Work Extraction from Fluid Flow: The Analog of Carnot's Efficiency

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Aiming to explore physical limits of wind turbines, we develop a model for determining the work extractable from a compressible fluid flow. The model employs conservation of mass, energy, and entropy and leads to a universal bound for the efficiency of the work extractable from kinetic energy. The bound is reached for a sufficiently slow, weakly forced quasi-one-dimensional, dissipationless flow. In several respects the bound is similar to the Carnot limit for the efficiency of heat engines. More generally, we show that the maximum work-extraction demands a contribution from the enthalpy, and is reached for sonic output velocities and strong forcing.

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How much work can be extracted from the kinetic energy of a fluid flow? The question is old [1–3], but it is still of obvious practical importance for wind energy usage [4]; e.g., it is relevant for shaping renewable energy policies [5]. Wind turbines cannot extract the whole kinetic energy, otherwise the flow will stall. The question is of fundamental importance, since it asks about the operational meaning of energy stored in a continuous medium.

No satisfactory answer to the above question is known. A popular model developed by Betz [2] (and independently by Lanchester [1] and Joukowsky [3]) studies a quantity ζ_B, which is smaller than the efficiency of work extracted from kinetic energy and proposes for it an upper bound $\zeta_B \le (16/27)$; see [4–8] for reviews. Betz's model makes an unwarranted assumption about the pressure distribution [9,10]. The proper efficiency in the model is bounded by 1; see § 1 and § 2 of [11]. Hence Betz's model does not answer the question.

We study work extraction due to an external force, using integral conservation laws of mass, entropy, and energy for a dissipationless, stationary fluid. The flow model is realistic, since the force is general, no incompressibility is assumed, etc. Our main assumption is that the axial component of the flow velocity is homogeneous in y and z directions at the initial and final cross sections of the flow; see Fig. 1.

We derive a new upper bound for the efficiency of work-extraction from the kinetic energy. We focus on this form of work extraction because it is relevant for wind turbines [1–10], and also because it is similar to heat-engine physics. The bound is attained for a weakly forced, subsonic, quasi-1D flow, where the fluid undergoes a cyclic process: its density and pressure after action of the force are equal to their initial values. This resembles Carnot's bound for heat engines that is also reached for cyclic, slow, and dissipationless processes [21]. We also determine the maximal work extracted from flow, without demanding that it necessarily comes from the kinetic energy. The maximum is reached for sonic output velocities and strong forcing. In this regime the work comes from enthalpy and can relate to increasing kinetic energy.

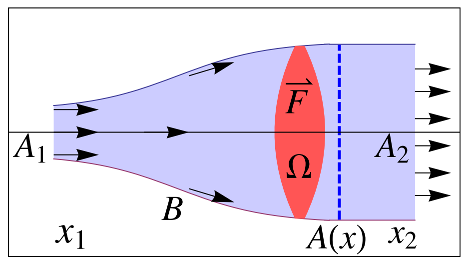

FIG. 1. The model. The flow (denoted by blue) goes from x_1 (input) to x_2 (output). $\vec{F}$ is the external force. The control volume B is blue filled. $A(x)$ (dashed line) is the cross section. $A_1 = A(x_1)$ and $A_2 = A(x_2)$ are, respectively, input and output surfaces. $\vec{F}$ is localized within the red-filled domain Ω and is negligible out of Ω. B includes Ω; see (2). Arrows denote stationary flow velocities; cf. (1) and (4).

The model.—The filled domain in Fig. 1 shows the stationary flow model. Here are our assumptions about it. (1) The fluid is dissipationless and compressible. (2) The work-extracting part of the turbine is modeled by a stationary space-dependent force $\vec{F}(\vec{x})$; see Fig. 1. As necessary for any turbine, $\vec{F}(\vec{x})$ varies smoothly and is negligible out of a finite domain Ω; see Fig. 1. (3) Homogeneous input flow: at the input $\vec{r}_1 \equiv (x_1, y, z)$, which is far from Ω (to the left in Fig. 1), the pressure p, velocity $\vec{v}$ and density ρ do not depend on (y, z), and transverse velocities are absent:

$$\vec{v}(\vec{r}_1) = (v_1, 0, 0), \qquad p(\vec{r}_1) = p_1, \qquad \rho(\vec{r}_1) = \rho_1. \quad (1)$$

Our consideration will use integral conservation laws based on the control volume B in Fig. 1. Now B is defined along the flow lines via two additional conditions: (i) B is used to calculate the total work done by $\vec{F}$:

$$\int_{\Omega} dV(-\vec{v}\cdot\vec{F}) = \int_{B} dV(-\vec{v}\cdot\vec{F}). \quad (2)$$

(ii) The area $a(x_1)$ of the input surface $A_1 = A(x_1)$ is possibly small, as needed for ensuring assumption 5 below, and for calculating the efficiency; see (16), (35) below. Hence B encircles Ω; cf. Fig. 1.

(4) The cross section $A(x)$ of B grows with x from input $A(x_1)$ to output $A(x_2)$. This assumption is needed for achieving work extraction. The general bounds (20), (36) on the efficiency of work-extraction demand a weaker condition $a(x_2) > a(x_1)$, where $a(x)$ is the area of $A(x)$.

(5) v_x is constant along the output surface A_2:

$$\vec{v}(\vec{r}_2) = [v_2, v_y(\vec{r}_2), v_z(\vec{r}_2)], \qquad \vec{r}_2 \equiv (x_2, y, z). \quad (3)$$

At the output $\vec{r}_2$ we apply the following notation

$$p(x_2, y, z) = p_2\tilde{p}(y, z), \qquad \rho(x_2, y, z) = \rho_2\tilde{\rho}(y, z), \quad (4)$$

where $\tilde{p}(y, z)$ and $\tilde{\rho}(y, z)$ are defined so as to hold

$$\langle\tilde{p}\rangle \equiv \int_{A_2} \frac{dydz\tilde{p}(y, z)}{a_2} = 1, \qquad \langle\tilde{\rho}\rangle = 1. \quad (5)$$

Equation (3) is a weak form of the plug-flow assumption done in hydraulics and quasi-1D motion [22,23]; see [24–26] for reviews that explore limits of plug-flows. (6) The fluid is an ideal gas with constant heat-capacities c_V and c_p. This implies for the entropy density s and internal energy density ε [22]:

$$\frac{s}{c_V} = \ln p - \gamma\ln\rho, \qquad \varepsilon = \frac{1}{\gamma-1}\frac{p}{\rho}, \qquad \gamma \equiv \frac{c_p}{c_V} > 1, \quad (6)$$

where the integration constant in s was fixed as in [22]. For air $\gamma = 1.4$ in agreement with the thermodynamic bound $\gamma > 1$ [22]. The local speed of sound reads [22]

$$v_s^2 = (\partial p/\partial\rho)|_s = \gamma p/\rho. \quad (7)$$

The setup is a generalization of Betz's model [1–10], because we do not assume that flow is incompressible, and we do not restrict $\vec{F}$ to be localized in a thin surface. Limitations of the setup are discussed in § 3 of [11].

Solving the full fluid dynamics equations for given $\vec{F}$ and boundary conditions (1) is out of reach. Instead we employ conservation laws of mass, entropy and energy that read for stationary flow [22] [$\vec{\nabla} = [\partial/\partial x, \partial/\partial y, \partial/\partial z]$]:

$$\vec{\nabla}(\rho\vec{v}) = 0, \qquad \vec{\nabla}(\rho\vec{v}s) = 0, \quad (8)$$

$$\vec{\nabla}\{(\rho\vec{v}^2\vec{v}/2) + \rho[\varepsilon + (p/\rho)]\vec{v}\} = \vec{v}\cdot\vec{F}, \quad (9)$$

where $\varepsilon + (p/\rho)$ is the enthalpy density, and where the external force $\vec{F}$ enters into stationary Euler's equation as:

$$\rho d\vec{v}/dt = \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + \vec{F}. \quad (10)$$

The momentum conservation is not employed, since it is useless without restrictive assumptions; see § 1 of [11].

We apply (8) and (9) to the control domain B in Fig. 1. Integrate $\vec{\nabla}\cdot(\rho\vec{v}) = 0$ in (8) over the volume B (cf. Fig. 1), and employ Gauss theorem to get three integrals over the surface of B: $(\int_{A_1} + \int_{A_2} + \int_{\mathcal{B}})d\vec{n}\cdot\vec{v}\rho = 0$, where $d\vec{n}$ points outward. Boundary conditions for a dissipationless fluid imply $d\vec{n}\cdot\vec{v}|_{\mathcal{B}} = 0$ [22]. Then employ (1)–(5) in $(\int_{A_1} + \int_{A_2})d\vec{n}\cdot\vec{v}\rho$. Other two relations in (8) and (9) are treated in the same way, also using (6):

$$a_1\rho_1 v_1 = a_2\rho_2 v_2, \qquad (p_2/p_1) = (\rho_2/\rho_1)^\gamma e^\sigma, \quad (11)$$

$$\frac{-\int dV\vec{v}\cdot\vec{F}}{a_1\rho_1 v_1} = \frac{v_1^2 - v_2^2 - v_{\mathrm{tr}}^2}{2} + \frac{\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right), \quad (12)$$

where $\int dV$ goes over volume B (colored blue in Fig. 1), a_k is the area of A_k, and where [cf. (3)–(5)]

$$v_{\mathrm{tr}}^2 \equiv \langle\tilde{\rho}[v_y^2(\vec{r}_2) + v_z^2(\vec{r}_2)]\rangle, \quad (13)$$

$$\sigma \equiv \langle\tilde{\rho}\ln[\tilde{\rho}/\tilde{p}]\rangle + (\gamma-1)\langle\tilde{\rho}\ln\tilde{\rho}\rangle \geq 0. \quad (14)$$

The lhs of (12) is the extracted work that amounts to the kinetic energy + enthalpy difference between input and output; cf. (2). Here v_{tr}^2 is the output transverse velocity contribution including vorticity. Both terms in (14) are non-negative [27] due to spatial inhomogenuities at the final surface A_2. Now σ corresponds to an effective entropy production [cf. (5)]. If we include dissipative effects by introducing in (8) a nonzero entropy production $\vec{\nabla}(\rho\vec{v}s) = s_{\mathrm{prod}}$, then above formulas will hold upon $\sigma \to \sigma + \int dV s_{\mathrm{prod}}/(c_V a_1\rho_1 v_1)$. Thus even for a dissipationless fluid, the inhomogeneity of the output plays the role of an effective entropy production $\sigma > 0$.

To simplify (11) and (12), employ dimensionless parameters:

$$\bar{a}_2 = \frac{a_2}{a_1}, \qquad \bar{v}_2 = \frac{v_2}{v_1}, \qquad \bar{p}_2 = \frac{p_2}{p_1}, \qquad \bar{v}_{\mathrm{tr}} = \frac{v_{\mathrm{tr}}}{v_1}, \quad (15)$$

$$M_1^2 = \frac{\rho_1 v_1^2}{\gamma p_1}, \qquad \bar{w} \equiv -\frac{\int dV\vec{v}\cdot\vec{F}}{\frac{1}{2}a_1\rho_1 v_1^3}, \quad (16)$$

where M_1 (Mach number) is ratio of the input velocity to the speed of sound (7) at the input, and $\bar{w}$ is the dimensionless work defined as the ratio of the work to the inflow $\frac{1}{2}a_1\rho_1 v_1^3$ of kinetic energy. Equation (11) lead to $\bar{p}_2\bar{v}_2^\gamma\bar{a}_2^\gamma = e^\sigma$ to be used together with (15) and (16) in (12):

$$\bar{w} = 1 - \bar{v}_2^2 - \bar{v}_{\rm tr}^2 + \frac{2}{M_1^2(\gamma-1)}(1 - e^{\sigma}\bar{a}_2^{1-\gamma}\bar{v}_2^{1-\gamma}). \quad (17)$$

Our purpose is to extract work, hence to achieve $\bar{w} > 0$.

Work extraction from kinetic energy.—We demand in (17) that the work is extracted from kinetic energy only:

$$1 = e^{\sigma}\bar{a}_2^{1-\gamma}\bar{v}_2^{1-\gamma}. \quad (18)$$

Due to $\sigma > 0$ and $\gamma > 1$, condition (18) can be achieved for $\bar{v}_2 < 1$ (smaller kinetic energy) only for $\bar{a}_2 > 1$ (cf. assumption 4). Using (18) and $\bar{a}_2 > 1$ we get from (17) and (15)

$$\bar{w} = 1 - \bar{a}_2^{-2} e^{\frac{2\sigma}{\gamma-1}} - \bar{v}_{\rm tr}^2 \quad (19)$$

$$\leq 1 - \bar{a}_2^{-2} = 1 - (a_1/a_2)^2, \quad (20)$$

where in deriving (20) we employed $\sigma \geq 0$, $\gamma > 1$ and $\bar{v}_{\rm tr}^2 \geq 0$. Both in thermodynamics and fluid dynamics the efficiency is defined as the work (result) divided over the inflow of energy (effort) [5,21]. For example, in heat engines the working body moves cyclically, hence the work per cycle comes from the inflow of heat only, and the efficiency is defined as the work divided over this inflow [21]. In our situation (19) the work is extracted from kinetic energy only and hence the efficiency is defined as the work over the inflow of kinetic energy [5]. Thus for (19) the efficiency coincides with $\bar{w}$, and hence (20) bounds the efficiency. [Generally, $\bar{w}$ and efficiency differ; see (35).]

Necessary conditions for attaining the bound (20) are $\bar{v}_{\rm tr}^2 = 0$ (no tangential velocity) and $\sigma = 0$ (no effective entropy production). The latter relation means $\tilde{p} = \tilde{\rho} = 1$ [cf. (14), (4)]; § 4 of [11] shows that (20) holds for nonideal gases. Below we demonstrate that bound (20) is attained for quasi-1D motion, where $\sigma = \bar{v}_{\rm tr} = 0$ and $\tilde{\rho} = \tilde{p} = 1$ take place naturally. Then as (18) and (11) show, work extraction from kinetic energy demands cyclicality:

$$\rho_1 = \rho_2, \qquad p_1 = p_2. \quad (21)$$

Note that only requiring $\rho_1 = \rho_2$ in (11) we get $\bar{a}_2\bar{v}_2 = 1$, and establish the bound (20) from (17) and $\sigma > 0$. The shape of efficiency bound (20), cyclicality condition (21), and no entropy production $\sigma = 0$ [needed for attaining (20)] make an analogy between (20) and Carnot's bound for heat-engines.

Work maximization over the final velocity.—The work (17) is formally maximized over $\bar{v}_2$ (for fixed values of other parameters) via $(\partial\bar{w}/\partial\bar{v}_2) = 0$ and $(\partial^2\bar{w}/\partial\bar{v}_2^2) < 0$. The second relation holds always, while the first one produces:

$$\bar{v}_2 = \bar{v}_m = (e^{\sigma}\bar{a}_2^{1-\gamma}M_1^{-2})^{\frac{1}{\gamma+1}}, \quad (22)$$

$$\bar{w}_m = \bar{w}(\bar{v}_m) = 1 - \bar{v}_m^2 - \bar{v}_{\rm tr}^2 + \frac{2(1 - M_1^2\bar{v}_m^2)}{(\gamma-1)M_1^2}, \quad (23)$$

The output velocity that corresponds to $\bar{v}_m$ equals to the speed of sound, as seen by starting from (7), (11), (15):

$$\frac{v_s^2(x_2)}{v_1^2} = \frac{1}{M_1^2}\frac{(p_2/p_1)}{(\rho_2/\rho_1)} = \frac{e^{\sigma}\bar{a}_2^{1-\gamma}\bar{v}_m^{1-\gamma}}{M_1^2} = \bar{v}_m^2, \quad (24)$$

noting that the last equality amounts to (22). The maximal work $\bar{w}_m$ can be attained, as seen below.

Work-extraction in quasi-1D flow.—Equations (11) and (12) are useful for bounding the work, but they cannot determine it, since v_2, σ, and $v_{\rm tr}$ are unknown. A more specific and informative approach is needed that allows us to address the attainability of bounds. Since the flow (shown in Fig. 1) has a smooth and slowly varying cross section $A(x)$, we apply the quasi-1D approach that comes from a coarse-graining of 3D fluid dynamics [22,23]. It assumes a stationary flow with the axial flow velocity $\vec{v} = (v, 0, 0)$, pressure p, density ρ, and the external force $\vec{F} = (F, 0, 0)$ depending only on the axial variable x. Hence transverse velocities and effective entropy production nullify: $v_{\rm tr} = \sigma = 0$; cf. (13) and (14). In other words, two hindrances for reaching (20) from (19) are absent for the quasi-1D model. Below we assume that $F(x)$ and the area $a(x)$ of $A(x)$ are known.

We use scaled functions of x [cf. (15)]:

$$\bar{v} = v/v_1, \bar{\rho} = \rho/\rho_1, \bar{p} = p/p_1, \bar{a} = a/a_1, \bar{F} = F/p_1. \quad (25)$$

Conservation laws of mass and entropy [22,23] are to be taken from volume integrals of (8) [cf. (11), (25)]

$$\bar{\rho}(x)\bar{a}(x)\bar{v}(x) = 1, \qquad \bar{p}(x) = \bar{\rho}^{\gamma}(x). \quad (26)$$

Equations (26) go together with the stationary Euler equation (10) written with the 1D assumption [cf. (1), (16)]:

$$\rho v v' = -p' + F, \quad \text{or} \quad \gamma M_1^2\bar{\rho}\,\bar{v}\,\bar{v}' = -\bar{p}' + \bar{F}, \quad (27)$$

where $(d\mathcal{X}/dx) \equiv \mathcal{X}'$ for any $\mathcal{X}$. Equations (27) and (26) lead to

$$\left[\frac{\gamma M_1^2\bar{v}^2}{2} + \frac{\gamma\bar{\rho}^{\gamma-1}}{\gamma-1}\right]' = \frac{\bar{F}}{\bar{\rho}}. \quad (28)$$

The work will be directly calculated from its definition (12), (16) by employing (26) and (28):

$$\frac{\bar{w}\gamma M_1^2}{2} = -\frac{\int dV\vec{v}\cdot\vec{F}}{a_1 p_1 v_1} = -\int_{x_1}^{x_2} dx\bar{a}(x)\bar{v}(x)\bar{F}(x) \quad (29)$$

$$= -\int_{x_1}^{x_2} dx\frac{\bar{F}(x)}{\bar{\rho}(x)} = \int_{x_2}^{x_1} dx\left[\frac{\gamma M_1^2\bar{v}^2}{2} + \frac{\gamma\bar{\rho}^{\gamma-1}}{\gamma-1}\right]'. \quad (30)$$

Equation (30) recovers the general formula (17) with $\sigma = \bar{v}_{\rm tr} = 0$, as a consequence of the quasi-1D approach.

To understand the physics of this problem, let us note that (26) can be written as, respectively,

$$\frac{\bar{\rho}'}{\bar{\rho}} + \frac{\bar{v}'}{\bar{v}} + \frac{\bar{a}'}{\bar{a}} = 0, \qquad \gamma\frac{\bar{\rho}'}{\bar{\rho}} = \frac{\bar{p}'}{\bar{p}}. \tag{31}$$

We take the derivative in (28) and work it out in two different ways using (31) and $\bar{p}(x) = \bar{\rho}^{\gamma}(x)$:

$$\frac{p'}{p}\left(1 - \frac{v^2}{v_s^2}\right) = \frac{F}{p} + \frac{\gamma v^2}{v_s^2}\frac{a'}{a}, \tag{32}$$

$$\frac{v'}{v}\left(\frac{v^2}{v_s^2} - 1\right) = \frac{a'}{a} + \frac{F}{\gamma p}, \tag{33}$$

where $v_s = v_s(x)$ is the speed of sound defined in (7). In the subsonic case $v^2 < v_s^2$, consider first (32) and (33) for $F = 0$ [22,23]. Now $a'(x) > 0$ implies expected trends: $p'(x) > 0$ and $v'(x) < 0$. Equations (32) and (33) show that a $F < 0$ can reverse those trends for $a'(x) > 0$. This reversing will be seen to be the mechanism of work extraction.

Figure 2 exemplifies the first scenario of work extraction, where $\bar{F}$ is weak. The velocity $\bar{v}(x)$ decays with x; its behavior is close to the case $\bar{F} = 0$ in (27). But the density $\bar{\rho}(x)$ does feel the weak force, since it changes cyclically returning to the initial value once the force ceases to act. We define x_2 such that $\bar{\rho}(x_1) = \bar{\rho}(x_2)$; see (21) and Fig. 2. Hence the work is extracted from the kinetic energy only, and the efficiency equals its maximal value (20).

Equations (32) and (33) explain why the weak force changes qualitatively the behavior of $\bar{p}(x) = \bar{\rho}^{\gamma}(x)$, but does not change the behavior of $\bar{v}(x)$: the geometric factor $\bar{a}'/\bar{a}$ in (32) is multiplied by a factor $\gamma v^2/v_s^2$, which is small for the subsonic flow, and which is lacking in (33).

Figure 2 shows that the change of density $\bar{\rho}(x)$ is small. Hence we can put $\bar{\rho}(x) \simeq 1$ in (29) and (26) obtaining

$$\bar{w} \simeq -\frac{2}{\gamma M_1^2}\int_{x_1}^{x_2} dx \bar{F}(x). \tag{34}$$

For the parameters of Fig. 2, both work and efficiency can be maximized simultaneously. But generally there is a conflict between these two maximizations, since involving the contribution from enthalpy can result in a more work at a smaller efficiency; see below and § 5 of [11].

Maximal work extraction.—Equations (22) and (23) show that, in the quasi-1D case ($\sigma = \bar{v}_{\rm tr} = 0$), the maximal work extraction $\bar{w}_m > 0$ demands a positive contribution $\propto 1 - M_1^2\bar{v}_m^2$ from enthalpy due to $M_1^2 < 1$ (subsonic input) and $\bar{a}_2 > 1$ (expanding area). Figure 3 shows that $\bar{w}_m$ is attained in a strongly forced quasi-1D case with a non-monotonic $\bar{v}(x)$ [cf. (33)] that reaches the sonic value $\bar{v}(x_2) = \bar{v}_m > 1$ in (22). Hence the kinetic energy increases, a typical scenario of attaining $\bar{w}_m$ under subsonic

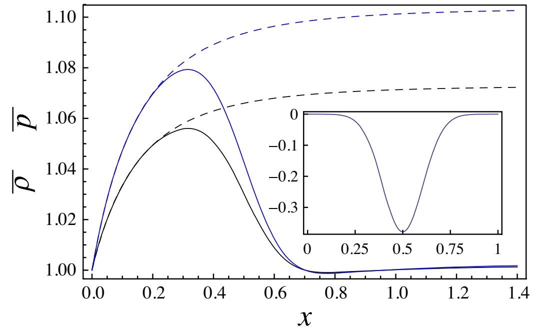

FIG. 2. Dimensionless density $\bar{\rho}$ (black curve) and pressure $\bar{p}$ (blue curve) versus x obtained from solving (27), (26) with an external force $\bar{F}(x) = -(f/L\sqrt{\pi})\exp[-(x - x_0)^2/L^2]$. The force is shown in the inset. Its magnitude is $f = 0.1$, center is at $x_0 = 0.5$, and $L = 0.15$ (we can take $L < 0.15$ without serious changes). Other parameters: $x_1 = 0$, $\bar{a}(x) = (1 + x \times 1.5)^2$, $\gamma = 1.4$ (air), $M_1^2 = 1/7 < 1$ (subsonic input flow). Here $\bar{a}(x) = \{1 + (x/x_2)[(r_2 - r_1)/r_1]\}^2$ refers to a truncated-cone shape of B in Fig. 1 with maximal and minimal radii, respectively, r_2 and r_1. This is the simplest shape for our ends. The black dashed curve and blue dashed curve show, respectively, $\bar{\rho}$ and $\bar{p}$ for $\bar{F} = 0$. The dimensionless velocity $\bar{v}(x)$ decays (not shown) reaching value $1/\bar{a}(x_2)$ for $x = x_2 = 0.955$; cf. (18) with $\sigma = 0$. The dimensionless work $\bar{w}(x)$ (not shown) grows and saturates at (34) for $x \geq 0.8$. We choose $x_2 = 0.955$, since the enthalpy contribution to the work is zero. (This contribution also nullifies for $x_2 = 0.702$, but there the work is smaller.) The efficiency of work extraction from kinetic energy equals to its maximal value (20), which is 0.971 for the present case.

input; see § 6 of [11]. Since $\bar{w}_m > 0$ is extracted from enthalpy only, the efficiency is redefined by normalizing the work to enthalpy input [cf. (6), (9), (16)]:

$$\eta = \frac{-\int dV\vec{v}\cdot\vec{F}}{a_1(\rho_1\varepsilon_1 + p_1)v_1} = \frac{\bar{w}_m(\gamma - 1)M_1^2}{2}. \tag{35}$$

Using $\bar{v}(x_2) = \bar{v}_m \geq 1$, $\sigma > 0$ and $\gamma > 1$ we bound from (23) and (35) the efficiency η at the maximal work extraction from enthalpy [cf. § 6 of [11] and (20)]:

$$\eta \leq 1 - [M_1^2\bar{a}_2^{-2}(x_2)]^{\frac{\gamma-1}{\gamma+1}}. \tag{36}$$

This bound is smaller than one, because we consider the initially subsonic regime $M_1^2 < 1$, and because $\bar{a}_2 > 1$.

For Fig. 3 the efficiency at the maximal work is $\eta(x_2 = 0.6714) = 0.4833$. The work extraction (18)–(21) from kinetic energy can be defined here at a smaller value $x_2 = x_2' = 0.3547$, where cyclic condition (21) holds, and hence the bound (20) is reached and reads: $\bar{w}(x_2') = 0.8185$. This is larger than 0.4833, but the work extracted at efficiency 0.8185 [cf. (34)] is smaller than the maximal value (23); see Fig. 3. This conflict between maximizing the work versus efficiency resembles that for heat engines, where Carnot's efficiency is larger than the

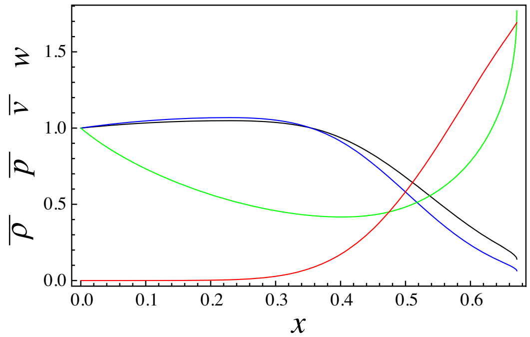

FIG. 3. Dimensionless density $\bar{\rho}$ (black curve), pressure $\bar{p}$ (blue curve), velocity $\bar{v}$ (green curve), and work $w = \bar{w}/10$ (red curve) versus x obtained from solving (27) and (26) for $x \in [0, 0.6714]$ under the same parameters as in Fig. 2 but stronger force $f = 1$. The dimensionless velocity $\bar{v}(x)$ increases for $x > 0.45$ reaching the sonic value (22) at the end point $x = x_2 = 0.6714$, where the work attains its maximum (23). Cyclic value (21), where bound (20) is attained: $\bar{\rho}(x_2' = 0.3547) = 1$.

efficiency at the maximal work, which for certain models has Curzon-Ahlborn's shape [21]; see § 6 of [11].

Outlook.—Our results show that the problem of work-extraction in fluid dynamics is far from being closed and has analogies with heat-engine physics. Possible future directions for this research are quantum windmills [28,29] and work-extraction from quantum flows.

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[11] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.125.064503 consists of 9 chapters. § 1 discusses momentum conservation. § 2 studies Betz's model in detail. § 3 discusses limitations of the present model. § 4 shows that bound (20) applies to non-ideal gases. § 5 and § 6 discuss the work versus efficiency. § 7 explains details of the sonic limit. § 8 considers implications of the Bernoulli equation. § 9 studies relations with d'Alembert's paradox, which includes Refs. [12–20].
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[27] Indeed, $\langle \tilde{\rho} \ln(\tilde{\rho}/\tilde{p}) \rangle \geq 0$, because it is a relative entropy between two distributions. To show $L \equiv \langle \tilde{\rho} \ln \tilde{\rho} \rangle \geq 0$ from (5), minimize L over $\tilde{\rho}$ as independent functional variable keeping track of (5) via a Lagrange multiplier: L is minimized for $\tilde{\rho} = 1$, and $L = 0$ at the minimum.

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