# Periodic Table of the Ordinary and Supersymmetric Sachdev-Ye-Kitaev Models

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We develop a unified minimal scheme to classify quantum chaos in the Sachdev-Ye-Kitaev (SYK) and supersymmetric (SUSY) SYK models and also work out the structure of the energy levels in one periodic table. The SYK with even $q$-body or SUSY SYK with odd $q$-body interaction, with $N$ even or odd number of sites, are put on an equal footing in the minimal Hilbert space; $N$ (mod 8), $q$ (mod 4) double Bott periodicity, and a reflection condition are identified. Exact diagonalizations (EDs) are performed to study both the bulk energy level statistics and hard-edge behaviors. Excellent agreements between the ED results and the symmetry classifications are demonstrated. Our compact and systematic methods can be transformed to map out more complicated periodic tables of SYK models with more degrees of freedom, tensor models, or symmetry protected topological phases.

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*Introduction.*—Extensive research efforts are devoted to investigating quantum chaos and quantum information scramblings in the Sachdev-Ye-Kitaev (SYK) model [1–6] and its supersymmetric (SUSY) generalizations [7]. In the early time, an exponential growth in out-of-time-order correlators (OTOCs) of the SYK models defines the quantum Lyapunov exponent $\lambda_L = 2\pi/\beta$ saturating the quantum chaos bound [8–15]. This remarkable feat suggests that the SYK models may correspond to a boundary theory of the 2D Jackiw-Teitelboim dilaton gravity [4,16].

In the late time, the OTOCs are known to saturate and become featureless, while the quantum chaos needs to be described by the random matrix theory (RMT). Despite its sparse nature in randomness, SYK models can still be described precisely by the RMT [17–25]. The RMT classifications of the SYK models have been applied to $q\,(\mathrm{mod}\,4) = 0$ for all $N$ [19,20,23,25] and $q\,(\mathrm{mod}\,4) = 1$, 2, 3 for the even $N$ case only [21,22]. For $q\,(\mathrm{mod}\,4) = 0$, Refs. [19,25,26] developed a special procedure to treat the $N$ odd case by adding an extra decoupled Majorana fermion into the system, while such a procedure is not necessary in the $N$ even case. So in all previous works, the SYK and $\mathcal{N} = 1$ SUSY SYK, the odd $N$ and even $N$ cases were treated differently and separately.

Here, by employing the Clifford algebra representation of Majorana fermions in a minimal Hilbert space, we develop a systematic and unified scheme to classify random matrix behaviors of the SYK models with generic $q$-body interaction and $N$ site. After identifying complete set of conserved (or chiral) quantities and antiunitary (or unitary) symmetry operators, we also work out the fine structure of the energy levels explicitly. The SYK with even $q$ and SUSY SYK with odd $q$, $N$ even or odd are treated on the same footing. There are 9 classes (except class AIII) with $N\,(\mathrm{mod}\,8)$ and $q\,(\mathrm{mod}\,4)$ double Bott periodicity and a reflection condition identified in one periodic Table I. A new moment ratio of the smallest positive eigenvalue is introduced to determine hard-edge index efficiently. Our systematic approach not only reproduces the previously known results in a much more efficient and compact way, but also lead to new results on $q\,(\mathrm{mod}\,4) = 1, 2, 3$ with odd $N$, thus completes the whole periodic table for the SYK models. It provides additional deep and global insights into the family of SYK models showing maximal quantum chaos and their corresponding quantum gravity models in the bulk.

*A unified minimal scheme.*—Consider a generic all-to-all $q$-body Majorana interaction

$$Q_q = i^{\lfloor q/2 \rfloor} \sum_{i_1 < \cdots < i_q} C_{i_1 \cdots i_q} \chi_{i_1} \chi_{i_2} \cdots \chi_{i_q}, \tag{1}$$

where $C_{i_1 \cdots i_q}$ is real and satisfies the Gaussian distributions with mean $\langle C_{i_1 \cdots i_q} \rangle = 0$ and variance $\langle C^2_{i_1 \cdots i_q} \rangle = J^2_{\mathrm{SYK}}(q-1)!/N^{q-1}$ for even $q$ or $\langle C^2_{i_1 \cdots i_q} \rangle = J_{\mathcal{N}=1}(q-1)!/N^{q-1}$ for odd $q$. The prefactor $i^{\lfloor q/2 \rfloor}$ ensures the Hermitian of $Q_q$, and $\lfloor q/2 \rfloor$ (integer floor) denotes the biggest integer smaller than $q/2$. When $q$ is even, $Q_q$ is a bosonic operator which is the SYK Hamiltonian with $q$-body interactions [3,4,9]; when $q$ is odd, $Q_q$ is a fermionic operator which is the supercharge of $\mathcal{N} = 1$ SUSY SYK Hamiltonian $H = Q_q^2$ [7,21,22].

Equation (1) contains $N$ Majorana fermions satisfying the Clifford algebra $\{\chi_i, \chi_j\} = 2\delta_{ij}$ and $\chi_i^\dagger = \chi_i$, thus

 

TABLE I. The periodic table of SYK for even $q > 2$ and SUSY SYK for odd $q > 1$. 3 classes for SYK and 8 classes for $\mathcal{N} = 1$ SUSY SYK. All the results in the table were achieved in a unified minimum scheme. In the listed degeneracy, $1 + 1$ means the two degenerate energy levels have opposite parities.

| $N \pmod 8$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P^2$ value | + | + | + | − | − | − | − | + |
| $R^2$ value | + | + | − | − | − | − | + | + |
| $q \pmod 4 = 0$ | AI | AI | A | AII | AII | AII | A | AI |
| $H = Q_q$ degeneracy | 1 | 1 | 1 + 1 | 2 | 2 | 2 | 1 + 1 | 1 |
| $q \pmod 4 = 2$ | D | D | A | C | C | C | A | D |
| $H = Q_q$ degeneracy | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $q \pmod 4 = 1$ | BDI | AI | CI | C | CII | AII | DIII | D |
| $Q_q$ degeneracy | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 1 |
| $H = Q_q^2$ degeneracy | 2 | 1 | 2 | 2 | 4 | 2 | 4 | 2 |
| $q \pmod 4 = 3$ | BDI | D | DIII | AII | CII | C | CI | AI |
| $Q_q$ degeneracy | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 |
| $H = Q_q^2$ degeneracy | 2 | 2 | 4 | 2 | 4 | 2 | 2 | 1 |

admitting a $2^{\lfloor N/2 \rfloor}$-dimensional matrix representation [27]. In this representation, one can choose $\chi_i$ with odd $i$ to be real and symmetric, and $\chi_i$ with even $i$ to be pure imaginary and skew symmetric [28]. By collecting real and imaginary represented Majorana fermions, one can define two antiunitary particle-hole symmetry operators: $P = K \prod_{i=1}^{\lceil N/2 \rceil} \chi_{2i-1}$ and $R = K \prod_{i=1}^{\lfloor N/2 \rfloor} i\chi_{2i}$, where $K$ is the complex conjugation operator, $\lceil N/2 \rceil$ (integer ceiling) denotes the smallest integer greater than $N/2$, therefore $\lfloor N/2 \rfloor + \lceil N/2 \rceil = N$ holds for any integer $N$.

It can be shown that $P\chi_i P^{-1} = -(-1)^{\lceil N/2 \rceil}\chi_i$ and $R\chi_i R^{-1} = (-1)^{\lfloor N/2 \rfloor}\chi_i$, thus

$$PQ_qP^{-1} = (-1)^{\lceil q/2 \rceil}(-1)^{q\lceil N/2 \rceil}Q_q, \tag{2a}$$

$$RQ_qR^{-1} = (-1)^{\lfloor q/2 \rfloor}(-1)^{q\lfloor N/2 \rfloor}Q_q. \tag{2b}$$

One can also find their squared values

$$P^2 = (-1)^{\lfloor \lceil N/2 \rceil /2 \rfloor}, \qquad R^2 = (-1)^{\lceil \lfloor N/2 \rfloor /2 \rceil}. \tag{3}$$

Multiplying two equations in Eq. (2) leads to the following unified classification: When $q(1 + N)$ is even, $P$ and $R$ either both commute with $Q_q$ or both anticommute with $Q_q$, thus $\Lambda = PR$ is a conserved quantity (fermion number parity) satisfying $[\Lambda, Q_q] = 0$ if $N$ is even, or an identity operator if $N$ is odd. Then one only needs one of the two operators acting as commuting operators $T_+$ or anticommuting operators $T_-$ [29]. When $q(1 + N)$ is odd, one of $P$, $R$ commutes with $Q_q$ and the other anticommutes with $Q_q$, thus $\Lambda = PR$ is a unitary chirality operator satisfying $\{\Lambda, Q_q\} = 0$. Then one needs both operators, one acting as $T_+$, the other acting as $T_-$ [29].

The classification can be systematically done by evaluating Eqs. (2), (3) and their commutation relations with $\Lambda$. The double Bott periodicity in $(q, N)$ can be directly inferred from Eqs. (2) and (3), which are invariant under $q \to q + 4$ or $N \to N + 8$. Furthermore, one can also identify a reflection symmetry: changing $(q, N) \to (-q, -N)$ and exchanging $P$ and $R$ keep Eqs. (2) and (3) invariant [29].

*ELS from RMT.*—For an ordered set of energy levels $\{\lambda_n\}$, the Wigner surmises [30–32] for the distribution of the consecutive energy level spacings $r_n = (\lambda_{n+2} - \lambda_{n+1})/(\lambda_{n+1} - \lambda_n)$ were derived $P_W(r) \propto (r + r^2)^\beta / (1 + r + r^2)^{1+3\beta/2}$, where $\beta = 1$, 2, 4 correspond to the Gaussian orthogonal, unitary, and symplectic ensemble (GOE-GUE-GSE), respectively. The bulk index $\beta$ can be extracted by comparing ED data with $P_W(r)$, or calculating the expectation $\langle r \rangle$ or its cousin $\langle \tilde{r} \rangle$ with $\tilde{r}_n = \min(r_n, 1/r_n)$. It was documented that $\langle \tilde{r} \rangle = 0.5359$, 0.6027, 0.6762 for $\beta = 1$, 2, 4, respectively. In contrast, the distribution of the ratio $r_n$ from independent random energy levels yields the Poisson statistics with $P_P(r) = 1/(1 + r)^2$ and $\langle \tilde{r} \rangle = 0.386$.

For the 7 random matrix ensembles with a spectral mirror symmetry, the hard-edge universality can be unveiled [33–40] from the distribution function of the smallest positive level $\lambda_1$, which vanishes as $P(\lambda_1) \sim \lambda_1^\alpha$ when $\lambda_1 \to 0$. In practice, to extract the edge exponent $\alpha$, it is more efficient to introduce a new ratio $A = \langle \lambda_1^2 \rangle / \langle \lambda_1 \rangle^2 \approx (1.60, 1.58)$, 1.27, 1.17, 1.13 for the ensembles with $\alpha = 0$, 1, 2, 3 (the two values at $\alpha = 0$ correspond to $\beta = 1$, 2 respectively) [29]. There is one-to-one mapping among the RMT classes, the two indices $(\beta, \alpha)$, and the two ratios $(\tilde{r}, A)$ [41]. We first use our new minimum scheme to reproduce the known results on $q \pmod 4 = 0$ case in the Supplemental Material [29], then apply it to study the other cases in the following.

*$q \pmod 4 = 2$.*—Since $q(1 + N)$ is even, two operators always anticommute with Hamiltonian $\{P, Q_q\} = \{R, Q_q\} = 0$, and $\Lambda = PR$ is the conserved parity if $N$ is even or $\Lambda = 1$ if $N$ is odd. The commutation relation between $P$ and $\Lambda$, the value of $P^2$ lead to the following classification: When $N \pmod 8 = 2, 6$, both operators swap the parity, thus $Q_q$ is in Class A(GUE); when $N \pmod 8 \neq 2, 6$, either $P$ preserves parity [$N \pmod 8 = 0, 4$] or no conserved quantity ($N$ is odd), thus $P^2 = \pm 1$ means $Q_q$ belongs to Class D(BdG), C(BdG), respectively. There is no degeneracy in all cases. These results are listed in the 6th and 7th row of Table I.

The classification and level degeneracy with even $N$ are consistent with those in Ref. [22]. For new results on odd $N$, we present an ED study of ELS [42] for $q = 6$ SYK model with $N = 17$, 19, 21, 23 in Fig. 1. The RMT indices $\beta$ and $\alpha$ are extracted from the probability distribution function $P(r)$ and $P(\lambda_1)$, or ratios $\langle \tilde{r} \rangle$ and $A$, respectively. Both methods lead to $(\beta, \alpha) = (2, 0), (2, 2), (2, 2), (2, 0)$ for $N = 17$, 19, 21, 23, respectively. We also checked the

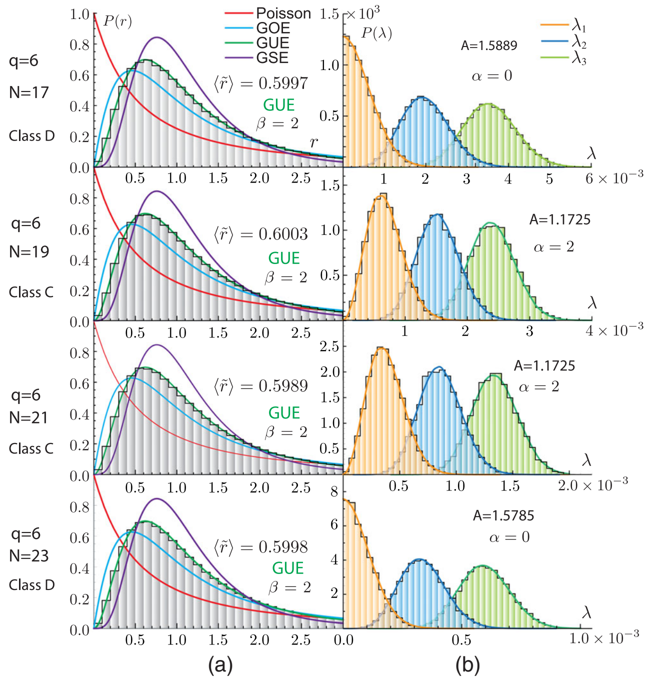


FIG. 1. The bulk energy level statistics and edge exponents of the $q = 6$ SYK model. (a) Distributions of the consecutive level spacing ratio $r$. The smooth curves are Wigner surmises corresponding to Poisson, GOE, GUE, and GSE statistics. (b) Distributions $P(\lambda_k)$, $k = 1, 2, 3$ of the smallest 3 energy levels. The smooth curves for $\lambda_1$, $\lambda_2$, $\lambda_3$ are obtained from numerical diagonalization of corresponding random matrix ensembles with size $10^3$ averaged over $10^6$ samples.

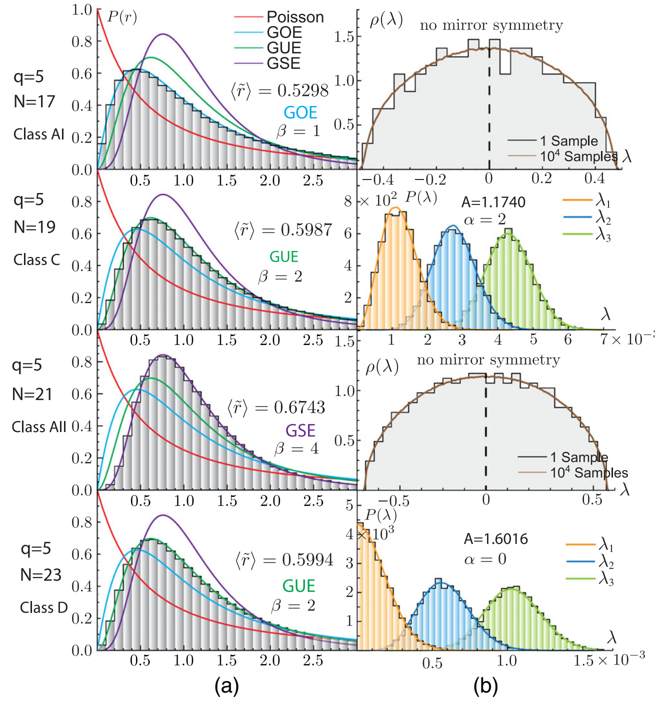


FIG. 2. The same notation as Fig. 1, but for $q = 5$ SYK supercharge. The class AI and AII have $\beta = 1, 4$, but no mirror symmetry [43], thus no well-defined $\alpha$ exponent.

degeneracy is always 1. These results match the classification and degeneracy listed in Table I.

$q\,(\mathrm{mod}\,4) = 1$.—Now, $q(1 + N)$ is odd for even $N$ and even for odd $N$, the former leads to the chiral operator $\Lambda = (-1)^F$, the latter leads to $\Lambda = 1$, so neither will lead to a conserved quantity.

When $N\,(\mathrm{mod}\,8) = 0, 4$, $\{P, Q_q\} = [R, Q_q] = 0$ and $P^2 = R^2 = \pm 1$. Since $P$ ($R$) anticommutes (commutes) with $Q_q$, their squared values lead to $N\,(\mathrm{mod}\,8) = 0$ is class BDI(chGOE), thus $Q_q$ has no degeneracy, but a mirror symmetry, so $H = Q_q^2$ has twofold degeneracy; $N\,(\mathrm{mod}\,8) = 4$ is class CII(chGSE), thus $Q_q$ has double degeneracy and also a mirror symmetry, so $H = Q_q^2$ has fourfold degeneracy. When $N\,(\mathrm{mod}\,8) = 2, 6$, $[P, Q_q] = \{R, Q_q\} = 0$ and $P^2 = -R^2 = \pm 1$. Since $P$ ($R$) commutes (anticommutes) with $Q_q$, their squared values lead to $N\,(\mathrm{mod}\,8) = 2$ is class CI(BdG), thus $Q_q$ has no degeneracy, but a mirror symmetry, so $H = Q_q^2$ has twofold degeneracy; $N\,(\mathrm{mod}\,8) = 6$ is class DIII(BdG), thus $Q_q$ has double degeneracy and also a mirror symmetry, so $H = Q_q^2$ has fourfold degeneracy.

When $N\,(\mathrm{mod}\,8) = 1, 5$, $[P, Q_q] = [R, Q_q] = 0$ and $P^2 = R^2 = \pm 1$. Since $P$ and $R$ commute with $Q_q$, their squared values lead to the following: $N\,(\mathrm{mod}\,8) = 1$ is class AI(GOE), thus $Q_q$ has no degeneracy, no mirror symmetry either, so $H = Q_q^2$ has no degeneracy; $N\,(\mathrm{mod}\,8) = 5$ is class AII(GSE), thus $Q_q$ has double degeneracy and no mirror symmetry, so $H = Q_q^2$ have twofold degeneracy. When $N\,(\mathrm{mod}\,8) = 3, 7$, $\{P, Q_q\} = \{R, Q_q\} = 0$ and $P^2 = R^2 = \mp 1$. Since $P$ and $R$ anticommute with $Q_q$, their squared values lead to suggest $N\,(\mathrm{mod}\,8) = 3, 7$ is class C(BdG), D(BdG). Both cases $Q_q$ has no degeneracy, but a mirror symmetry, so $H = Q_q^2$ has twofold degeneracy.

The classification for $q\,(\mathrm{mod}\,4) = 1$ and the level degeneracy are summarized in the 8 th–10 th rows of Table I. The results with even $N$ are consistent with those in Ref. [22]. For new results on odd $N$, we present an ED study of ELS for $q = 5$ SYK supercharge with $N = 17$, 19, 21, 23 in Fig. 2. Since the index $\alpha$ is only defined for cases with a spectral mirror symmetry, when the mirror symmetry is absent, we plot the spectral density $\rho(\lambda)$ averaged from 1 sample and $10^4$ samples [43,44]. These data lead to $(\beta, \alpha) = (1, -), (2, 2), (4, -), (2, 0)$ for $N = 17$, 19, 21, 23, respectively.

$q\,(\mathrm{mod}\,4) = 3$.—The situation is similar to $q\,(\mathrm{mod}\,4) = 1$ case. In fact, the reflection relation [29] in the operator algebra Eqs. (2) and (3) hints the SYK model with $(q, N)$ and $(4n - q, 8m - N)$ are in the same class, where $n$ and $m$ are integers. The classification and degeneracy of

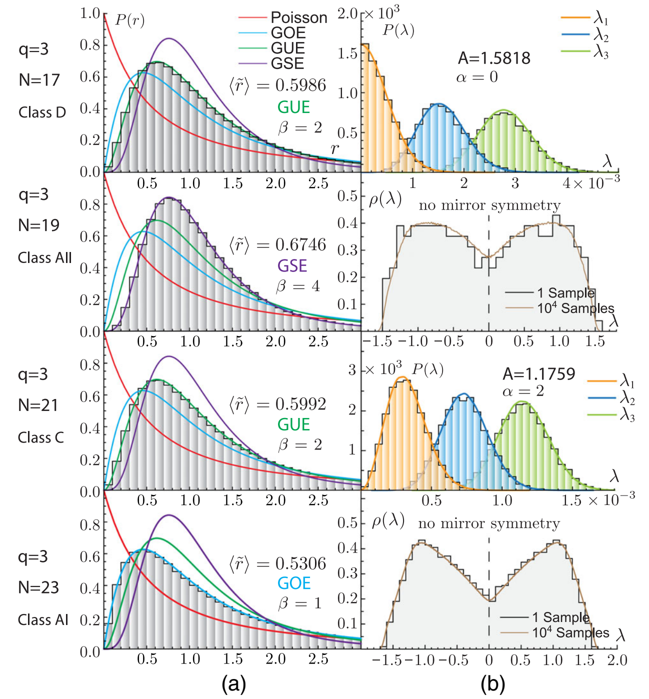


FIG. 3. The same notation as Fig. 2 but for $q = 3$ SYK supercharge. Note that the RMT classes appear in a reversed order than in the $q\,(\mathrm{mod}\,4) = 1$ case [45,46].

$q\,(\mathrm{mod}\,4) = 3$ case can be obtained from $q\,(\mathrm{mod}\,4) = 1$ results by replacing $N \to 8 - N$, which are summarized in the 11 st–13 th rows of Table I. Classifications and degeneracy at even $N$ reproduce those in Ref. [22]. For the new results at odd $N$, we present the ED study of $q = 3$ SYK supercharge for $N = 17, 19, 21, 23$ in Fig. 3. These data show $(\beta, \alpha) = (2, 0), (4, -), (2, 2), (1, -)$ for $N = 17$, 19, 21, 23, respectively.

*Discussions.*—It is instructive to compare Table I with the periodic tables of topological insulators and superconductors [47,48]. Despite using the same Cartan labels and sets of antiunitary operators, it is on topological equivalent classes of noninteracting electrons' gapped bulk states with gapless edge modes. Its Bott periodicity is in the space dimension $d\,(\mathrm{mod}\,8)$. Later, adding more symmetries such as translational symmetries, point group symmetries, or nonisomorphic symmetries leads to richer periodic tables for topological crystalline insulators [49].

The periodic Table I is on quantum chaos in $0 + 1$ dimensional gapless quantum spin liquids of interacting fermions. The most compact and systematic classification method developed here can be applied to all kinds of generalizations of SYK models such as the colored-SYK model [10,50], SYK models with a global $U(1)$ or $O(M)$, $U(M)$ symmetry [11,51], and the two indices SYK model [6]. It may also be applied to study quantum chaos in the colored [52,53] or uncolored tensor model [54] and cavity QED [55]. This is just putting more symmetries to lead to more conserved quantities, also more operators, therefore generating more complicated periodic tables. Its advantages over the extended schemes may be even more evident and dramatic as the periodic tables get more complicated.

As mentioned in the introduction, there are two complementary ways to characterize the quantum chaos. One way is to evaluate the OTOC function at a finite temperature $1 \ll \beta J \ll N$ by the $1/N$ expansion [56] to extract the Lyapunov exponent at an early time $\beta < t < t_E = \beta \log N$, where $t_E$ is called the Ehrenfest or scrambling time. The other way is to use the RMT to characterize the ELS or spectral form factor in a tenfold way at a finite and large enough $N$. The RMT describes the energy level correlations at the Heisenberg timescale $t_H \sim 1/\Delta \sim e^N/NJ$, where $\Delta$ is the mean many-body energy level spacing, but breaks down when the energy level spacing is beyond the Thouless energy $E_{\mathrm{Th}} \sim N^2\Delta$. So it fails when $t < t_{\mathrm{Th}} \sim 1/E_{\mathrm{Th}} \sim t_H/N^2$. Although the $1/N$ expansion has been pushed from the scrambling time $t_s$ to the longer timescale $N/J$ in Ref. [12], it remains unknown how to push it to the Thouless time $t_{\mathrm{Th}}$. Because the wide separation between the two timescales $N/J$ and $t_{\mathrm{Th}}$, it remains challenging to explore the double periodicity and the reflection symmetry in Table I by $1/N$, $1/q$, or $N/q^2$ expansions. The possible impacts on the classifications of quantum black holes in the bulk [57] are being studied.

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*Note added.*—Recently, we found the possible implications of our periodic table on the bulk gravity has been thoroughly presented in Ref. [58]. For example, the 8 distinct random matrix classes of the supersymmetric SYK models listed in Table I correspond to 8 variants of Jackiw-Teitelboim supergravity theories in the bulk.

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[27] The irreducible matrix representation of the Clifford algebra can be faithfully realized by gamma matrices. The $d$-dimensional gamma matrices are be constructed from Pauli matrices ($d = 2$ case) $\gamma_1^{(2)} = \sigma_x$, $\gamma_2^{(2)} = \sigma_y$, $\gamma_3^{(2)} = \sigma_z$, and then iteratively build $d + 2$ matrices by taking tensor product $\gamma_k^{(d+2)} = \sigma_x \otimes \gamma_k^{(d)}$, $k = 1, \ldots, d + 1$, $\gamma_{d+2}^{(d+2)} = \sigma_y \otimes I_{2^{d/2}}$, $\gamma_{d+3}^{(d+2)} = \sigma_z \otimes I_{2^{d/2}}$, where $d$ is an even number. The $N$ Majorana can be represented by $\chi_k = (1/\sqrt{2})\gamma_k^{(2\lfloor N/2 \rfloor)}$ with $k = 1, 2, \ldots, N$.

[28] Note that in the minimal scheme used in the main text, one cannot choose odd $i$ to be pure imaginary and even $i$ to be real when $N$ is odd, because the number of real matrices equals the number of imaginary matrices plus one. However, in the extended scheme used in Refs. [19,25,26], one can use either way.

[29] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.124.244101 for more information. Section A has details on (i) proving two relations in the periodic table of SYK models, (ii) reproducing the previously known classification at $q\,(\mathrm{mod}\,4) = 0$ by the minimum scheme developed in the main text, and (iii) re-deriving the results at $q\,(\mathrm{mod}\,4) = 3$ achieved in the main text using the reflection relation by directly checking its operator algebra. Section B provides a review of (i) the tenfold way classification in random matrix theory, (ii) the probability distribution function $P(r)$ for the bulk ELS and the bulk index $\beta$, the universal ratio of the nearest energy level spacing $\langle \tilde{r} \rangle$ to determine the bulk exponent $\beta$ effectively, and (iii) the distribution functions $P(\lambda_k)$, $k = 1$, 2, 3 for the lowest eigenvalues for the 7 classes with a mirror symmetry (3 chiral classes and 4 BdG classes) and the edge exponent $\alpha$, a new moment ratio $A$ to determine the edge exponent $\alpha$ efficiently

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[41] The RMT classes are often named by its Cartan label of corresponding symmetric spaces. Here, we use their Cartan names followed with GOE-GUE-GSE, etc., in parentheses, only use the Cartan names in the tables and figures. The relation between RMT classes and $(\beta, \alpha)$ indices are 3 Wigner-Dyson classes: AI, A, AII (which has no mirror symmetry) correspond $(\beta, \alpha) = (1, -), (2, -), (4, -)$; 3 chiral classes: BDI, CII, AIII, correspond $(\beta, \alpha) = (1, 0), (4, 3), (2, 1)$; 4 BdG classes: CI, D, C, DIII, correspond $(\beta, \alpha) = (1, 1), (2, 0), (2, 2), (4, 1)$.

[42] Below we choose $J_{\mathrm{SYK}} = 1$ and $J_{\mathcal{N}=1} = 1$ when performing numerical calculations.

[43] It shows the mirror symmetry is absent for every single realization of $C_{i_1,\ldots,i_q}$, but still emerges after making disorder averages. Despite the absence of the exact mirror symmetry at any given random distribution, there is still an approximate mirror symmetry due to the self-averaging property as $N \to \infty$. It will be promoted to the exact mirror symmetry after the average over many random realizations as shown in Figs. 2 and 3.

[44] The spectral density is also known as many-body density of states (DOS), which seems to deviate from the RMT semicircle law in some cases, i.e., Fig. 3(b). The many-body DOS calculated by $1/N$ expansion at $q = 4$ in Ref. [9] also deviates from the RMT semicircle law. But it was believed that the DOS is less universal than the bulk ELS and edge exponent in the RMT. See also Ref. [46].

[45] The definition $H = Q_q^2$ immediately leads to the simple relation between the many body DOS of $H$ and that of $Q_q$: $\rho_H(E) = (1/\sqrt{E})\rho_{Q_q}(\sqrt{E}), E \geq 0$. So as long as $\rho_{Q_q}(\sqrt{E} \to 0) \neq 0$, then $\rho_H(E \to 0) \sim 1/\sqrt{E}$. So the dip in Fig. 3(b) at $q = 3$ for $Q_q$ does not affect this low-energy asymptotic behavior for $H = Q_q^2$. This divergence was also found in the $1/N$ expansion of $\mathcal{N} = 1$ SUSY [7,13,14].

[46] There is an evident dip at $E = 0$ for the AI, AII class at $N = 23$, 19, respectively. In fact, the dip always exists for $q = 3$ at any $N \,(\mathrm{mod}\, 8)$ (data not shown). However, it disappears and gets back to the semicircle law for $q = 7$ at any $N \,(\mathrm{mod}\, 8)$ (data not shown). The physical origin of this large deviation from the expected semicircle law only at $q = 3$ is expected. It was known that the DOS may be close to Gaussian when $q \ll \sqrt{N}$, but close to being a semicircle when $q \gg \sqrt{N}$, could take a complicated form when $q \sim \sqrt{N}$. This fact could explain why the DOS gets back to the semicircle when $q = 7$ or larger.

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