Minimal Tradeoff and Ultimate Precision Limit of Multiparameter Quantum Magnetometry under the Parallel Scheme

Zhibo Hou,* Zhao Zhang, Guo-Yong Xiang,† Chuan-Feng Li, and Guang-Can Guo
CAS Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei 230026, People's Republic of China
and CAS Center For Excellence in Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei 230026, People's Republic of China

Hongzhen Chen,* Liqiang Liu,* and Haidong Yuan‡
Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, Shatin, Hong Kong

(Received 9 September 2019; accepted 12 May 2020; published 8 July 2020)

The precise measurement of a magnetic field is one of the most fundamental and important tasks in quantum metrology. Although extensive studies on quantum magnetometry have been carried out over past decades, the ultimate precision that can be achieved for the estimation of all three components of a magnetic field under the parallel scheme remains unknown. This is largely due to the lack of understandings on the incompatibility of the optimal probe states for the estimation of the three components. Here we provide an approach to characterize the minimal tradeoff among the precisions of multiple parameters that arise from the incompatibility of the optimal probe states, which leads to the identification of the ultimate precision limit for the estimation of all three components of a magnetic field under the parallel scheme. The optimal probe state that achieves the ultimate precision is also explicitly constructed. The obtained precision sets a benchmark on the precision of the multiparameter quantum magnetometry under the parallel scheme, which is of fundamental interest and importance in quantum metrology.

DOI: 10.1103/PhysRevLett.125.020501

Many applications of quantum metrology can be reduced to the measurement and estimation of a magnetic field. For example, various applications in quantum biosensing with nitrogen vacancy (NV) centers are achieved by measuring the magnetic field of the targeted biomolecules [1]. Quantum magnetometry under the parallel scheme that utilizes entangled probe states, as shown in Fig. 1, has been studied over many decades since the pioneer work of Helstrom and Holevo [2,3]. The ultimate precision, however, is only well understood for the single-parameter quantum magnetometry. An example extensively studied is the estimation of the magnitude of a magnetic field. In this case, the ultimate precision for the local estimation, where the experiment needs to be repeated for sufficient number of times, is achieved by the GHZ-type state as $[(|00...0\rangle + |11...1\rangle)/\sqrt{2}]$ [4,5]. For the Bayesian estimation, the minimal Holevo covariance is achieved with the Berry-Wisemen type of states [6]. For the estimation of all three components of the magnetic field, the answer is only known for special cases. For the Bayesian estimation, the optimal performance for the estimation of the generated unitary rotation has been studied under the assumption of uniform prior distribution [7–13]. For the local estimation, the optimal performance is only known when the unitary rotation generated is close to the identity and the figure of merit is taken as the sum of equally weighted variance [14–23], for which it has been shown that the best precision is achieved by 2-anticoherent states [14]. For general unitary rotations, a heuristic state, which is also 2-anticoherent, is provided in [15] with the achieved precision matching the optimal performance in the weak limit, i.e., when the generated unitary is close to the identity. In general, however, the optimal performance of

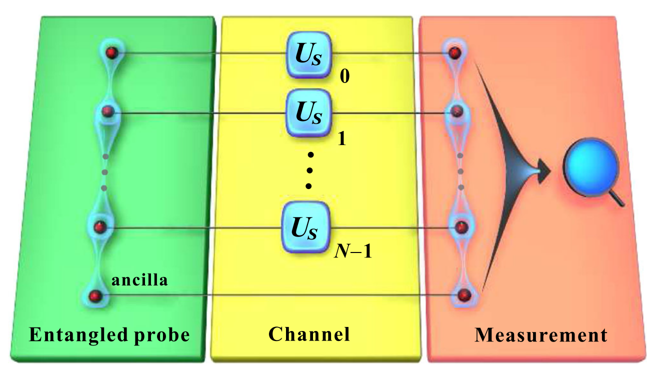

FIG. 1. Parallel scheme for multiparameter quantum magnetometry. Here $U_s = e^{-i\boldsymbol{B}\cdot\boldsymbol{\sigma}t}$ describes the unitary dynamics on the spin due to the interaction with the magnetic field. An additional ancillary system may be used.

the multiparameter quantum magnetometry under the parallel scheme remains unknown.

This is related to a main research theme in multiparameter quantum estimation, which is to quantify the tradeoff among the precisions of multiple parameters [24–42]. Over the past decades there have been extensive studies on this theme, however, the minimal tradeoff remains only known for very limited cases [24–27]. The study on the tradeoff induced by the incompatibility of the measurements has made much progress [3,32–40,43–48]. However, the tradeoff induced by the incompatibility of the optimal probe states is still poorly understood. Here we present an approach to quantify the tradeoff induced by the incompatibility of the optimal probe states. With this approach we obtain the minimal tradeoff for the multiparameter quantum magmetometry under the parallel scheme, where the figure of merit can be taken as the sum of arbitrarily weighted variance and the generated unitary does not need to be close to the identity operator. This enables the identification of the ultimate precision limit for multiparameter quantum magnetometry under the parallel scheme, which can also be used to calibrate the ultimate performances of the quantum reference frame alignment, quantum gyroscope, etc. We note that additional controls during the evolution are not considered in the parallel scheme. Controlled schemes have been studied in [26,49,50], where it has been shown that optimal controls may further improve the precision.

We first use spin-1/2 as the probe. The dynamics for a spin-1/2 in a magnetic field can be described by the Hamiltonian $H = \boldsymbol{B}\cdot\boldsymbol{\sigma} = B_1\sigma_1 + B_2\sigma_2 + B_3\sigma_3$, where σ_1, σ_2, and σ_3 are Pauli matrices. This can be equivalently written as $H = B\boldsymbol{n}\cdot\boldsymbol{\sigma}$ with $B = \sqrt{B_1^2 + B_2^2 + B_3^2}$ as the magnitude and $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ as the direction of the magnetic field. After an evolution time t, the dynamics generates a SU(2) operator as $U_s = e^{-i\alpha\boldsymbol{n}\cdot\boldsymbol{\sigma}}$ with $\alpha = Bt$. As we allow the figure of merit taken as the sum of arbitrarily weighted variance, we can just consider the precision for the simultaneous estimation of α, θ and ϕ with the figure of merit as $w_1\delta\hat{\alpha}^2 + w_2\delta\hat{\theta}^2 + w_3\delta\hat{\phi}^2$, here $w_1, w_2, w_3 > 0$ are the weights and $\delta\hat{x}^2 = E[(\hat{x} - x)^2]$ denotes the variance for an unbiased estimator. The precision of various other parameters can be expressed in terms of α, θ, and ϕ with different weights. For example, the precision for the estimation of B is related to α as $\delta\hat{B}^2 = (\delta\hat{\alpha}^2/t^2)$, thus the sum of equally weighted variance for (B, θ, ϕ) can be written as $\delta\hat{B}^2 + \delta\hat{\theta}^2 + \delta\hat{\phi}^2 = (1/t^2)\delta\hat{\alpha}^2 + \delta\hat{\theta}^2 + \delta\hat{\phi}^2$. Similarly the sum of equally weighted variance for the estimation of (B_1, B_2, B_3) can be expressed as $\delta\hat{B}_1^2 + \delta\hat{B}_2^2 + \delta\hat{B}_3^2 = (1/t^2)(\delta\hat{\alpha}^2 + \alpha^2\delta\hat{\theta}^2 + \alpha^2\sin^2\theta\delta\hat{\phi}^2)$. This differs from most previous studies which take the figure of merit as the sum of equally weighted variance under a specific parametrization [15–19,22, 51,52].

The precision limit for a parameter x can be calibrated by the quantum Crámer-Rao bound (QCRB)

$$\delta\hat{x}^2 \geq \frac{1}{mJ_x}, \tag{1}$$

here $J_x = 4\langle\Delta H_x^2\rangle$ is the quantum Fisher information (QFI), m is the number of repetition (which we will neglect as it accounts the classical effect). H_x is the generator of the corresponding parameter x [2,3,53], which is defined as $H_x \equiv iU_s^\dagger(\partial_x U_s)$, $U_s = e^{-i\alpha\boldsymbol{n}\cdot\boldsymbol{\sigma}}$ is the generated unitary [40,54–57], $\langle\Delta H_x^2\rangle = \langle\Phi|H_x^2|\Phi\rangle - \langle\Phi|H_x|\Phi\rangle^2$ is the variance of H_x with respect to the initial probe state $|\Phi\rangle$. For $x \in \{\alpha, \theta, \phi\}$, the corresponding generator can be obtained as

$$\begin{aligned} H_\alpha &= c_\alpha\boldsymbol{n}_\alpha\cdot\boldsymbol{\sigma}, \\ H_\theta &= c_\theta\boldsymbol{n}_\theta\cdot\boldsymbol{\sigma}, \\ H_\phi &= c_\phi\boldsymbol{n}_\phi\cdot\boldsymbol{\sigma}, \end{aligned} \tag{2}$$

with $c_\alpha = 1$, $\boldsymbol{n}_\alpha = \boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $c_\theta = \sin\alpha$, $\boldsymbol{n}_\theta = \cos\alpha\boldsymbol{n}_1 - \sin\alpha\boldsymbol{n}_2$, $c_\phi = \sin\alpha\sin\theta$ and $\boldsymbol{n}_\phi = \cos\alpha\boldsymbol{n}_2 + \sin\alpha\boldsymbol{n}_1$, respectively, here $\boldsymbol{n}_1 = \partial_\theta\boldsymbol{n} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$, $\boldsymbol{n}_2 = \boldsymbol{n}\times\boldsymbol{n}_1 = (-\sin\phi, \cos\phi, 0)$.

With N spins interacting with the field, aided with an ancilla, the generator for each parameter is $H_x^{(N)} = \sum_{k=0}^{N-1} H_x^{[k]}$, where $H_x^{[k]} = I\otimes\cdots\otimes I\otimes H_x\otimes I\cdots\otimes I\otimes I_A$ denotes the generator on the kth spin, I denotes the identity operator and I_A denotes the identity operator on the ancilla. The variance of $H_x^{(N)}$ is given by

$$\langle\Delta[H_x^{(N)}]^2\rangle = \langle(H_x^{(N)})^2\rangle - \langle H_x^{(N)}\rangle^2, \tag{3}$$

where the first term can be expanded as

$$\begin{aligned} \langle(H_x^{(N)})^2\rangle &= \sum_{k=0}^{N-1}\langle(H_x^{[k]})^2\rangle + \sum_{j\neq k}\langle H_x^{[j]}H_x^{[k]}\rangle, \\ &= c_x^2\left(N + \sum_{j\neq k} r_{xx}^{(j,k)}\right), \end{aligned} \tag{4}$$

and the second term as $\langle H_x^{(N)}\rangle^2 = c_x^2(\sum_{k=0}^{N-1} r_x^{(k)})^2$, here $r_{xx}^{(j,k)} = \mathrm{tr}[\rho^{(j,k)}(\boldsymbol{n}_x\cdot\boldsymbol{\sigma}\otimes\boldsymbol{n}_x\cdot\boldsymbol{\sigma})] \leq 1$, $r_x^{(k)} = \mathrm{tr}(\rho^{(k)}\boldsymbol{n}_x\cdot\boldsymbol{\sigma})$ with $\rho^{(j,k)}$ as the reduced density matrix for the jth and kth spin and $\rho^{(k)}$ as the reduced density matrix for the kth spin [15,22,52]. It is easy to see the same formula holds without the ancillary system (which corresponds to taking $I_A = 1$), however, for finite N the ancillary system provides more room on the choices of $\rho^{(j,k)}$, which can be seen in the analysis of the optimal states below.

It is easy to see $\langle\Delta[H_x^{(N)}]^2\rangle \leq N^2 c_x^2$, where the equality can be reached if and only if $\sum_{k=0}^{N-1} r_x^{(k)} = 0$ and $r_{xx}^{(j,k)} = 1$ for all j, k. For a single parameter this can be achieved by choosing the probe state as the GHZ-type state, $|\Phi_x\rangle = (1/\sqrt{2})(|+_x\rangle^{\otimes N} + |-_x\rangle^{\otimes N})$, where $|\pm_x\rangle$ are the eigenstates of H_x. It is easy to check that the reduced two-spin state is $\rho^{(j,k)} = \frac{1}{2}(|+_x+_x\rangle\langle+_x+_x| + |-_x-_x\rangle\langle-_x-_x|) = \frac{1}{4}(I^{(j,k)} + \boldsymbol{n}_x\cdot\boldsymbol{\sigma}\otimes\boldsymbol{n}_x\cdot\boldsymbol{\sigma})$ for all (j,k) and the reduced single spin state is $\rho^{(k)} = I^{(k)}/2$ for all k, thus $r_{xx}^{(j,k)} = 1$ and $r_x^{(k)} = 0$. The highest precision for a single parameter can thus be achieved.

For the estimation of multiple parameters, however, the issue is much more complicated. A main research theme in multiparameter quantum metrology is to determine whether it is possible to achieve the highest precisions for all parameters simultaneously and calibrate the minimal tradeoff among the precisions of different parameters when it is not possible.

To calibrate the tradeoff, we write a general two-qubit state as

$$\rho^{(j,k)} = \frac{1}{4}\bigg[I^{(j,k)} + \sum_l r_l^{(j)}\sigma_l^{(j)}\otimes I^{(k)} + \sum_p r_p^{(k)} I^{(j)}\otimes\sigma_p^{(k)} + \sum_{l,p} r_{lp}^{(j,k)}\sigma_l^{(j)}\otimes\sigma_p^{(k)}\bigg], \tag{5}$$

here $l,p\in\{\alpha,\theta,\phi\}$, and we have denoted $\sigma_\alpha = \boldsymbol{n}_\alpha\cdot\boldsymbol{\sigma}$, $\sigma_\theta = \boldsymbol{n}_\theta\cdot\boldsymbol{\sigma}$ and $\sigma_\phi = \boldsymbol{n}_\phi\cdot\boldsymbol{\sigma}$. Now let $U = e^{i(\alpha/2)\boldsymbol{n}\cdot\boldsymbol{\sigma}}e^{-i(\phi/2)\sigma_3}e^{-i(\theta/2)\sigma_2}$, which is the unitary that satisfies $U\sigma_1U^\dagger = \sigma_\theta$, $U\sigma_2U^\dagger = \sigma_\phi$ and $U\sigma_3U^\dagger = \sigma_\alpha$, and let $|\Phi_-^{(j,k)}\rangle = (U/\sqrt{2})(|01\rangle - |10\rangle)$, then $|\Phi_-^{(j,k)}\rangle\langle\Phi_-^{(j,k)}| = \frac{1}{4}[I^{(j,k)} - \sum_{x\in\{\alpha,\theta,\phi\}}\sigma_x^{(j)}\otimes\sigma_x^{(k)}]$. As $\rho^{(j,k)}\geq 0$, we have $\langle\Phi_-^{(j,k)}|\rho^{(j,k)}|\Phi_-^{(j,k)}\rangle = \mathrm{tr}(\rho^{(j,k)}|\Phi_-^{(j,k)}\rangle\langle\Phi_-^{(j,k)}|)\geq 0$, which gives a constraint as $r_{\alpha\alpha}^{(j,k)} + r_{\theta\theta}^{(j,k)} + r_{\phi\phi}^{(j,k)} \leq 1$. This clearly shows that $r_{\alpha\alpha}^{(j,k)}$, $r_{\theta\theta}^{(j,k)}$, $r_{\phi\phi}^{(j,k)}$ can not equal to 1 simultaneously and the tradeoff among the precisions of different parameters is unavoidable. It turns out such constraint fully calibrates the tradeoff among the precisions.

We consider the figure of merit as $w_\alpha\delta\hat{\alpha}^2 + w_\theta\delta\hat{\theta}^2 + w_\phi\delta\hat{\phi}^2$, where $w_i > 0$ are weights that can be chosen arbitrarily according to specific needs. Under the constraint $r_{\alpha\alpha}^{(j,k)} + r_{\theta\theta}^{(j,k)} + r_{\phi\phi}^{(j,k)} \leq 1$, the sum of weighted variance is then bounded below as [58]

$$w_\alpha\delta\hat{\alpha}^2 + w_\theta\delta\hat{\theta}^2 + w_\phi\delta\hat{\phi}^2 \geq \frac{\left(\sqrt{w_\alpha} + \frac{\sqrt{w_\theta}}{|\sin\alpha|} + \frac{\sqrt{w_\phi}}{|\sin\alpha\sin\theta|}\right)^2}{4N(N+2)}. \tag{6}$$

This bound can be saturated when the reduced two-qubit state takes the form as $\rho^{(j,k)} = \frac{1}{4}[I^{(j,k)} + \tilde{r}_{\alpha\alpha}\sigma_\alpha^{(j)}\otimes\sigma_\alpha^{(k)} + \tilde{r}_{\theta\theta}\sigma_\theta^{(j)}\otimes\sigma_\theta^{(k)} + \tilde{r}_{\phi\phi}\sigma_\phi^{(j)}\otimes\sigma_\phi^{(k)}]$ for all $0\leq j<k\leq N-1$ with

$$\begin{aligned}
\tilde{r}_{\alpha\alpha} &= \frac{(N+1)\sqrt{w_\alpha} - \frac{\sqrt{w_\theta}}{|\sin\alpha|} - \frac{\sqrt{w_\phi}}{|\sin\alpha\sin\theta|}}{(N-1)\left(\sqrt{w_\alpha} + \frac{\sqrt{w_\theta}}{|\sin\alpha|} + \frac{\sqrt{w_\phi}}{|\sin\alpha\sin\theta|}\right)},\\
\tilde{r}_{\theta\theta} &= \frac{(N+1)\frac{\sqrt{w_\theta}}{|\sin\alpha|} - \sqrt{w_\alpha} - \frac{\sqrt{w_\phi}}{|\sin\alpha\sin\theta|}}{(N-1)\left(\sqrt{w_\alpha} + \frac{\sqrt{w_\theta}}{|\sin\alpha|} + \frac{\sqrt{w_\phi}}{|\sin\alpha\sin\theta|}\right)},\\
\tilde{r}_{\phi\phi} &= \frac{(N+1)\frac{\sqrt{w_\phi}}{|\sin\alpha\sin\theta|} - \sqrt{w_\alpha} - \frac{\sqrt{w_\theta}}{|\sin\alpha|}}{(N-1)\left(\sqrt{w_\alpha} + \frac{\sqrt{w_\theta}}{|\sin\alpha|} + \frac{\sqrt{w_\phi}}{|\sin\alpha\sin\theta|}\right)},
\end{aligned} \tag{7}$$

and the quantum Fisher information matrix is diagonal. The problem now is to identify the states whose reduced two-spin states are of this form which leads to the minimal tradeoff among the precisions. We first consider the case with ancilla, then without ancilla.

By employing a qutrit (or three levels of two additional spin-1/2) as the ancillary system we can prepare the probe state as

$$|\Phi_{\mathrm{SA}}\rangle = s_\alpha|\Phi_\alpha\rangle\otimes|0\rangle + s_\theta|\Phi_\theta\rangle\otimes|1\rangle + s_\phi|\Phi_\phi\rangle\otimes|2\rangle, \tag{8}$$

here $|\Phi_x\rangle = (1/\sqrt{2})(|+_x\rangle^{\otimes N} + |-_x\rangle^{\otimes N})$ with $|\pm_x\rangle$ as the eigenstates of H_x, $x\in\{\alpha,\theta,\phi\}$, N is the number of spins that interact with the magnetic field. The reduced two-spin state of this state is

$$\rho^{(j,k)} = \frac{1}{4}[I^{(j,k)} + |s_\alpha|^2\sigma_\alpha^{(j)}\otimes\sigma_\alpha^{(k)} + |s_\theta|^2\sigma_\theta^{(j)}\otimes\sigma_\theta^{(k)} + |s_\phi|^2\sigma_\phi^{(j)}\otimes\sigma_\phi^{(k)}] \tag{9}$$

for all $0\leq j<k\leq N-1$ and the reduced single spin state is $\rho^{(k)} = I^{(k)}/2$ for all $0\leq k\leq N-1$. For multiple parameters the QCRB is given by $\mathrm{Cov}(\hat{x})\geq J^{-1}$, where J is the quantum Fisher information matrix whose entries can be obtained from the generators as $J_{lp} = 4[\frac{1}{2}\langle\Phi_{\mathrm{SA}}|\{H_l^{(N)}, H_p^{(N)}\}|\Phi_{\mathrm{SA}}\rangle - \langle\Phi_{\mathrm{SA}}|H_l^{(N)}|\Phi_{\mathrm{SA}}\rangle\times\langle\Phi_{\mathrm{SA}}|H_p^{(N)}|\Phi_{\mathrm{SA}}\rangle]$, $l,p\in\{\alpha,\theta,\phi\}$ and $|\Phi_{\mathrm{SA}}\rangle$ is the initial probe state. For the state in Eq. (8), it is straightforward to check (see the Supplemental Material [58]) that J is a diagonal matrix. If the optimal $\tilde{r}_{\alpha\alpha}$, $\tilde{r}_{\theta\theta}$, and $\tilde{r}_{\phi\phi}$ in Eq. (7) are all non-negative, then by choosing $s_\alpha = \sqrt{\tilde{r}_{\alpha\alpha}}$, $s_\theta = \sqrt{\tilde{r}_{\theta\theta}}$, and $s_\phi = \sqrt{\tilde{r}_{\phi\phi}}$, the ultimate lower bound in Eq. (6) is saturated. For sufficiently large N, this is always the case. It is also straightforward to check the weak commutativity condition, $\langle\Phi(\alpha,\theta,\psi)|[L_l,L_p]|\Phi(\alpha,\theta,\psi)\rangle = 0$, holds for all $l,p\in\{\alpha,\theta,\phi\}$ [60], here $\Phi(\alpha,\theta,\psi)$ is the final state and L_p is the symmetric logarithmic derivatives (SLD) for parameter $p\in\{\alpha,\theta,\phi\}$, which is the solution to

the equation $\partial_p \rho = \frac{1}{2}(L_p\rho + \rho L_p)$. This condition, which can be simplified as $\mathrm{Im}[\langle \partial_l \Phi(\alpha,\theta,\psi)|\partial_p\Phi(\alpha,\theta,\psi)\rangle] = 0$, ensures the existence of a measurement saturating the QCRB [27,45,60]. We provide an explicit construction of the optimal measurement in the Supplemental Material [58]. The lower bound in Eq. (6) can thus always be achieved for sufficiently large N, which is then the ultimate precision limit that can be achieved under the parallel scheme.

If some $\tilde{r}_{xx}$ in Eq. (7), $x \in \{\alpha,\theta,\phi\}$, are negative for small N, then the lower bound in Eq. (6) can not be saturated by the probe states of this form. The best precision achieved by these states can be obtained by optimizing the coefficients s_α, s_θ and s_ϕ [58]. In Fig. 2 we plotted the precisions that can be achieved for different weights and N, it can be seen that the obtained precision is already close to the ultimate bound even for small N, and it saturates the ultimate bound when N gets large.

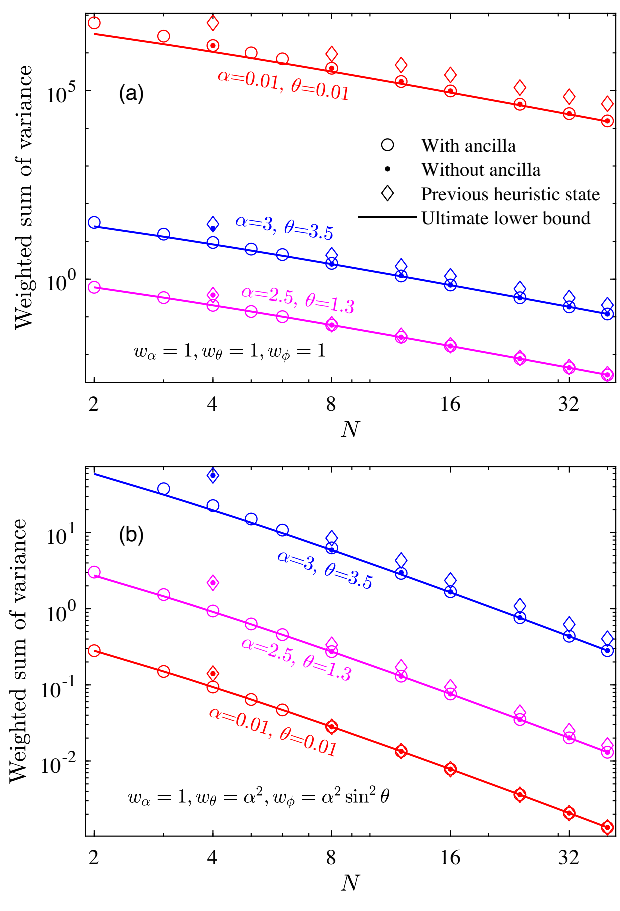

FIG. 2. (a) Weighted sum of variance with $w_\alpha = 1$, $w_\theta = 1$ and $w_\phi = 1$, which corresponds to $\delta\hat{\alpha}^2 + \delta\hat{\theta}^2 + \delta\hat{\phi}^2$. (b) Weighted sum of variance with $w_\alpha = 1, w_\theta = \alpha^2$ and $w_\phi = \alpha^2\sin^2\theta$, which corresponds to $\delta\hat{B}_1^2 + \delta\hat{B}_2^2 + \delta\hat{B}_3^2$. Three typical sets of values as specified in the figure for each case. The time has been normalized, i.e., $t = 1$.

The ultimate lower bound in Eq. (6) can also be achieved without the ancillary system by preparing the probe state as $|\Phi_o\rangle = \sqrt{\tilde{r}_{\alpha\alpha}}|\Phi_\alpha\rangle + \sqrt{\tilde{r}_{\theta\theta}}|\Phi_\theta\rangle + \sqrt{\tilde{r}_{\phi\phi}}|\Phi_\phi\rangle$ when $N\to\infty$, as it is easy to verify that in this case the reduced two-qubit state, when $N\to\infty$, is the same as the optimal reduced two-qubit state required to saturate the ultimate lower bound [58].

We now compare the obtained precision with previous results. For the estimation of the three components of the magnetic field, $B_1 = (\alpha/t)\sin\theta\cos\phi$, $B_2 = (\alpha/t)\sin\theta\sin\phi$, $B_3 = (\alpha/t)\cos\theta$, we have $\delta\hat{B}_1^2 + \delta\hat{B}_2^2 + \delta\hat{B}_3^2 = (\delta\hat{\alpha}^2 + \alpha^2\delta\hat{\theta}^2 + \alpha^2\sin^2\theta\delta\hat{\phi}^2)/t^2$, which corresponds to $w_\alpha = 1/t^2$, $w_\theta = \alpha^2/t^2$, $w_\phi = \alpha^2\sin^2\theta/t^2$. The ultimate precision is then given by

$$\delta\hat{B}_1^2 + \delta\hat{B}_2^2 + \delta\hat{B}_3^2 \geq \frac{(1+2|\frac{\alpha}{\sin\alpha}|)^2}{4N(N+2)t^2}. \tag{10}$$

While the best precision obtained previously with the heuristic state [15] is $\delta\hat{B}_1^2 + \delta\hat{B}_2^2 + \delta\hat{B}_3^2 \geq 3[1+2(\alpha^2/\sin^2\alpha)]/[4N(N+2)t^2]$. They are equivalent only at the weak limit when $\alpha = Bt\to 0$, as shown in Fig. 2(b). The difference between them, which is $2(|(\alpha/\sin\alpha)|-1)^2/[4N(N+2)t^2]$, can be large particularly when $\alpha\to m\pi$.

This approach can also be applied to general spin-S, where the Hamiltonian is $H = B\boldsymbol{n}\cdot\boldsymbol{S} = B_1S_1 + B_2S_2 + B_3S_3$, here $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, S_1, S_2, and S_3 are spin-S operators that satisfy $[S_1,S_2] = iS_3$, $[S_2,S_3] = iS_1$, $[S_3,S_1] = iS_2$ (with this commutation relation, $S_i = (\sigma_i/2)$ when $S = 1/2$, which has an extra factor of $\frac{1}{2}$ comparing with the Pauli matrices). For $S > 1/2$, $S_i^2 \not\propto I$, instead $S_1^2 + S_2^2 + S_3^2 = S(S+1)I$. The generators for α, θ and ϕ can be similarly obtained as $H_\alpha = c_\alpha S_\alpha$, $H_\theta = c_\theta S_\theta$, and $H_\phi = c_\phi S_\phi$, where $c_\alpha = 1$, $S_\alpha = \boldsymbol{n}_\alpha\cdot\boldsymbol{S}$, $\boldsymbol{n}_\alpha = \boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $c_\theta = 2\sin(\alpha/2)$, $S_\theta = \boldsymbol{n}_\theta\cdot\boldsymbol{S}$, $\boldsymbol{n}_\theta = \cos(\alpha/2)\boldsymbol{n}_1 - \sin(\alpha/2)\boldsymbol{n}_2$, $c_\phi = 2\sin(\alpha/2)\sin\theta$, $S_\phi = \boldsymbol{n}_\phi\cdot\boldsymbol{S}$, $\boldsymbol{n}_\phi = \sin(\alpha/2)\boldsymbol{n}_1 + \cos(\alpha/2)\boldsymbol{n}_2$ and $\boldsymbol{n_1} = \partial_\theta\boldsymbol{n}$, $\boldsymbol{n_2} = \boldsymbol{n}\times\boldsymbol{n_1}$. Similarly we can obtain the lower bound for the weighted sum of variance as

$$w_\alpha\delta\hat{\alpha}^2 + w_\theta\delta\hat{\theta}^2 + w_\phi\delta\hat{\phi}^2 \geq \frac{1}{4}\frac{\left(\sqrt{w_\alpha} + \frac{\sqrt{w_\theta}}{|2\sin\frac{\alpha}{2}|} + \frac{\sqrt{w_\phi}}{|2\sin\frac{\alpha}{2}\sin\theta|}\right)^2}{[\sum_{k=0}^{N-1}\sum_{x\in\{\alpha,\theta,\phi\}} r_{xx}^{(k)} + \sum_{j\neq k}\sum_{x\in\{\alpha,\theta,\phi\}} r_{xx}^{(j,k)}]}, \tag{11}$$

here $r_{xx}^{(k)} = \mathrm{tr}(\rho^{(k)}S_x^2)$, $r_{xx}^{(j,k)} = \mathrm{tr}(\rho^{(j,k)}S_x\otimes S_x)$, $\forall\, x\in\{\alpha,\theta,\phi\}$. It is easy to get $\sum_{x\in\{\alpha,\theta,\phi\}} r_{xx}^{(k)} = \mathrm{tr}(\rho^{(k)}\sum_{x\in\{\alpha,\theta,\phi\}} S_x^2) = S(S+1)$. The constrains on $r_{xx}^{(j,k)}$, however, are much harder to obtain compared to spin-1/2. In the Supplemental Material [58], we show that $\sum_{x\in\{\alpha,\theta,\phi\}} S_x\otimes S_x \leq S^2 I$, thus $\sum_{x\in\{\alpha,\theta,\phi\}} r_{xx}^{(j,k)} \leq S^2$. With these constraints we obtain the ultimate lower bound [58]

$$w_\alpha \delta\hat{\alpha}^2 + w_\theta \delta\hat{\theta}^2 + w_\phi \delta\hat{\phi}^2 \geq \frac{\left(\sqrt{w_\alpha} + \frac{\sqrt{w_\theta}}{|2\sin\frac{\alpha}{2}|} + \frac{\sqrt{w_\phi}}{|2\sin\frac{\alpha}{2}\sin\theta|}\right)^2}{4NS(NS+1)}. \quad (12)$$

With an ancillary qutrit, this ultimate lower bound can be saturated for sufficiently large NS with the state $|\Phi_{\mathrm{SA}}\rangle = s_\alpha|\Phi_\alpha\rangle \otimes |0\rangle + s_\theta|\Phi_\theta\rangle \otimes |1\rangle + s_\phi|\Phi_\phi\rangle \otimes |2\rangle$, here $|\Phi_x\rangle = (1/\sqrt{2})(|+_x\rangle^{\otimes N} + |-_x\rangle^{\otimes N})$ with $|\pm_x\rangle$ as the eigenstates of S_x corresponding to the eigenvalue $\pm S$, respectively, $\forall\, x \in \{\alpha, \theta, \phi\}$, and the coefficients should satisfy

$$|s_\alpha|^2 = \frac{(2NS+1)\sqrt{w_\alpha} - \frac{\sqrt{w_\theta}}{2|\sin\frac{\alpha}{2}|} - \frac{\sqrt{w_\phi}}{2|\sin\frac{\alpha}{2}\sin\theta|}}{(2NS-1)\left(\sqrt{w_\alpha} + \frac{\sqrt{w_\theta}}{2|\sin\frac{\alpha}{2}|} + \frac{\sqrt{w_\phi}}{2|\sin\frac{\alpha}{2}\sin\theta|}\right)},$$

$$|s_\theta|^2 = \frac{(2NS+1)\frac{\sqrt{w_\theta}}{2|\sin\frac{\alpha}{2}|} - \frac{\sqrt{w_\phi}}{2|\sin\frac{\alpha}{2}\sin\theta|} - \sqrt{w_\alpha}}{(2NS-1)\left(\sqrt{w_\alpha} + \frac{\sqrt{w_\theta}}{2|\sin\frac{\alpha}{2}|} + \frac{\sqrt{w_\phi}}{2|\sin\frac{\alpha}{2}\sin\theta|}\right)},$$

$$|s_\phi|^2 = \frac{(2NS+1)\frac{\sqrt{w_\phi}}{2|\sin\frac{\alpha}{2}\sin\theta|} - \sqrt{w_\alpha} - \frac{\sqrt{w_\theta}}{2|\sin\frac{\alpha}{2}|}}{(2NS-1)\left(\sqrt{w_\alpha} + \frac{\sqrt{w_\theta}}{2|\sin\frac{\alpha}{2}|} + \frac{\sqrt{w_\phi}}{2|\sin\frac{\alpha}{2}\sin\theta|}\right)}, \quad (13)$$

which always have solutions when NS is sufficiently large. It is also straightforward to check that the weak commutativity condition holds, the ultimate lower bound can thus always be saturated for sufficiently large NS.

Summary.—The ultimate precision of quantum magnetometry under the parallel scheme is of fundamental interest and importance in quantum metrology. It can also be directly used as the benchmark for the performance of quantum gyroscope and quantum reference frame alignment. Our approach connects the tradeoff directly to the constraints on the probe states and the generators, which makes the tradeoff transparent and deepens the understandings on the incompatibility of the optimal probe states. We expect this approach can lead to many useful (may not always be achievable, nevertheless nontrivial) bounds in various scenarios of multiparameter quantum estimation. Future studies can include measurements suitable for specific physical settings and generalization to noisy dynamics via the purification approach [61–66]

The work at U.S. T. C. is supported by the National Key Research and Development Program of China (Grant No. 2018YFA0306400), the National Natural Science Foundation of China under Grants (No. 61905234, No. 11974335, No. 11574291, and No. 11774334), Key Research Program of Frontier Sciences, CAS (No. QYZDY-SSW-SLH003), Anhui Initiative in Quantum Information Technologies. The work at CUHK is supported by the Research Grants Council of Hong Kong (GRF No. 14207717).

[*]These authors contributed equally to this work.
[†]gyxiang@ustc.edu.cn
[‡]hdyuan@mae.cuhk.edu.hk

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