Symmetries and Dualities in the Theory of Elasticity

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Microscopic symmetries impose strong constraints on the elasticity of a crystalline solid. In addition to the usual spatial symmetries captured by the tensorial character of the elastic tensor, hidden nonspatial symmetries can occur microscopically in special classes of mechanical structures. Examples of such nonspatial symmetries occur in families of mechanical metamaterials where a duality transformation relates pairs of different configurations. We show on general grounds how the existence of nonspatial symmetries further constrains the elastic tensor, reducing the number of independent moduli. In systems exhibiting a duality transformation, the resulting constraints on the number of moduli are particularly stringent at the self-dual point but persist even away from it, in a way reminiscent of critical phenomena.

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Classical elasticity describes how rigid objects respond to deformations [1–5]. New facets of this time-honored subject continue to emerge in often unexpected guises and contexts. Recent examples range from quantum elasticity [6–8] and fractons [9–11] to nonorientable elasticity [12], the odd elasticity of active solids [13], and topological elasticity [14–26].

The very existence of rigid objects would seem rather mysterious if we were not so used to them in daily life: it is a consequence of the spontaneous breaking of translational invariance that occurs when a fluid condenses into a solid [27]. This spontaneously broken symmetry guarantees the existence of excitations with arbitrarily low energies called Nambu-Goldstone modes [5,28–31]. In mechanics, the Goldstone modes are familiar objects: phonons of arbitrarily large wavelength [32,33]. Elasticity can be viewed as the effective field theory of such Goldstone modes: a continuum description that ignores irrelevant microscopic details and instead focuses on the behavior at large scales relevant to our direct interactions with elastic bodies.

The coarse-graining procedure that goes from a microscopic description to a continuum elastic theory should discard irrelevant details, but must crucially preserve symmetries [34,35]. The spatial symmetries of a crystal can be gathered in a space group, containing all spatial transformations that leave the crystal invariant [36,37]. The space group of a crystal puts strong constraints on its elasticity, e.g., on the number of independent moduli [2,38,39]. For instance, the elasticity of a two-dimensional crystal with triangular symmetry is isotropic (i.e., it is the same for all orientations) and, as a consequence, can display at most two independent elastic moduli.

In addition to spatial symmetries, additional nonspatial symmetries can occur microscopically. A symmetry is simply a transformation of the system that leaves it invariant. Symbolically, we can write $T(\mathcal{S}) = \mathcal{S}$, where $\mathcal{S}$ represents the system and T the symmetry transformation. Spatial transformations such as rotations or translations can certainly be symmetries, but they do not exhaust all the possibilities. Recent studies revealed that hidden nonspatial symmetries can emerge, for instance, in families of mechanical metamaterials where a duality transformation relates pairs of *distinct* configurations [40]. Symbolically, two dual systems $\mathcal{S}_1$ and $\mathcal{S}_2$ related by the duality transformation T satisfy $\mathcal{S}_2 = T(\mathcal{S}_1)$ and $\mathcal{S}_1 = T(\mathcal{S}_2)$. The duality transformation has no reason to be a spatial transformation. In self-dual systems (mapped onto themselves by the duality) the duality transformation can then become an additional hidden symmetry distinct from spatial ones.

In this Letter, we seek to determine the consequences of these additional constraints on the linear elasticity of a material. More precisely, we consider the following question: how do microscopic symmetries affect the coarse-grained tensor of elastic moduli? Formally, we will determine the relation between the elastic tensors $c_{ijk\ell}$ and $\tilde{c}_{ijk\ell}$ of two systems $\mathcal{S}$ and $\tilde{\mathcal{S}} = T(\mathcal{S})$, respectively described by the momentum-space force-constant matrix $S(q)$ and the transformed one $\tilde{S}(q) = U(q)S(\mathcal{O}\cdot q)U(q)^{-1}$. Here, $S(q)$ relates the microscopic forces and displacements, while $U(q)$ and $\mathcal{O}$ define the transformation (see next section for precise definitions). For standard spatial symmetries, the answer is simply contained in the fact that $c_{ijk\ell}$ must transform as a tensor. Our analysis goes beyond this simple case and allows us to analyze the effect of additional hidden (nonspatial) symmetries of the force-constant matrix, which can result in even stronger constraints. In addition, it applies to the case of dualities whereby the force-constant matrices of two different systems are related to each other by a nontrivial transformation.

We apply our general formulas to the example of twisted kagome lattices (see Fig. 1), a family of two-dimensional

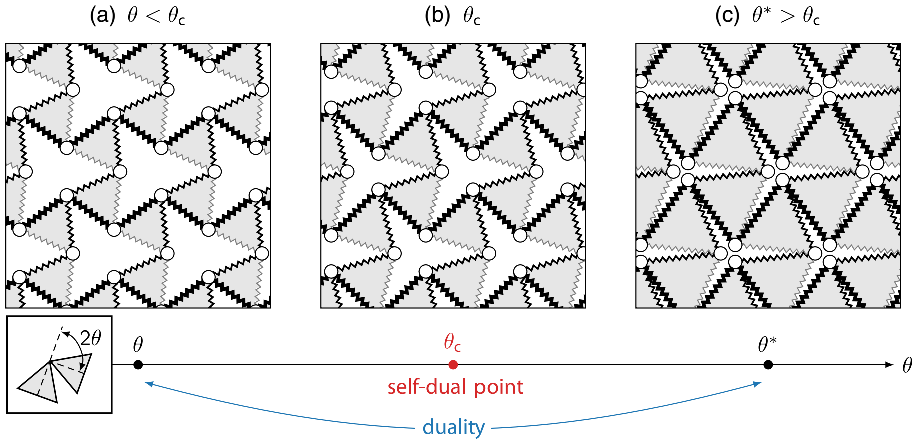

FIG. 1. Twisted kagome lattice. Examples of twisted kagome lattices with different twisting angles. The duality maps structures with a twisting angle θ to ones with a twisting angle $\theta^* = 2\theta_c - \theta$. The critical structure with twisting angle θ_c is mapped to itself: it is self-dual. All these structures have the same space group, including the self-dual one. Here, inequivalent springs have different stiffnesses, as represented by their thicknesses in the figure, to remove any point group symmetry. (a) Below the critical angle. (b) At the critical angle. (c) Above the critical angle. Inset: definition of the twisting angle θ.

crystals exhibiting a duality with a self-dual point where a nonspatial symmetry emerges [40]. When all point group symmetries are lifted, six independent elastic moduli are expected in the continuum description of such systems. Yet, the self-dual twisted kagome lattices have isotropic elasticity with only one elastic modulus, despite not having any microscopic symmetry beyond Bravais lattice translations. Most strikingly, the elastic tensor is also constrained away from the self-dual point, reducing the number of elastic moduli to three. Our theory explains these counterintuitive properties and casts them in a general formalism applicable beyond this concrete example.

Linear elasticity.—Linear elasticity describes the relation between the stress tensor σ_{ij} and the displacement gradient (or strain) tensor $\epsilon_{k\ell}$, respectively, representing the long-wavelength forces and deformations in a solid. More precisely, the displacement tensor is $\epsilon_{k\ell} = \partial u_\ell / \partial x_k$, where $u(x)$ represents the displacement of the point originally located at x and now located at $X(x) = x + u(x)$, while the stress tensor is defined such that its divergence is the surface force $f_i = \partial_j \sigma_{ij}$ acting on an infinitesimal patch of material continuum. We choose to work with the nonsymmetrized tensors to encompass extensions of elasticity where the antisymmetric components are relevant [13,41]. Hooke's law in continuum form,

$$\sigma_{ij} = c_{ijk\ell}\epsilon_{k\ell}, \tag{1}$$

linearly relates σ_{ij} and $\epsilon_{k\ell}$ through the elastic tensor $c_{ijk\ell}$, whose entries are the static elastic moduli of the solid.

Spatial symmetries put strong constraints on the material properties of a crystal such as its elastic tensor $c_{ijk\ell}$. This is because the elastic tensor $c_{ijk\ell}$ unsurprisingly transforms as a tensor under a spatial transformation $T \in O(d)$:

$$c_{ijk\ell} \mapsto \tilde{c}_{ijk\ell} = T_{ii'}T_{jj'}T_{kk'}T_{\ell\ell'}c_{i'j'k'\ell'}. \tag{2}$$

Hence, there is only a certain number of entries in $c_{ijk\ell}$ (i.e., of elastic moduli) that can be independent of each other, and those are prescribed by the symmetry of the material (we refer the reader to the Supplemental Material [41] for a short summary). Yet, nothing guarantees that all of these moduli *must* be independent, especially when additional constraints not originating from purely spatial symmetries exist.

Microscopically, we describe the elastic material as a set of massive particles arranged on a d-dimensional crystal and ruled by Newton equations $M\partial_t^2 u = F$, where $u = x - x_{\text{eq}}$ are the displacements of the masses with respect to their equilibrium positions x_{eq}, and M is a mass matrix describing the inertia of the particles. The forces F between the particles are given in the harmonic approximation by $F = -Su$ where the force-constant matrix S is essentially the matrix of second derivatives of the potential in the absence of prestress [56]. Hooke's law (1) is the macroscopic version of the relation $F = -Su$ between forces and displacements. Hence, the elastic tensor $c_{ijk\ell}$ can, in principle, be computed explicitly from the force-constant matrix S, see Ref. [57] (also Refs. [58–64]).

Here, we specialize to the case of a crystal, where particles are arranged in a spatially periodic fashion. Hence, we can use the Bloch theorem to block diagonalize Newton equations and to write $M\partial_t^2 u(q) = F(q) = -S(q)u(q)$, where q is the quasimomentum vector. Because of the original translation invariance of the system (that is spontaneously broken), a global translation of the particles in any direction cannot induce any restoring force. We assume that there is no other soft mode. Hence, the kernel of the force-constant matrix $S(q = 0)$ consists of the rigid-body translations of all the particles (i.e., the translations of the center of mass of the unit cell). Elasticity describes the long-wavelength modes $q \to 0$ (acoustic phonons) projected onto rigid-body translations with the constraint that the projection of the force $F(q)$ on fast modes must relax (i.e., the projection on modes with a finite frequency at $q = 0$, that span the orthogonal complement of the kernel, is zero). The result of integrating out these irrelevant modes is in agreement with the zero temperature limit of finite-temperature elasticity [59]. The elastic tensor can then be obtained from the momentum-space force-constant matrix $S(q)$ near zero momentum as [13,56,57]

$$\frac{c_{ijk\ell}}{\rho} = \left[\frac{\partial^2 S}{\partial q_i \partial q_k} - \frac{\partial S}{\partial q_i}[S^{-1}]\frac{\partial S}{\partial q_k}\right]_{j\ell} \quad (3)$$

where ρ is the density. (This expression is taken at momentum $q = 0$ and the inverse S^{-1} is computed in the orthogonal complement of the kernel of the matrix.) When all masses are equal, the force-constant matrix S can be replaced by the more familiar dynamical matrix $D = M^{-1/2} S M^{-1/2}$.

It is convenient to decompose the stress and deformation tensors in irreducible components. Hooke's law (1) then reads [13,65]

$$\sigma^a = K^{ab}\epsilon^b. \quad (4)$$

In two dimensions, for instance, the four components of the stress (deformation) vector σ^a (ϵ^b) correspond to compression, rotation, and two linearly independent shear stresses (strains) (see Supplemental Material [41] for a visual representation). More generally, a and b label basis matrices τ^a that span irreducible representations of the rotation group $\mathrm{SO}(d)$. The elastic matrix $K^{ab} = \frac{1}{4}\sum_{ijk\ell} \tau^a_{ij} c_{ijk\ell} \tau^b_{k\ell}$ contains exactly the same information as the elastic tensor $c_{ijk\ell}$, only ordered in a different way.

Symmetries and dualities and their effect on the elastic tensor.—We now consider a situation where a momentum-space force-constant matrix $\tilde{S}(q)$ is related to another force-constant matrix $S(q)$ by a relation of the form

$$\tilde{S}(q) = (USU^{-1})(\mathcal{O} \cdot q), \quad (5)$$

where U is unitary and $\mathcal{O}$ is orthogonal. We stress that the matrices U and $\mathcal{O}$ act on different spaces: U acts on the displacements u of the masses, while $\mathcal{O}$ acts on the spatial coordinates x (or, equivalently, momenta q). This relation describes a symmetry when we impose $\tilde{S} = S$, i.e., the transformed system is identical to the original one. It also describes situations where $\tilde{S}$ and S are distinct, and in particular systems related by duality transformations [40]. The two force-constant matrices $S(q)$ and $\tilde{S}(q)$ define two elastic tensors $c_{ijk\ell}$ and $\tilde{c}_{ijk\ell}$ (equivalently, two elastic matrices K^{ab} and $\tilde{K}^{ab}$) through Eq. (3). We now proceed to determine the relation between $c_{ijk\ell}$ and $\tilde{c}_{ijk\ell}$ imposed by Eq. (5). Using Eqs. (3) and (5), one obtains by a direct calculation (see Supplemental Material [41])

$$\tilde{c}_{ijk\ell} = \mathcal{O}_{i'i} R_{jj'} \mathcal{O}_{k'k} R_{\ell\ell'} c_{i'j'k'\ell'}, \quad (6)$$

where the orthogonal matrix R is the projection of $U(0)$ on solid-body translations [the kernel of $S(0)$]. In terms of the elastic matrix K in Eq. (4), the relation (6) can be cast in the more compact form,

$$\tilde{K} = VKV^{\dagger}, \quad (7)$$

where

$$V^{ab} = \frac{1}{2}\mathrm{tr}[\tau^a R[\tau^b]^T \mathcal{O}]. \quad (8)$$

The standard result Eq. (2) is recovered from Eq. (6) in the case of spatial symmetries, for which $R = \mathcal{O}^T \equiv T$. However, this particular case does not exhaust Eq. (6) as the relation (5) is not necessarily the representation of a spatial symmetry, i.e., of an element of the space group of the crystal. As such, the matrix R needs not be related to $\mathcal{O}$ [66]. In the next section, we shall present a concrete example where such hidden nonspatial symmetries occur in elasticity.

Twisted kagome lattices.—Consider the family of mechanical structures called twisted kagome lattices [14,15,18,67–69]. These are two-dimensional periodic structures composed of three particles per unit cell on a triangular lattice, with each particle connected to four neighbors, as represented in Fig. 1. We consider a situation where inequivalent bonds (i.e., those not related by Bravais lattice translations) have different spring stiffnesses k_i, $i = 1, 2, 3$ (see Fig. 1). This family is parametrized by a simple geometric parameter: the twisting angle θ between two connected triangles, see the inset of Fig. 1. It was shown in Ref. [40] that a duality relates the dynamical matrices of the structures with θ and $\theta^* = 2\theta_c - \theta$ (with $\theta_c = \pi/4$) through the relation [40]

$$\mathcal{U}(k) D(\theta^*, -k) \mathcal{U}^{-1}(k) = D(\theta, k), \quad (9)$$

where $\mathcal{U}(k) = \mathrm{diag}(i\varsigma_y, i\varsigma_y e^{-ik\cdot a_2}, i\varsigma_y e^{ik\cdot a_1})$. In this expression, the matrices $i\varsigma_y$ act on the displacements (x, y) of each of the three masses in the unit cell of the crystal, ς_i are Pauli matrices, and $a_i = [\cos((i-1)2\pi/3), \sin((i-1)2\pi/3)]^T$ are primitive vectors of the triangular Bravais lattice. The duality (9) typically relates different systems, with different twisting angles, such as the mechanical networks represented in Figs. 1(a) and 1(c). However, there is a particular self-dual angle $\theta_c = \pi/4$ such that $\theta_c^* = \theta_c$ (see Fig. 1), where the duality relation becomes an additional nonspatial symmetry of the dynamical matrix.

From Eq. (9), one finds that $R = i\varsigma_y$ and $\mathcal{O} = -\mathrm{Id}$. Upon substituting these results in Eq. (8), we obtain

$$V = \varsigma_3 \otimes i\varsigma_2, \tag{10}$$

where $\otimes$ is the Kronecker product and ς_i are Pauli matrices. It is instructive to write the most general form of the elastic matrix for a standard material (i.e., energy and angular momentum are conserved and solid-body rotations do not change the elastic energy). In this situation, $K^{a0} = 0 = K^{0b}$ and $K^{ab} = K^{ba}$ (see Ref. [13] and Supplemental Material [41] for details), so we have

$$K = \begin{pmatrix} K^{00} & 0 & K^{02} & K^{03} \\ 0 & 0 & 0 & 0 \\ K^{02} & 0 & K^{22} & K^{23} \\ K^{03} & 0 & K^{23} & K^{33} \end{pmatrix}. \tag{11}$$

The elastic matrices $K(\theta)$ and $K(\theta^*)$ of two twisted kagome lattices must indeed have the form (11). Following the preceding analysis, the duality relation (9) implies an additional set of constraints

$$VK(\theta)V^\dagger = K(\theta^*) \tag{12}$$

with the transformation matrix V defined in Eq. (10). As a consequence, we find that

$$K(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K^{22}(\theta) & K^{23}(\theta) \\ 0 & 0 & K^{23}(\theta) & K^{33}(\theta) \end{pmatrix} \tag{13}$$

with

$$K^{22}(\theta) = K^{33}(\theta^*), \tag{14a}$$

$$K^{33}(\theta) = K^{22}(\theta^*), \tag{14b}$$

$$K^{23}(\theta) = -K^{23}(\theta^*). \tag{14c}$$

In particular, the constraint $VK(\theta_c)V^\dagger = K(\theta_c)$ at the critical angle $\theta_c = \theta_c^*$ leads to $K^{22}(\theta_c) = K^{33}(\theta_c)$ while $K^{23}(\theta_c) = 0$.

Hence, the duality relation (12) implies two striking consequences. First, twisted kagome lattices have only shear moduli: the coefficients K^{00}, K^{02}, and K^{03} always vanish [see Eq. (13)]. Crucially, the duality constrains the elastic moduli everywhere along the duality line (not only at the self-dual point). Physically, the lack of bulk moduli is related to the existence of a Guest-Hutchinson mechanism [14,15,18,67,68], see in particular Ref. [14]. Second, a stronger constraint occurs at the self-dual point where the elastic tensor becomes isotropic and characterized by a single shear modulus, despite no change in symmetry in the lattice. The occurrence of an isotropic elastic tensor holds even when all point group symmetries are lifted (i.e., the space group is p1). A direct computation of the elastic tensor from the dynamical matrix shown in Fig. 2, using either Eq. (3) or the real-space equivalent [57] confirms all our results [40].

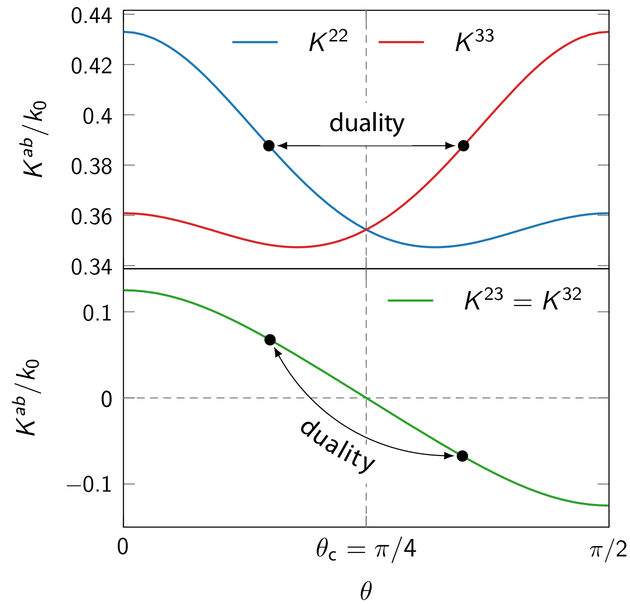

FIG. 2. Elastic constants for an anisotropic kagome lattices. The elastic moduli K^{22}, K^{33}, and $K^{23} = K^{32}$ computed from the microscopic description of kagome lattices according to Eq. (3) are plotted as a function of the twisting angle θ for a generic situation where all inequivalent springs in the unit cell have different stiffnesses (see Fig. 1) [40]. The duality (represented by black arrows) exchanges K^{22} and K^{33}, as well as K^{23} and $-K^{23}$. We have set $k_1 = k_0$, $k_1 = 2k_0$, $k_3 = 3k_0$.

To illustrate the effect of dualities, we consider the spectrum of elastic waves in anisotropic twisted kagome

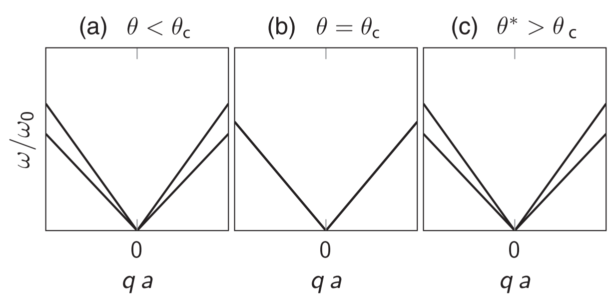

FIG. 3. Effect of dualities on elastic waves in an anisotropic kagome lattice. The dispersion relations of elastic waves in an anisotropic kagome lattice are plotted for (a) $\theta = 0.1\pi$, (b) θ_c, and (c) the dual angle θ^* of case (a). The dispersion relations in (a) and (c) are identical, because of the duality between the corresponding systems. We distinguish the two acoustic branches in (a) or (c), but not in the self-dual system (b), where they share the same slope. We have set $k_1 = k_0$, $k_1 = 2k_0$, $k_3 = 3k_0$, a is the lattice constant, and $\omega_0^2 = k_0/m$. The x and y axes have identical length.

lattices. The dynamics of elastic waves is described by the equation $\rho\ddot{u} = \nabla \cdot \sigma$. A Fourier transform of this equation gives $\omega^2 u_i(q) = h_{i\ell}(q) u_\ell(q)$ with $h_{i\ell}(q) = c_{ijk\ell} q_j q_k / \rho$. The dispersion relations obtained by diagonalizing the matrix $h(q)$ are plotted in Figs. 3(a)–3(c). We observe that the dual structures with twisting angles θ and θ^* [(a) and (c)] have identical spectra. Besides, the two branches are degenerate in the self-dual structure (b), as expected from the form of the elastic tensor.

Conclusions.—We have shown how hidden nonspatial symmetries (originating, for instance, from dualities) strongly constrain the elastic moduli of a solid. Our results suggest a general mechanism not limited to elasticity by which microscopic dualities and nonspatial symmetries impose constraints on generalized rigidities and response functions. These subtle effects are not captured by an analysis based on the spatial symmetry (i.e., the point group or space group) of the underlying structure. They are therefore likely to be overlooked in analyses performed purely within macroscopic continuum theories.

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