sigma_2^z\sigma_3^z\sigma_4^z, \tag{8}$$

where C_0, C_2, and C_4 are constants that depend on J and Γ. This expression follows from expanding the square root in Eq. (7) in powers of the σ_i^z and using $(\sigma_i^z)^2 = 1$ and $(W_{ai})^2 = 1$; the binary polynomial inside the square root terminates and the only terms that remain are of the form in Eq. (8). While the expansion is useful in proving the identity between Eqs. (7) and (8), we remark that the result is exact (nonperturbative), because both expressions only take values in discrete sets.

The low energy manifold of states corresponds to the sum over the lowest eigenvalues, $H_{\text{eff}}^{\text{star}} = \sum_{a=1}^4 E_a^{(-)}$, which is separated from the next levels by a gap of size at least $2|\Gamma|$. We thus arrive at the following simple effective Hamiltonian for a single star:

$$H_{\text{eff}}^{\text{star}} = \gamma - \lambda\sigma_1^z\sigma_2^z\sigma_3^z\sigma_4^z, \tag{9}$$

where the coefficients γ and λ are functions of Γ and J are explicitly given in the Supplemental Material [10]. These relations follow from the consistency between Eqs. (7) and (8). The parity $P \equiv \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z$ for the ground state of Eq. (9) is $P = +1$, since $\lambda > 0$. By modifying the matrix W, we could flip the sign of λ and have instead the $P = -1$ parity sector as the ground state (for example, by flipping the sign of any one column of W).

Let us now turn to the low energy effective model for the whole lattice. In the limit $|\Gamma| \gg J$, we find the effective Hamiltonian

$$H_{\text{eff}} = -\lambda \sum_s \prod_{i \in s} \sigma_i^z - \tilde{\Gamma} \sum_i \sigma_i^x. \tag{10}$$

This Hamiltonian is exactly that of the $\mathbb{Z}_2$ quantum gauge theory, which supports a topological phase for $\tilde{\Gamma}/\lambda$ below a threshold. To get the toric or surface code limit, one only has to notice that the lowest order term that survives in a perturbation theory in $\tilde{\Gamma}/\lambda$ is the term that flips all spins around a plaquette [11–13].

Taking $|\Gamma| \to \infty$, while keeping λ fixed, opens an infinite gap to the excited sectors, where at least one $E_a^{(-)}$ becomes $E_a^{(+)}$. The splitting $2|\lambda|$ between the two parity states within the lowest energy sector remains finite. The expansion of λ in the regime of $J \ll \Gamma$ yields $\lambda = 12J^4/\Gamma^3 + \mathcal{O}(J^6/\Gamma^5)$. (Note that terms of order Γ vanish.) To access this regime we would fix λ and tune J such that $J = |\lambda \Gamma^3/12|^{1/4}$. Physically, in this limit the matter fields μ can be "integrated out" to obtain the exact four-spin effective Hamiltonian.

We corroborate the above analytical features with numerical studies in the Supplemental Material [10]. All degeneracies are confirmed to machine precision.

Combinatorial gauge symmetry for both electric and magnetic loops.—So far we used the combinatorial gauge symmetry to construct a model with $\mathbb{Z}_2$ plaquette operators. Here we shall construct a model with both the $\mathbb{Z}_2$ plaquette and star operators, just as in the toric code, but still using *only* at most two-body interactions.

We add another four spin-1/2 degrees of freedom to the center of all plaquettes in addition to the ones on the star. We denote these additional matter spins on the dual lattice as τ, shown in Fig. 3(a). Furthermore, the pairwise interaction couples τ^x and σ^x, i.e., an XX interaction. The full Hamiltonian is

$$H = -\sum_s \left[J \sum_{\substack{a \in s \\ i \in s}} W_{ai} \sigma_i^z \mu_a^z + \Gamma \sum_{a \in s} \mu_a^x \right] - \sum_p \left[J \sum_{\substack{b \in p \\ j \in p}} W_{bj} \sigma_j^x \tau_b^x + \Gamma \sum_{b \in p} \tau_b^z \right]. \tag{11}$$

In other words, on the dual lattice the spin components are transformed by $X \leftrightarrow Z$ relative to the original lattice. There is no need for a transverse field on the gauge σ spins in this model; quantum dynamics is already present through the presence of both XX and ZZ interactions.

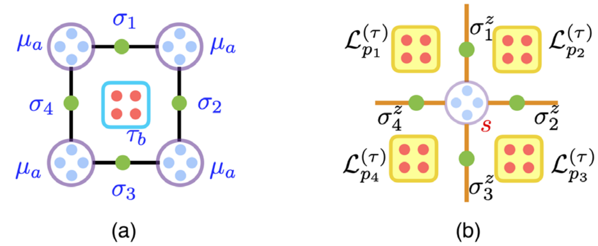

FIG. 3. (a) A single plaquette of the $\mathbb{Z}_2$ gauge theory, with 4 gauge spins σ_i on the links, 4 matter spins μ_a on the site, and 4 additional matter spins τ_b at the center of the plaquette. (b) A single star operator F_s.

By analogy with the plaquette operators G_p in Eq. (6), there is a set of star operators F_s, according to combinatorial gauge symmetry, which exists on the dual lattice [see Fig. 3(b)]:

$$F_s = \prod_{p \in s} \mathcal{L}_p^{(\tau)} \prod_{i \in s} \sigma_i^z. \tag{12}$$

The dual, "left" operators $\mathcal{L}_p^{(\tau)}$ flip τ spins in the x basis just like the operators $\mathcal{L}_s^{(\mu)}$ in Eq. (6) flip μ spins in the z basis. By construction, these two operators commute: $[\mathcal{L}_s^{(\mu)}, \mathcal{L}_p^{(\tau)}] = 0$. Therefore, we have a star and a plaquette operator that also commute: $[G_p, F_s] = 0$, exactly as in the toric code. It is easy to check that the Hamiltonian commutes with both stars and plaquettes: $[H, G_p] = [H, F_s] = 0$.

Given the commuting set of star and plaquette operators, the Hamiltonian in Eq. (11) is equivalent to the toric code in the asymptotic limit of large Γ, except that it contains *only* two-body interactions and fields. This is a direct result of the combinatorial gauge symmetry.

Extension to other topological states.—Fracton topological phases [6,14–16] (for a review, see Ref. [17]) are novel phases of matter with a robust subextensive ground state degeneracy and with excitations that are strictly immobile, or constrained to move within a subdimensional manifold. Apart from theoretical interest such as classifications of phases of matter and formulations in terms of higher-rank gauge theories [18], fracton systems are also believed to hold promise for fault-tolerant quantum computation, as well as robust quantum memory [16]. In spite of the intensive theoretical investigations on fractonic models, experimental realizations directly in terms of spins have barely been discussed [19].

The building blocks of our $\mathbb{Z}_2$ gauge theory can also be used to construct 3D models, such as one of the simplest fractonic model, the X cube [5,6]. The construction with matter and gauge spins parallels closely that in 2D, and we provide details for the construction of both the 3D toric code and the X cube model in the Supplemental Material [10].

Summary and outlook.—We have argued that many-body spin states admit a combinatorial gauge symmetry and we have used it to construct quantum spin liquids out of only two-body and single-body terms. The symmetry holds exactly for all ranges of parameters in the Hamiltonians that we have constructed. This presents an alternative path to explore quantum spin liquids in systems without four-body (or higher) interaction terms. Our approach may prove useful in the quest for topological qubits (via surface codes), which can potentially be hosted by certain quantum spin liquids.

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