Quantum Thermodynamic Uncertainty Relation for Continuous Measurement

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We use quantum estimation theory to derive a thermodynamic uncertainty relation in Markovian open quantum systems, which bounds the fluctuation of continuous measurements. The derived quantum thermodynamic uncertainty relation holds for arbitrary continuous measurements satisfying a scaling condition. We derive two relations; the first relation bounds the fluctuation by the dynamical activity and the second one does so by the entropy production. We apply our bounds to a two-level atom driven by a laser field and a three-level quantum thermal machine with jump and diffusion measurements. Our result shows that there exists a universal bound upon the fluctuations, regardless of continuous measurements.

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Introduction.—Uncertainty relations distinguish the possible from the impossible and have played fundamental roles in physics. Recently, thermodynamic uncertainty relations (TURs) have been found in stochastic thermodynamics, showing that the fluctuation of time-integrated observables is lower bounded by thermodynamic costs, such as entropy production and dynamical activity [1–23] (see Ref. [24] for a review). TURs predict the fundamental limit of biomolecular processes and thermal machines, and they have been applied to infer the entropy production [25–27].

In contrast to classical systems, studies of TURs in the quantum regime are in very early stages. One of the distinguishing properties of quantum systems is how they behave under measurement. In stochastic thermodynamics, it is naturally assumed that we can measure the stochastic trajectories of the system. In quantum systems, output is obtained through measurements, but the measurements themselves alter the system state. Moreover, in addition to the freedom of how we compute the current in stochastic thermodynamics, we have an extra degree of freedom based on how we measure the quantum systems. Although TURs have been recently studied in quantum systems [28–32], these works have not considered the measurement effects explicitly, or specified a type of measurement in advance.

In this Letter, we derive a quantum thermodynamic uncertainty relation (QTUR) for Markovian open quantum dynamics using quantum estimation theory [33–35]. In Ref. [18], we have derived a TUR for Langevin dynamics via the Cramér-Rao inequality. Extending this line of reasoning to quantum dynamics, we derive a QTUR for continuous measurements with the quantum Cramér-Rao inequality. The quantum Cramér-Rao inequality holds for arbitrary measurements, while the classical one is satisfied for specific measurements, indicating that the quantum version is more general. By virtue of this generality, the obtained QTUR holds for arbitrary continuous measurements satisfying a scaling condition [cf. Eq. (5)]. Our QTUR has two variants; the first relation is bounded by the dynamical activity, and the second by the entropy production. We demonstrate the QTUR with a two-level atom and a quantum thermal machine under jump and diffusion measurements.

Methods.—The TURs in classical stochastic thermodynamics consider the fluctuation of currents, which are time integrals of the stochastic trajectories. Analogously, we wish to bound the fluctuation of the time-integrals of continuous measurements in quantum dynamics.

In continuous measurements, we consider a principal system S and an environment E. Consider a Kraus operator $\mathcal{V}_m$ acting on the principal system, which satisfies $\sum_m \mathcal{V}_m^\dagger \mathcal{V}_m = \mathbb{I}$ ($\mathbb{I}$ denotes the identity operator). We can describe the time evolution induced by the Kraus operator $\mathcal{V}_m$ on the principal system by a unitary operator U acting upon the composite system $S+E$. Let $|e_k\rangle$ be an orthonormal basis for E. We can define the unitary operator U such that [36]

$$|\psi'\rangle = U|\psi_S\rangle \otimes |e_0\rangle = \sum_m \mathcal{V}_m|\psi_S\rangle \otimes |e_m\rangle, \qquad (1)$$

where $|e_0\rangle$ is some standard state of the environment and $|\psi_S\rangle$ is the initial state of the principal system. When applying the measurement $|e_m\rangle$ to the environment, the principal system becomes $|\psi'_S\rangle \propto \mathcal{V}_m|\psi_S\rangle$. Therefore, the operator $\mathcal{V}_m$ is associated with the output m and constitutes a measurement operator. We sequentially repeat this procedure to describe the continuous measurement [35]. We consider a continuous measurement during a time interval $[0,T]$. We discretize time by dividing the interval $[0,T]$ into N equipartitioned intervals, where the time resolution is $\Delta t \equiv T/N$. At each time interval, we consider Eq. (1). Then the state of the composite system at time $t=T$ is [35]

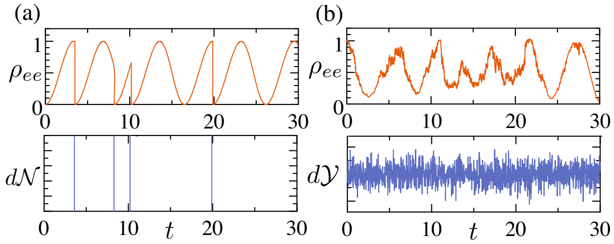

FIG. 1. Quantum trajectories and measurements of (a) jump measurement (photon counting) and (b) diffusion measurement (homodyne detection) in a two-level atom. Upper panels are quantum trajectories of $\rho_{ee} \equiv \langle \epsilon_e|\rho|\epsilon_e\rangle$ and lower panels are measurement outputs.

$$|\psi(T)\rangle = \sum_{\boldsymbol{m}} \mathcal{V}_{m_{N-1}} \ldots \mathcal{V}_{m_0}|\psi_S\rangle \otimes |e_{m_{N-1}}, \ldots, e_{m_0}\rangle, \quad (2)$$

where $\boldsymbol{m} \equiv [m_0, \ldots, m_{N-1}]$. In Eq. (2), we assume that $\mathcal{V}_m$ is time independent, leading to Markovian dynamics. We hereafter consider the limit of $N \to \infty$, where $\boldsymbol{m}$ becomes a record of the continuous measurement. For instance, in the case of a jump measurement, m_i corresponds to either "detection" or "no detection" of a jump within Δt. Depending upon $\boldsymbol{m}$, the state of the principal system $|\psi_S(T)\rangle \propto \mathcal{V}_{m_{N-1}} \ldots \mathcal{V}_{m_0}|\psi_S\rangle$ is determined and is referred to as a quantum trajectory. For example, in Fig. 1, we show quantum trajectories and their corresponding measurement records for the jump [Fig. 1(a)] and diffusion [Fig. 1(b)] measurements.

The time evolution of the density operator ρ is $\dot{\rho} = [\sum_m \mathcal{V}_m \rho \mathcal{V}_m^\dagger - \rho]/dt$, which obeys the Lindblad equation:

$$\dot{\rho} = \mathcal{L}(\rho) \equiv -i[H, \rho] + \sum_c \mathcal{D}(\rho, L_c), \quad (3)$$

where $\mathcal{L}$ is the Lindblad operator, $[\cdot, \cdot]$ is the commutator, H is a Hamiltonian, $\mathcal{D}(\rho, L) \equiv L\rho L^\dagger - \{L^\dagger L, \rho\}/2$ is the dissipator with $\{\cdot, \cdot\}$ being the anticommutator, and L_c is a jump operator. Although the Kraus operator $\mathcal{V}_m$ depends on measurements, the Lindblad equation does not depend on the continuous measurements performed. In Eq. (3), the first and the second terms are referred to as coherent dynamics and dissipation, respectively. We assume that the Hamiltonian H and the jump operators L_c are parametrized by $\theta \in \mathbb{R}$; we express these expressions by H_θ and $L_{c,\theta}$, respectively. We define $\mathcal{L}_\theta$, which is the Lindblad operator consisting of H_θ and $L_{c,\theta}$. We consider the estimation of the parameter θ from the continuous measurement. Let Θ be an observable and $\mathbb{E}_\theta[\Theta]$ be the expectation of Θ with a parameter θ. According to the quantum Cramér-Rao inequality, the following inequality holds [37,38]: $\mathrm{Var}_\theta[\Theta]/(\partial_\theta \mathbb{E}_\theta[\Theta])^2 \geq 1/\mathcal{I}_Q(\theta)$, where $\mathrm{Var}_\theta[\Theta]$ is the variance of Θ and $\mathcal{I}_Q(\theta)$ is the quantum Fisher information (see Refs. [33,34] for its review). This expression is a generalization of the conventional quantum Cramér-Rao inequality [38]. Let $\mathcal{I}_C(\theta; \mathcal{M}_m)$ be the classical Fisher information obtained through positive-operator valued measure elements $\mathcal{M}_m$; then $\mathcal{I}_Q(\theta) = \max_{\mathcal{M}_m} \mathcal{I}_C(\theta; \mathcal{M}_m)$, indicating that the quantum Cramér-Rao inequality is satisfied by any quantum measurements [33,34].

Recently, Ref. [35] obtained the quantum Fisher information for continuous measurements. For $T \to \infty$, Ref. [35] showed that $\mathcal{I}_Q(\theta) = 4T\partial_{\theta_1}\partial_{\theta_2}\tilde{\lambda}_{\theta_1,\theta_2}|_{\theta_1=\theta_2=\theta}$, where $\tilde{\lambda}_{\theta_1,\theta_2}$ is a dominant eigenvalue for $T \to \infty$ of a modified Lindblad operator $\tilde{\mathcal{L}}_{\theta_1,\theta_2}(\rho) \equiv -iH_{\theta_1}\rho + i\rho H_{\theta_2} + \sum_c L_{c,\theta_1}\rho L_{c,\theta_2}^\dagger - (1/2)\sum_c [L_{c,\theta_1}^\dagger L_{c,\theta_1}\rho + \rho L_{c,\theta_2}^\dagger L_{c,\theta_2}]$ (see Refs. [35,39] for derivation). For $\theta_1 \to \theta$ and $\theta_2 \to \theta$, $\tilde{\mathcal{L}}_{\theta_1,\theta_2} \to \mathcal{L}_\theta$ and $\tilde{\lambda}_{\theta_1,\theta_2} \to 0$.

QTUR of dynamical activity.—We now derive a QTUR using the quantum Cramér-Rao inequality. We hereafter assume that the density operator of the system is in a single steady state ρ^{ss} and only consider the limit of $T \to \infty$. In Ref. [18], a TUR was derived via the classical Cramér-Rao inequality by considering a virtual perturbation [16], which affects only the timescale of the dynamics while keeping the steady-state distribution unchanged. Analogously, we consider the following modified Hamiltonian and jump operator in Eq. (3):

$$H_\theta = (1 + \theta)H, \qquad L_{c,\theta} = \sqrt{1 + \theta}L_c. \quad (4)$$

Since the Lindblad operator corresponding to Eq. (4) is given by $\mathcal{L}_\theta = (1 + \theta)\mathcal{L}_{\theta=0}$, the dynamics of $\mathcal{L}_\theta$ are identical to the unmodified dynamics (i.e., the dynamics of $\theta = 0$), except for the time scale. Let us consider a time-integrated observable $\Theta(\boldsymbol{m})$ satisfying

$$\mathbb{E}_\theta[\Theta(\boldsymbol{m})] = h(\theta)\mathbb{E}_{\theta=0}[\Theta(\boldsymbol{m})], \quad (5)$$

where $h(\theta)$ is a scaling function independent of $\boldsymbol{m}$ [$h(0) = 1$ should be satisfied]. Typically, it is given by $h(\theta) = 1 + \theta$. $\Theta(\boldsymbol{m})$ may be an arbitrary function of $\boldsymbol{m}$ as long as Eq. (5) is satisfied. For instance, suppose an estimator counts the number of photons emitted during $[0, T]$; because the system is assumed to be in a steady state, the average number of photons emitted for $\mathcal{L}_\theta$ is $1 + \theta$-times larger than that of $\mathcal{L}_{\theta=0}$, and hence this observable satisfies Eq. (5) with $h(\theta) = 1 + \theta$. Combining the quantum Cramér-Rao inequality and Eq. (5), we find $\mathrm{Var}[\Theta]/\mathbb{E}[\Theta]^2 \geq h'(0)^2/\mathcal{I}_Q(0)$. $\mathcal{I}_Q(\theta)$ can be calculated by differentiation of a dominant eigenvalue of $\tilde{\mathcal{L}}_{\theta_1,\theta_2}$. Using eigenvalue differentiation [35,40], we obtain

$$\frac{\mathrm{Var}[\Theta]}{\mathbb{E}[\Theta]^2} \geq \frac{h'(0)^2}{T(\Upsilon + \Psi)}. \quad (6)$$

Here,

$$\Upsilon \equiv \sum_c \mathrm{Tr}[L_c \rho^{\mathrm{ss}} L_c^\dagger], \tag{7}$$

$$\Psi \equiv -4\mathrm{Tr}[\mathcal{K}_1 \circ \mathcal{L}_{\mathbb{P}}^+ \circ \mathcal{K}_2(\rho^{\mathrm{ss}}) + \mathcal{K}_2 \circ \mathcal{L}_{\mathbb{P}}^+ \circ \mathcal{K}_1(\rho^{\mathrm{ss}})], \tag{8}$$

where $\mathcal{K}_1(\rho) \equiv -iH\rho + (1/2)\sum_c (L_c \rho L_c^\dagger - L_c^\dagger L_c \rho)$ and $\mathcal{K}_2(\rho) \equiv i\rho H + (1/2)\sum_c (L_c \rho L_c^\dagger - \rho L_c^\dagger L_c)$, $\circ$ is function composition, and $\mathcal{L}_{\mathbb{P}}^+$ is a subspace of $\mathcal{L}^+$ that is complementary to the steady-state subspace, with $\mathcal{L}^+$ being the Moore-Penrose pseudoinverse of $\mathcal{L}$ (see Ref. [41] for an explicit expression). Equation (6) is the first result of this Letter, which holds for arbitrary continuous measurements satisfying Eq. (5) in Markovian open quantum systems.

For simplicity, let us consider the following case:

$$L_{ji} = \sqrt{\eta_{ji}} |\epsilon_j\rangle\langle\epsilon_i|, \qquad \rho_{ij}^{\mathrm{ss}} = 0 \quad (i \neq j), \tag{9}$$

where $|\epsilon_i\rangle$ is the eigenbasis of the Hamiltonian H, η_{ji} is a transition rate from $|\epsilon_i\rangle$ to $|\epsilon_j\rangle$ (we redefined the subscript of the jump operator from L_c to L_{ji}), and $\rho_{ij}^{\mathrm{ss}} \equiv \langle\epsilon_i|\rho^{\mathrm{ss}}|\epsilon_j\rangle$. The off-diagonal elements of the steady-state density matrix in the energy eigenbasis are zero. These assumptions are often satisfied for quantum thermal machines [42]. We obtain $\Upsilon = \sum_{i\neq j} \rho_{ii}^{\mathrm{ss}} \eta_{ji}$, corresponding to the dynamical activity in a classical Markov process, implying that Υ is a quantum analogue of the dynamical activity [41]. Moreover, we can obtain a simpler lower bound by scaling the jump operator alone [41]. In this case, Ψ in Eq. (6) becomes 0, which re-derives classical TUR. This shows that Ψ quantifies the degree of the coherent dynamics in the Lindblad equation, which is also shown in a two-level atom. Therefore, Eq. (6) is a quantum generalization of a TUR [10,17], which is bounded by dynamical activity. In classical Markov processes, a TUR bounded by the dynamical activity was derived only for discrete space systems because the dynamical activity is not well defined for continuous space. Contrastingly, Eq. (6) holds for both discrete jump and continuous diffusion cases. Recently, Ref. [30] proved a similar bound for quantum jump processes. The bound of Ref. [30] was derived for given quantum trajectories. Therefore, their bound is obtained for a specified continuous measurement. Reference [32] derived a TUR in a quantum nonequilibrium steady state using the *classical* Cramér–Rao inequality; since their TUR bounds the fluctuation of instantaneous currents (i.e., current-measurement operators), measurement effects are not explicitly incorporated.

As an example of QTUR, we consider a two-level atom driven by a classical laser field. Let $|\epsilon_g\rangle$ and $|\epsilon_e\rangle$ denote the ground and excited states, respectively. A Hamiltonian is given by $H = \Delta|\epsilon_e\rangle\langle\epsilon_e| + (\Omega/2)(|\epsilon_e\rangle\langle\epsilon_g| + |\epsilon_g\rangle\langle\epsilon_e|)$, where Δ is a detuning between the laser-field and the atomic-transition frequencies, and Ω is the Rabi-oscillation frequency. A jump operator is $L = \sqrt{\kappa}|\epsilon_g\rangle\langle\epsilon_e|$, where κ is the decay rate, and it induces a jump from $|\epsilon_e\rangle$ to $|\epsilon_g\rangle$. We obtain the dynamical activity $\Upsilon = \kappa\rho_{ee}^{\mathrm{ss}} = \kappa\Omega^2/(4\Delta^2 + \kappa^2 + 2\Omega^2)$ and the coherent-dynamics contribution

$$\Psi = \frac{8\Omega^4[4\Delta^4 + \Delta^2(\kappa^2 + 8\Omega^2) + (\kappa^2 + 2\Omega^2)^2]}{\kappa(4\Delta^2 + \kappa^2 + 2\Omega^2)^3}. \tag{10}$$

We first consider a jump measurement (photon detection). The quantum trajectory is given by the stochastic Schrödinger equation (where the corresponding $\mathcal{V}_m$ is shown in Ref. [41]):

$$d\rho = \left(-i[H,\rho] - \frac{1}{2}\{L^\dagger L, \rho\} + \rho \mathrm{Tr}[L\rho L^\dagger]\right)dt + \left(\frac{L\rho L^\dagger}{\mathrm{Tr}[L\rho L^\dagger]} - \rho\right)d\mathcal{N}, \tag{11}$$

where $d\mathcal{N}$ is a noise increment and $d\mathcal{N} = 1$ when a photon is detected between t and $t + dt$ and $d\mathcal{N} = 0$ otherwise. $\boldsymbol{m} = [m_0, \ldots, m_{N-1}]$ in Eq. (2) corresponds to $[\Delta\mathcal{N}_0, \ldots, \Delta\mathcal{N}_{N-1}]$. The average of this quantity reads $\mathbb{E}[d\mathcal{N}] = \mathrm{Tr}[L\rho^{\mathrm{ss}}L^\dagger]dt$. We consider an observable $\Theta_\mathcal{N} \equiv \int_0^T d\mathcal{N}$, which counts the number of photons emitted within the interval $[0, T]$. Since $\mathbb{E}_\theta[d\mathcal{N}] = (1 + \theta)\mathbb{E}_{\theta=0}[d\mathcal{N}]$ and thus $\mathbb{E}_\theta[\Theta_\mathcal{N}] = (1 + \theta)\mathbb{E}_{\theta=0}[\Theta_\mathcal{N}]$, $\Theta_\mathcal{N}$ satisfies the QTUR of Eq. (6) with $h'(0) = 1$.

We next consider a diffusion measurement (homodyne detection). A quantum trajectory of the diffusion measurement is given by a quantum-state diffusion (the corresponding $\mathcal{V}_m$ is shown in Ref. [41]):

$$d\rho = \left(-i[H,\rho] - \frac{1}{2}\{L^\dagger L, \rho\} + L\rho L^\dagger\right)dt + (L\rho + \rho L^\dagger - \mathrm{Tr}[L\rho + \rho L^\dagger]\rho)dW, \tag{12}$$

where W is the standard Wiener process. The measurement result is given by $d\mathcal{Y} = \mathrm{Tr}[L\rho + \rho L^\dagger]dt + dW$ [43]. $\boldsymbol{m} = [m_0, \ldots, m_{N-1}]$ in Eq. (2) corresponds to $[\Delta\mathcal{Y}_0, \ldots, \Delta\mathcal{Y}_{N-1}]$. We consider an observable $\Theta_\mathcal{Y} \equiv \int_0^T d\mathcal{Y}$. Since $\mathbb{E}_\theta[\Theta_\mathcal{Y}] = \int_0^T \mathrm{Tr}[L_\theta\rho^{\mathrm{ss}} + \rho^{\mathrm{ss}}L_\theta^\dagger]dt = \sqrt{1+\theta}\,\mathbb{E}_{\theta=0}[\Theta_\mathcal{Y}]$ $[h(\theta) = \sqrt{1+\theta}$ in Eq. (5)], $\Theta_\mathcal{Y}$ satisfies the QTUR of Eq. (6) with $h'(0) = 1/2$. Therefore, the lower bound of the diffusion measurement is $1/4$ times smaller than that of the jump measurement.

We verify the QTUR of Eq. (6) for the two-level atom with a computer simulation [44,45]. We first plot $\mathcal{I}_Q(0) = T(\Upsilon + \Psi)$ (solid line), $T\Upsilon$ (dashed line), and $T\Psi$ (dotted line) as a function of κ in Fig. 2(a) [parameters are shown in the caption of Fig. 2(a)]. From Fig. 2(a), when κ becomes larger (i.e., more frequent jumps), the dynamical activity Υ is dominant in the quantum Fisher information $\mathcal{I}_Q(0)$. For $\kappa \to 0$, $\Upsilon \to 0$ and the coherent dynamics contribution Ψ becomes the major portion of $\mathcal{I}_Q(0)$. We numerically check

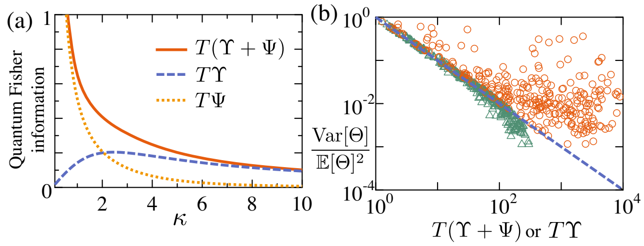

FIG. 2. Quantum Fisher information and the results of computer simulations of jump measurement. (a) The quantum Fisher information $\mathcal{I}_Q(0) = T(\Upsilon + \Psi)$ (solid line), $T\Upsilon$ (dashed line), and $T\Psi$ (dotted line) as a function of κ, where $T = 1$, $\Omega = 1$, and $\Delta = 1$. (b) $\mathrm{Var}[\Theta_\mathcal{N}]/\mathbb{E}[\Theta_\mathcal{N}]^2$ as a function of $T(\Upsilon + \Psi)$ (circles) and $T\Upsilon$ (triangles) for the jump measurement, where $\Delta \in [0.1, 10.0]$, $\Omega \in [0.1, 10.0]$, $\kappa \in [0.1, 10.0]$, and $T = 1000$. The dashed line corresponds to $1/[T(\Upsilon + \Psi)]$ for the circles and $1/[T\Upsilon]$ for the triangles.

the QTUR for the jump measurement by randomly generating κ, Ω, and Δ [the ranges of the parameters are shown in the caption of Fig. 2(b)] and calculate $\mathrm{Var}[\Theta_\mathcal{N}]/\mathbb{E}[\Theta_\mathcal{N}]^2$. In Fig. 2(b), the circles denote $\mathrm{Var}[\Theta_\mathcal{N}]/\mathbb{E}[\Theta_\mathcal{N}]^2$ as a function of $T(\Upsilon + \Psi)$ and the lower bound of Eq. (6) is shown by the dashed line. We confirm that all realizations satisfy the QTUR, which verifies Eq. (6). In a classical case [10,17], the lower bound arises from the dynamical activity alone (i.e., $T\Upsilon$). Thus, we also check whether $\mathrm{Var}[\Theta_\mathcal{N}]/\mathbb{E}[\Theta_\mathcal{N}]^2$ can be bounded only by $T\Upsilon$. In Fig. 2(b), the triangles denote $\mathrm{Var}[\Theta_\mathcal{N}]/\mathbb{E}[\Theta_\mathcal{N}]^2$ as a function of $T\Upsilon$, where the dashed line describes $1/(T\Upsilon)$. Clearly, some realizations are below $1/(T\Upsilon)$, indicating that the lower bound of the QTUR is below the classical bound [10,17]. Similar enhancement of precision has been reported for quantum jump processes [30], and for classical systems with periodic driving [13] or magnetic fields [11]. We also performed a computer simulation for the diffusion measurement and verified the bound (see Ref. [41]).

QTUR of entropy production.—Employing a scaling different from Eq. (4), we can bound $\mathrm{Var}[\Theta]/\mathbb{E}[\Theta]^2$ by the entropy production. Again, we assume that the system satisfies the conditions of Eq. (9). Moreover, we assume that whenever $\eta_{ji} > 0$, $\eta_{ij} > 0$ should be satisfied. Inspired by Ref. [19], we consider the following modified process instead of Eq. (4):

$$L_{ji,\theta} = \sqrt{\eta_{ji}\left[1 + \theta\left(1 - \sqrt{\frac{\eta_{ij}\rho_{jj}^{\mathrm{ss}}}{\eta_{ji}\rho_{ii}^{\mathrm{ss}}}}\right)\right]}|\epsilon_j\rangle\langle\epsilon_i|\ (i \neq j). \quad (13)$$

With Eq. (13), the steady-state density remains unchanged. Repeating a similar calculation to the dynamical-activity case (see Ref. [41] for details), an observable Θ satisfying Eq. (5) obeys

$$\frac{\mathrm{Var}[\Theta]}{\mathbb{E}[\Theta]^2} \geq \frac{2h'(0)^2}{T\Sigma}, \quad (14)$$

where $\Sigma \equiv \sum_{i\neq j}\rho_{ii}^{\mathrm{ss}}\eta_{ji}\ln[\rho_{ii}^{\mathrm{ss}}\eta_{ji}/(\rho_{jj}^{\mathrm{ss}}\eta_{ij})]$. Equation (14) is the second result of this Letter. The expression of Σ is identical to the entropy production rate in stochastic thermodynamics [46]; therefore, our approach rederives the classical TUR [1,3] but its applicability is broader than that of a classical counterpart, as detailed below.

As an example, we consider a quantum thermal machine. Such machines are the basis for quantum clocks and thus it is important to consider their precision [28,42]. Specifically, we employ a thermal machine with three levels $|\epsilon_A\rangle$, $|\epsilon_B\rangle$, and $|\epsilon_g\rangle$ powered by three heat reservoirs at different inverse temperatures β_r ($r = 1, 2, 3$) [42,47]. Each transition is coupled with each of the heat reservoirs [Fig. 3(a)]. The Hamiltonian is $H = \omega_3|\epsilon_B\rangle\langle\epsilon_B| + \omega_1|\epsilon_A\rangle\langle\epsilon_A|$, where ω_1, ω_2, and $\omega_3 = \omega_1 + \omega_2$ are energy gaps between $|\epsilon_A\rangle \leftrightarrow |\epsilon_g\rangle$, $|\epsilon_B\rangle \leftrightarrow |\epsilon_A\rangle$, and $|\epsilon_B\rangle \leftrightarrow |\epsilon_g\rangle$, respectively. Let $\dot{Q}_r$ be the heat current from the rth reservoir with temperature β_r. We assume that the dynamics of the density operator ρ obey the Lindblad equation $\dot{\rho} = -i[H, \rho] + \sum_{i\neq j}\mathcal{D}(\rho, L_{ji})$, where L_{ji} is defined in Eq. (9) with $\eta_{gA} = \gamma(n_1^{\mathrm{th}} + 1)$, $\eta_{Ag} = \gamma n_1^{\mathrm{th}}$, $\eta_{AB} = \gamma(n_2^{\mathrm{th}} + 1)$, $\eta_{BA} = \gamma n_2^{\mathrm{th}}$, $\eta_{gB} = \gamma(n_3^{\mathrm{th}} + 1)$, and $\eta_{Bg} = \gamma n_3^{\mathrm{th}}$ [$n_r^{\mathrm{th}} \equiv (e^{\beta_r\omega_r} - 1)^{-1}$ and γ is the decay rate]. The entropy production rate is $\Phi = -\sum_{r=1}^{3}\beta_i\dot{Q}_r$ [48,49], and satisfies $\Phi = \Sigma$ [41]. Therefore, the classical entropy production rate Σ corresponds to the entropy production rate in the quantum thermal machine Φ.

We first consider a standard jump measurement. The quantum trajectory is given by a stochastic Schrödinger equation:

$$d\rho = -i[H, \rho]dt + \sum_{i\neq j}\left(\rho\mathrm{Tr}[L_{ji}\rho L_{ji}^\dagger] - \frac{\{L_{ji}^\dagger L_{ji}, \rho\}}{2}\right)dt$$
$$+ \sum_{i\neq j}\left(\frac{L_{ji}\rho L_{ji}^\dagger}{\mathrm{Tr}[L_{ji}\rho L_{ji}^\dagger]} - \rho\right)d\mathcal{N}_{ji}, \quad (15)$$

where $d\mathcal{N}_{ji}$ is a noise increment as defined in Eq. (11). We consider the observable $\Theta_C \equiv \sum_{i\neq j}R_{ji}\int_0^T d\mathcal{N}_{ji}$, where $R_{ji} = -R_{ij}$ and $R_{ji} \in \mathbb{R}$. Θ_C satisfies the scaling condition of Eq. (5) and thus the QTUR of Eq. (14). Because the dynamics of Eq. (15) are jumps between energy eigenstates that are equivalent to classical dynamics, Θ_C trivially satisfies Eq. (14).

We next consider a transformed jump measurement [50]. The Lindblad equation is invariant under the transformation $L'_{ji} = L_{ji} + \zeta_{ji}\mathbb{I}$ and $H' = H - (i/2)\sum_{i\neq j}[\zeta_{ji}^* L_{ji} - \zeta_{ji}L_{ji}^\dagger]$, where $\zeta_{ji} \in \mathbb{C}$ is a parameter. $\zeta_{ji} = 0$ for all i and j recovers the standard jump measurement. Thus, we can consider a transformed stochastic Schrödinger equation, where H and L_{ji} are replaced with

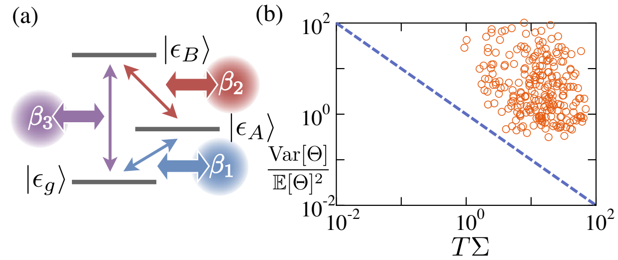

FIG. 3. Illustration of the model and results of computer simulation for the thermal machine. (a) Thermal machine consisting of three levels $|\epsilon_A\rangle$, $|\epsilon_B\rangle$, and $|\epsilon_g\rangle$. The transitions between each of the states are coupled with heat reservoirs with the inverse temperature β_r $(r = 1, 2, 3)$. (b) $\mathrm{Var}[\Theta'_C]/\mathbb{E}[\Theta'_C]^2$ (circles) as a function of $T\Sigma$ for the transformed jump measurement, where $\beta_1 \in [0.1, 1.0]$, $\beta_2 \in [0.01, 0.1]$, $\beta_3 \in [1.0, 10.0]$, $\omega_1 \in [1.0, 10.0]$, $\omega_2 \in [1.0, 10.0]$, $R'_{ij} \in [0.0, 1.0]$, $|\zeta_{ij}| \in [0.0, 0.2]$, $\gamma = 0.1$, and $T = 100$. R'_{ji} and ζ_{ji} are defined for $i \neq j$ and satisfy $R'_{ji} = -R'_{ij}$ and $|\zeta_{ji}| = |\zeta_{ij}|$. The dashed line corresponds to the lower bound $2/[T\Sigma]$.

H' and L'_{ji}, respectively, in Eq. (15), and we define $d\mathcal{N}'_{ji}$ as a noise increment in the transformed equation. Quantum trajectories are no longer simple jump processes between energy eigenstates [41]. We consider an observable $\Theta'_C \equiv \sum_{i\neq j} R'_{ji} \int_0^T d\mathcal{N}'_{ji}$, where $R'_{ji} = -R'_{ij}$, for the transformed equation. When $|\zeta_{ji}| = |\zeta_{ij}|$ for all i and j, Θ'_C satisfies the scaling condition of Eq. (5) and the QTUR of Eq. (14) holds [41].

We verify the QTUR of Eq. (14) for the transformed jump measurement (i.e., $\zeta_{ji} \neq 0$ and $|\zeta_{ij}| = |\zeta_{ji}|$) via a computer simulation by randomly generating β_r, ω_r, R'_{ji}, and ζ_{ji} [parameters are shown in the caption of Fig. 3(b)] and calculating $\mathrm{Var}[\Theta'_C]/\mathbb{E}[\Theta'_C]^2$. In Fig. 3(b), the circles denote $\mathrm{Var}[\Theta'_C]/\mathbb{E}[\Theta'_C]^2$ as a function of the entropy production $T\Sigma$ and the lower bound of Eq. (14) is shown by a dashed line. We confirm that all realizations satisfy the QTUR, verifying Eq. (14). Although the bound of Eq. (14) itself is identical to the classical TUR [1,3], our QTUR provides the lower bound for arbitrary measurements with the scaling condition. No matter how we measure the thermal machine, an observable satisfying the scaling relation [Eq. (5)] should obey the QTUR of Eq. (14), which cannot be deduced from classical TURs. We also note observables not satisfying the scaling condition of Eq. (5). As demonstrated in the example, although the scaling condition is satisfied for typical measurement schemes, such as jump and diffusion measurements, this is not the case for general continuous measurements. For such cases, inequalities of Eqs. (6) and (14) hold with $\mathbb{E}[\Theta]$ replaced by $\partial_\theta \mathbb{E}_\theta[\Theta]$.

Conclusion.—In this Letter, we have derived the QTUR from the quantum Cramér-Rao inequality. The QTUR holds for arbitrary continuous measurements satisfying the scaling condition. We expect the present study to form a basis for obtaining uncertainty relations in the quantum regime.

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