Supersymmetry in the Standard Sachdev-Ye-Kitaev Model

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Supersymmetry is a powerful concept in quantum many-body physics. It helps to illuminate ground-state properties of complex quantum systems and gives relations between correlation functions. In this Letter, we show that the Sachdev–Ye–Kitaev model, in its simplest form of Majorana fermions with random four-body interactions, is supersymmetric. In contrast to existing explicitly supersymmetric extensions of the model, the supersymmetry we find requires no relations between couplings. The type of supersymmetry and the structure of the supercharges are entirely set by the number of interacting Majorana modes and are thus fundamentally linked to the model's Altland–Zirnbauer classification. The supersymmetry we uncover has a natural interpretation in terms of a one-dimensional topological phase supporting Sachdev–Ye–Kitaev boundary physics and has consequences away from the ground state, including in q-body dynamical correlation functions.

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The Sachdev–Ye–Kitaev (SYK) model [1,2] is a toy model that provides insight into diverse physical phenomena ranging from the holographic principle [3–5] to quantum chaos [6–11] and non-Fermi liquid behavior of strongly correlated electron systems [12–18]. Similar to black holes, the model is believed to scramble quantum information with maximal efficiency [17,19].

The simplest variant of the SYK model describes k Majorana fermions that interact through a random four-body term [2]. The model's proposed physical realizations include mesoscopic systems based on Majorana fermions in vortices or quantum dots [20,21] or the ends of a one-dimensional topological phase [22].

Various generalizations of the SYK model have been considered, including models with n-body interactions [23,24] and supersymmetric extensions [25–29]. Typically, exact supersymmetry (SUSY) requires fine-tuning of the parameters [30–33]. In the supersymmetric SYK extensions, this fine-tuning corresponds to requiring certain relations between different couplings [25].

In this Letter, we show that already the simplest four-body SYK model, without any fine-tuning, is supersymmetric for all but two values of k mod 8. The type of SUSY depends only on k. The supercharges will be shown to relate to ramps and long-time plateaus in time-dependent correlation functions [34], which thus provide signatures of SUSY far from equilibrium. In particular, we find that the number of supercharges is linked to the presence and nature of time-reversal symmetry and is reflected in the ramp shape [35]. We also show that the number and structure of supercharges set the plateau features in q-body time-dependent correlation functions.

Throughout this Letter, we focus on SUSY in the sense of supersymmetric quantum mechanics [30,31,36–42]. SUSY is characterized by $\mathcal{N}$, the number of mutually anticommuting Hermitian fermionic supercharges that square to the Hamiltonian [36]

$$\{Q_a, Q_b\} = 2H\delta_{ab}, \qquad [H, Q_a] = 0. \tag{1}$$

We consider the following Hamiltonian, which describes four-body interactions between k Majorana modes [2],

$$H = \sum_{t=0}^{k-1}\sum_{s=0}^{t-1}\sum_{r=0}^{s-1}\sum_{q=0}^{r-1} J_{qrst}\gamma_q\gamma_r\gamma_s\gamma_t + E_0, \tag{2}$$

with real (as required by Hermiticity) but otherwise structureless couplings J_{qrst} and the constant E_0 that ensures positive energies. The Hermitian Majorana operators $\gamma_q = \gamma_q^\dagger$ satisfy the anticommutation relation $\{\gamma_q, \gamma_r\} = 2\delta_{qr}$ [43] and span an M-dimensional Hilbert space with $M = 2^{\lceil k/2 \rceil}$ [44]. Since each term in the Hamiltonian [Eq. (2)] contains an even number of Majoranas, it conserves fermion parity P, given by

$$P = \begin{cases} i^{k/2}\gamma_1\gamma_2, \ldots, \gamma_k & \text{even } k, \\ i^{(k+1)/2}\gamma_1\gamma_2, \ldots, \gamma_k\gamma_\infty & \text{odd } k. \end{cases} \tag{3}$$

To work in a Hilbert space with well-defined fermion parity, the additional Majorana γ_∞ "at infinity" must be included when k is odd [45]. The operator γ_∞ is not local to the SYK model; considering, e.g., a realization in a superconducting vortex [20], it represents a degree of freedom with support far away from the vortex where the SYK Majoranas $\gamma_{j\neq\infty}$ reside. Like the local Majoranas, γ_∞ is Hermitian and satisfies $\{\gamma_q, \gamma_r\} = 2\delta_{qr}$. Since $[H, P] = 0$, all eigenstates of H can by labeled by their parity eigenvalue $p = \pm 1$, giving $H|\psi_\mu^p\rangle = \varepsilon_\mu^p |\psi_\mu^p\rangle$ and $P|\psi_\mu^p\rangle = p|\psi_\mu^p\rangle$.

The number of interacting Majorana modes, specifically k mod 8, sets the model's antiunitary symmetries [22, 45–47]. These come in two variants $T_\pm$, antiunitary operators satisfying $T_\pm \gamma_{q\neq\infty} T_\pm^{-1} = \gamma_{q\neq\infty}$. They further satisfy $T_\pm P T_\pm^{-1} = \pm P$. We refer to T_+ as time-reversal symmetry because it commutes with fermion parity and hence sets correlations within a parity sector. Conversely, we call T_- particle-hole symmetry. Crucially for this work, since T_- flips fermion parity, its presence implies correlations *between* parity sectors.

The consideration of both T_+ and T_- implies [46] a classification with more structure than the threefold Wigner–Dyson way highlighted in Ref. [22]. In fact, as we now briefly review, it gives rise to the eight real Altland–Zirnbauer classes. For even k, either time-reversal symmetry T_+ or particle-hole symmetry T_- is present. For odd k, both T_+ and T_- are present—in this case, $T_+ \gamma_\infty T_+^{-1} = (-1)^{(k+1)/2} \gamma_\infty$. Their product, the unitary operator $Z = T_+ T_-$, equals the product of all local Majorana operators up to a complex phase and corresponds to a chiral symmetry [45,46]. A key feature of Z, which we will use repeatedly for diagnosing locality, is $[Z, \gamma_{q\neq\infty}] = 0$ and $\{Z, \gamma_\infty\} = 0$. The squares $T_\pm^2$ vary with k and label the eight real Altland–Zirnbauer classes [48]. While the Altland–Zirnbauer and Wigner–Dyson picture give the same level spacing statistics, the former also takes cross-parity correlations into account. We summarize the symmetry classification in Table I and review it in detail in the Supplemental Material [49].

SUSY is known to imply a degeneracy between the parity sectors [36]: the supercharges Q_a exchange bosonic states with parity eigenvalues $p = +1$ and fermionic states with parity eigenvalue $p = -1$ [36]. Thus, the supercharges anticommute with fermion parity, $\{P, Q_a\} = 0$. The presence of particle-hole symmetry also guarantees degeneracy between parity sectors, which as we now note, also implies SUSY. Parity degeneracy directly follows from particle-hole symmetry because $|\psi_\mu^p\rangle$ and $T_-|\psi_\mu^p\rangle$ have the same energy $\varepsilon_\mu = \varepsilon_\mu^p = \varepsilon_\mu^{-p}$ (since $[T_-, H] = 0$), but opposite parity ($\{T_-, P\} = 0$) [45,46]. Therefore, $|\psi_\mu^p\rangle\langle\psi_\mu^{-p}|$ is an odd-parity zero mode, i.e., an operator that commutes with the Hamiltonian but anticommutes with fermion parity [46]. This in turn implies SUSY: the operator $\tilde{Q}_\mu = \sqrt{\varepsilon_\mu}|\psi_\mu^+\rangle\langle\psi_\mu^-|$ satisfies $\tilde{Q}_\mu \tilde{Q}_\mu^\dagger = \varepsilon_\mu |\psi_\mu^+\rangle\langle\psi_\mu^+|$ and $\tilde{Q}_\mu^\dagger \tilde{Q}_\mu = \varepsilon_\mu |\psi_\mu^-\rangle\langle\psi_\mu^-|$, and hence the linear combinations $Q_{1,\mu} = \tilde{Q}_\mu + \tilde{Q}_\mu^\dagger$ and $Q_{2,\mu} = i(\tilde{Q}_\mu - \tilde{Q}_\mu^\dagger)$ are Hermitian, anticommute, and square to ε_μ times the projector on the two parity-degenerate states. Consequently, the two supercharges

$$Q_1 = \sum_\mu (\tilde{Q}_\mu + \tilde{Q}_\mu^\dagger), \qquad Q_2 = -i \sum_\mu (\tilde{Q}_\mu - \tilde{Q}_\mu^\dagger) \quad (4)$$

satisfy Eq. (1) and anticommute with P. Particle-hole symmetry is present unless $k = 4n$. Thus, all but two of the symmetry classes are supersymmetric.

TABLE I. Time-reversal symmetry T_+ and particle-hole symmetry T_- in the SYK model. The symmetries may be absent (denoted by 0), or present and square to -1 or $+1$. The second row gives the corresponding Cartan labels.

k mod 8	0	1	2	3	4	5	6	7
Label	AI	BDI	D	DIII	AII	CII	C	CI
T_+^2	+1	+1	0	−1	−1	−1	0	+1
T_-^2	0	+1	+1	+1	0	−1	−1	−1

Given the presence of six supersymmetric classes, there are a number of questions regarding the interplay of SUSY and the symmetry classification. How does $\mathcal{N}$ depend on the symmetry class? How do Q_j transform under $T_\pm$ and how does this translate to the structure of the supercharges? We next turn to these questions.

We start with counting $\mathcal{N}$. A direct approach is based on counting level degeneracies. This follows from the observation that the "spectrally flattened" Hermitian supercharges $\Gamma_j = Q_j/\sqrt{H}$ satisfy

$$\{\Gamma_j, \Gamma_k\} = 2\delta_{jk}, \qquad [H, \Gamma_k] = 0, \qquad \{P, \Gamma_k\} = 0. \quad (5)$$

They are thus many-body zero mode forms of Majorana fermions. An even $\mathcal{N}$ of such zero modes give rise to a $2^{\mathcal{N}/2}$-dimensional fermionic degeneracy space for each of the ε_μ with one of the $|\psi_\mu^p\rangle$ chosen as "vacuum." (With a suitable choice, the Γ_j-fermion parity of an eigenstate matches the state's physical fermion parity.) This procedure is similar in spirit to the standard construction of supermultiplets [51]. For the six supersymmetric SYK classes, a twofold degeneracy is guaranteed by T_- and a further twofold (Kramers) degeneracy is present whenever $T_+^2 = -1$, resulting in an overall fourfold degeneracy. This suggests $\mathcal{N} = 2$, except for classes DIII and CII where this count gives $\mathcal{N} = 4$. What this counting does not address is how many Γ_j (and hence Q_j) are local to the SYK model. Next, we investigate this to obtain the decomposition $\mathcal{N} = \mathcal{N}_{\rm loc} + \mathcal{N}_\infty$ with $\mathcal{N}_{\rm loc}$ counting the number of supercharges involving only $\gamma_{q\neq\infty}$. We first discuss the symmetry classes D and C before demonstrating the implications of locality in classes BDI and CI. For brevity, we derive the supercharges in classes DIII and CII with $T_-^2 = -1$ in [49] and only summarize the results here.

We begin with classes D and C. Here k is even; hence, all γ_q are local. Therefore, our argument above applies directly: we find $\mathcal{N} = \mathcal{N}_{\mathrm{loc}} = 2$. All the other supersymmetric classes have odd k; thus, potentially $\mathcal{N} \neq \mathcal{N}_{\mathrm{loc}}$ due to γ_∞. As we shall see, in all of these classes, $\mathcal{N} = \mathcal{N}_{\mathrm{loc}} + 1$ with $\mathcal{N}$ following its degeneracy-based value above. This is intuitive because $\gamma_\infty \equiv \Gamma_\infty$ automatically satisfies Eq. (5) (in particular, it anticommutes with any local parity-odd operator); thus, $\mathcal{N}_{\mathrm{loc}}$ is at most $\mathcal{N} - 1$. To formally establish $\mathcal{N}_{\mathrm{loc}}$ and the transformation of Γ_j under $T_\pm$, we work in the energy eigenbasis, $H = \mathrm{diag}(\{\varepsilon_\mu\}) \otimes \mathbb{1}_{2^{\mathcal{N}/2}}$, with $P = \mathbb{1}_{M/2} \otimes \tau_3$. (Here and below, τ_j and σ_j are Pauli matrices; τ_j act in parity grading and σ_j in the space of Kramers doublets, where applicable. We will often omit trivial tensor factors.) In this basis, class D (C) has particle-hole symmetry [up to a phase $\mathrm{diag}(\{\exp(i\varphi_\mu)\})$ omitted here and below] $T_- = \tau_{1(2)}\mathcal{K}$ (with $\mathcal{K}$ for complex conjugation); this follows from $T_-^2 = \pm 1$ and parity being the only degeneracy, $\mathbb{1}_{2^{\mathcal{N}/2}} = \tau_0$. We have $\Gamma_{1,2} = \tau_{1,2}$, which correspond to the two supercharges introduced in Eq. (4) [52].

To study classes BDI and CI, we focus on a degeneracy space with energy ε_μ and first establish the form of $T_\pm$ and thus Z in this space. $T_+^2 = +1$ implies that parity is again the only degeneracy, so $T_- = \tau_{1(2)}\mathcal{K}$ in class BDI (CI). $T_+|\psi_\mu^p\rangle \propto |\psi_\mu^p\rangle$ implies that the most general form is $T_+ = \exp(i\varphi_\mu\tau_3)\mathcal{K}$. With a suitable choice of the relative phases between the two parity sectors, we can thus use $Z = T_+T_- = \tau_1$; in this basis, $T_+ = \mathcal{K}$ ($T_+ = \tau_3\mathcal{K}$) for class BDI (CI). The two Γ_j satisfying Eq. (5) can again be chosen as $\Gamma_{1,2} = \tau_{1,2}$. However, checking the (anti)commutation with Z, we find that only Γ_1 is local. Conversely, we can identify $\Gamma_2 \equiv \Gamma_\infty \equiv \gamma_\infty$; this is consistent both with γ_∞ itself satisfying Eq. (5) and its transformation under T_+. We thus find $\mathcal{N}_{\mathrm{loc}} = 1$.

In classes DIII and CII, we find $\mathcal{N}_{\mathrm{loc}} = 3$ local supercharges, as we show in detail in [49]. The spectrally flattened supercharges can be written as Kronecker products $\Gamma_j = \tau_1\sigma_j$. Their product $\Gamma_4 = -i\Gamma_1\Gamma_2\Gamma_3 = \tau_1$ is also local but does not anticommute with $\Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}$; it does, however, contribute to correlation function, as we discuss below. As in classes BDI and CI, the nonlocal supercharge is $\Gamma_\infty = \tau_2$.

The values $\mathcal{N}_{\mathrm{loc}}$, together with the sign s in $T_\pm\Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}T_\pm^{-1} = s\Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}$, have a natural interpretation if one views the SYK model as arising at the end of a one-dimensional topological phase in class BDI [22,45]. These systems admit a $\mathbb{Z}_8$ classification: at one of their ends, they have k_s Majoranas satisfying $T_\pm\gamma_qT_\pm^{-1} = s\gamma_q$; the topological index is $\nu = (k_+ - k_-)$ mod 8. Thus, the eight topological classes can be labeled by $\nu = 0, 1, 2, 3, 4, -3, -2, -1$ with the integers counting the number and sign of unpaired Majoranas. In the SUSY classes, we find the same pattern for $s\mathcal{N}_{\mathrm{loc}}$ against k mod 8 ($T_\pm\gamma_{q\neq\infty}T_\pm^{-1} = \gamma_{q\neq\infty}$ implies $k_+ = k$, $k_- = 0$) [see Table II]. The $\mathcal{N}_{\mathrm{loc}}$ supercharges $\Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}$ can thus be viewed as the many-body emergence of the minimal number and type of unpaired Majoranas consistent with k.

TABLE II. The Dyson index β, number $\mathcal{N}_{\mathrm{loc}}$, signature $T_\pm\Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}T_\pm^{-1} = s\Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}$, and the Majorana fermion structure of $\Gamma_{j\neq\infty}$. (The supercharges Q_j have the same properties, since $T_\pm HT_\pm^{-1} = H$.) In the Majorana expansion of Γ_j, only those Υ_a with $n_a = 4n+1$ or $n_a = 4n+3$ contribute; the two options are shown in the last two rows of the table. The horizontal line visually distinguishes Γ_4 from the three supercharges because it does not anticommute with them. A blank entry indicates that Γ_4 does not exist in these classes.

k mod 8	1	2	3	5	6	7
Label	BDI	D	DIII	CII	C	CI
β	1	2	4	4	2	1
$s\mathcal{N}_{\mathrm{loc}}$	1	2	3	−3	−2	−1
$\Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}$	$4n+1$	$4n+1$	$4n+1$	$4n+3$	$4n+3$	$4n+3$
Γ_4			$4n+3$	$4n+1$		

Next, we turn to the structure of the supercharges in terms of the Majorana fermions γ_q. For this, we employ another operator basis of the Hilbert space, the products of n_a Majorana operators $\gamma_{q\neq\infty}$ [53]

$$\Upsilon_a = i^{n_a(n_a-1)/2}\gamma_{i_1(a)}\gamma_{i_2(a)}, \ldots, \gamma_{i_{n_a}(a)} \tag{6}$$

with $i_j(a) \neq i_{j'}(a)$. Υ_a are Hermitian, unitary, and orthonormal with respect to the trace, $tr[\Upsilon_a\Upsilon_b]/M = \delta_{ab}$. In total, there are 2^k local operators Υ_a [53]. As we aim to expand $\Gamma_{j\neq\infty}$, i.e., Hermitian odd-parity operators in terms of Υ_a, we use only those Υ_a with odd n_a and use only real expansion coefficients.

Both time-reversal and particle-hole symmetry have the same (anti-) commutation properties when acting on Υ_a. Since $T_\pm\gamma_qT_\pm^{-1} = \gamma_q$, only the phase of Υ_a [cf. Eq. (6)] may change when applying $T_\pm$, giving

$$T_\pm\Upsilon_aT_\pm^{-1} = (-1)^{n_a(n_a-1)/2}\Upsilon_a. \tag{7}$$

That is, $T_\pm$ and Υ_a commute when $n_a=4n+1$ and anticommute when $n_a = 4n+3$. This, together with $v_{j,a} \in \mathbb{R}$ below, implies that, when expanding the supercharges,

$$\Gamma_j = \sum_a v_{j,a}\Upsilon_a, \qquad \sum_a v_{j,a}^2 = 1, \tag{8}$$

only terms with $n_a = 4n+1$ contribute to Γ_j when $[T_\pm, \Gamma_j] = 0$ and only terms with $n_a = 4n+3$ contribute when $\{T_\pm, \Gamma_j\} = 0$. In classes DIII and CII, we also consider $\Gamma_4 = -i\Gamma_1\Gamma_2\Gamma_3$, whose transformation properties follow from those of $\Gamma_{1,2,3}$. The resulting expansion structure is summarized in Table II.

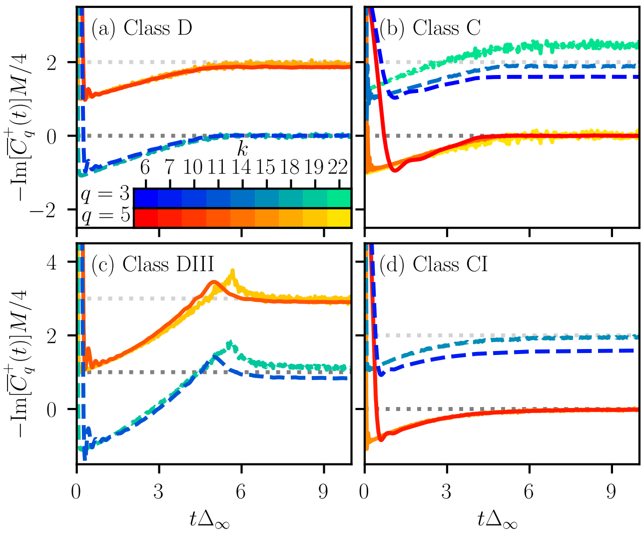

FIG. 1. q-body time-dependent correlation function at infinite temperature, averaged over an ensemble of up to 2^{16} Gaussian distributed J_{qrst}, for classes (a) D, (b) C, (c) DIII, and (d) CI. The different colors denote different k and q, cf. inset in (a), the dashed lines represent $q = 3$ and the solid lines $q = 5$. The ramp shape follows the Dyson index β and hence links to the number of supercharges. The long-time plateau $\bar{C}_{q,\infty}$ is studied in Fig. 2. Error bars are either smaller than the linewidth (for small k) or smaller than the disorder-induced fluctuations of the lines (for large k).

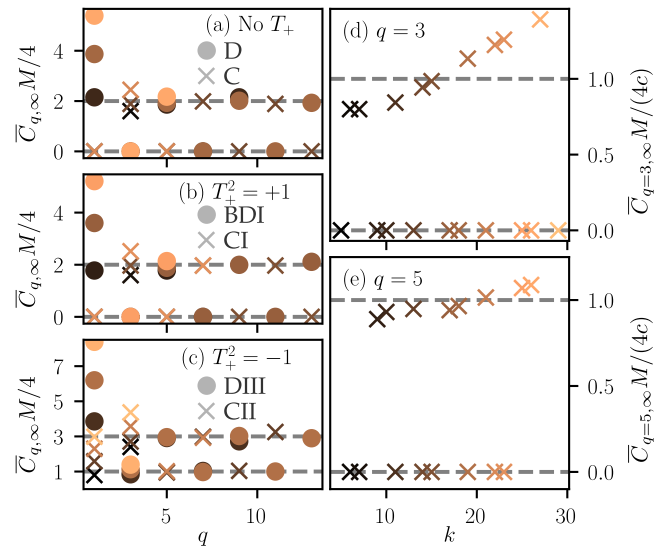

FIG. 2. Normalized plateau $\bar{C}_{q,\infty}M/4$ of the q-body correlation function, averaged over an ensemble of up to 2^{14} Gaussian distributed J_{qrst}. The color encodes the number k of Majoranas, cf. panels (d) and (e). In all classes, $\bar{C}_{\infty}M/4$ alternates with q approximately as predicted in Eq. (13); the agreement is excellent when $\binom{k}{q}/\binom{k}{\lfloor k/2\rfloor}$ is close to one. In panel (d), we show that $\bar{C}_{\infty}M/(4c)$ [with c the random matrix expectation based on Eq. (13)] increases as a function of k but with a rate that decreases upon increasing q [panel (e)]. Statistical error bars are smaller than the marker size.

Having discussed the interplay of SUSY and the symmetry classification, we now identify signatures of SUSY, $\mathcal{N}_{\mathrm{loc}}$, and the supercharge structure in various observables. A simple link between $\mathcal{N}_{\mathrm{loc}}$ and observables exists due to the fact that the number of different $\Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}$ and their linearly independent odd-parity products, i.e., including Γ_4 in classes CII and DIII, equals the degrees of freedom β (i.e., the Dyson index linked to T_+ [54]) of the Hamiltonian's off-diagonal matrix elements. In fact, the most general Hermitian linear combinations of these Γ_j have the same type of off diagonals, up to an imaginary unit, as the Hamiltonian: real for $\beta = 1$ (classes BDI and CI), complex for $\beta = 2$ (classes D and C), and real quaternion for $\beta = 4$ (classes DIII and CII).

In the SUSY classes, the value of β sets the energy level correlations, including the long-range spectral rigidity, across *opposite* parity sectors (these are uncorrelated without SUSY), which lead to "ramps" in time-dependent correlation functions of parity-*odd* observables. (For single-Majorana examples, see Refs. [34,46].) These ramps occur at time scales below 2π times the inverse mean level spacing $1/\Delta_\infty$ and have β-dependent shape [35]. In particular, the ramp connects to a long-time plateau smoothly when $\beta = 1$, sharply when $\beta = 2$, and with a kink when $\beta = 4$. In Fig. 1, we show ensemble-averaged q-body correlation functions [Eq. (10) below] in classes D, C, DIII, and CI. For completeness, we show the correlation function in the remaining symmetry classes, including those that do not support SUSY, in [49].

Besides this direct correspondence between the supercharges and ramp structure, we additionally find more subtle consequences of SUSY: the long-time ($t \gg 1/\Delta_\infty$) plateau in q-body correlation functions is also related to the number and structure of the supercharges, cf. Fig. 2. To quantify this relationship, we consider the retarded time-dependent q-body correlation function

$$C_q^+(t) = -i\Theta(t)\frac{1}{\binom{k}{q}}\sum_{a,n_a=q}\langle\{\Upsilon_a(t),\Upsilon_a(0)\}\rangle, \qquad (9)$$

where $\langle\ldots\rangle$ denotes thermal average. Although the signatures we reveal are present at any temperature, we find an especially transparent relationship at infinite temperature, where the correlation function reads

$$C_q^+(t) = -i\Theta(t)\frac{1}{\binom{k}{q}}\sum_{a,n_a=q}\frac{1}{M}\sum_{p\mu\nu}|\langle\psi_\mu^p|\Upsilon_a|\psi_\nu^{-p}\rangle|^2 \times 2\cos\left[t(\varepsilon_\mu^p-\varepsilon_\nu^{-p})\right]. \qquad (10)$$

When $t \gg 1/\Delta_\infty$, terms with $\varepsilon_\mu^p \neq \varepsilon_\nu^{-p}$ give a quickly oscillating contribution $\delta C_q^+(t)$ that averages to zero.

Only states with $\varepsilon_\mu^p = \varepsilon_\nu^{-p}$ give a time-independent contribution $C_{q,\infty}$. Thus, $C_q^+(t) = -i\Theta(t)[C_{q,\infty} + \delta C_q^+(t)]$ with

$$C_{q,\infty} = \frac{1}{\binom{k}{q}} \sum_{a,n_a=q} \frac{2}{M} \sum_\mu \mathrm{tr}\Upsilon_{a\mu}^2, \qquad \Upsilon_{a\mu} = P_\mu \Upsilon_a P_\mu, \quad (11)$$

where in converting the equal-energy sum to a trace, we introduced the projection P_μ to the eigenspace with energy ε_μ and used that Υ_a is Hermitian and parity odd.

Next, we convert Eq. (11) into a sum over $\Gamma_{j<\infty}$. We start by expanding $\Upsilon_{a\mu} = \sum_{j<\infty} y_j P_\mu \Gamma_j$ (with real y_j), which holds as within an eigenspace. The (projected) operators $P_\mu \Gamma_{j<\infty}$ form a basis for local, Hermitian, parity-odd operators. If Υ_a transforms the same (opposite) way to Γ_j under $T_\pm$, then generically $y_j \neq 0$ ($y_j = 0$). Now using the trace orthogonality of the $\Gamma_{j<\infty}$ and $\mathrm{tr}\Gamma_j^2 = 2^{\mathcal{N}/2} = \mathcal{N}$ (for $\mathcal{N} = 2, 4$), we find

$$C_{q,\infty} = \frac{1}{\binom{k}{q}} \sum_{a,n_a=q} \frac{2}{M\mathcal{N}} \sum_\mu \sum_{j<\infty} [\mathrm{tr}(P_\mu \Upsilon_a P_\mu \Gamma_j)]^2. \quad (12)$$

A simple estimate for $C_{q,\infty}$ can be given assuming that expanding $P_\mu \Gamma_{j<\infty} = \sum_a v_{\mu j,a} \Upsilon_a$ results in random coefficients $v_{\mu j,a}$ subject only to normalization and symmetry constraints. Denoting such a random vector average by $\overline{(\ldots)}$, we find

$$\frac{\bar{C}_{q,\infty} M}{4} = \begin{cases} \frac{\mathcal{N}}{\beta} \mathcal{N}_{\mathrm{loc}} & q: \Upsilon_a \triangleq \Gamma_{j\leq \mathcal{N}_{\mathrm{loc}}}, \\ \frac{\mathcal{N}}{\beta} \delta_{\beta,4} & \text{otherwise}, \end{cases} \quad (13)$$

where $\Upsilon_a \triangleq \Gamma_j$ here means that Υ_a transforms the same way as Γ_j under $T_\pm$. Thus, each $\Gamma_{j<\infty}$ gives the same contribution to $\bar{C}_{q,\infty}$ when they contain q-Majorana terms and zero otherwise. The nonzero value for $\beta = 4$ when $\Upsilon_a \not\triangleq \Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}$ arises due to Γ_4 since $\Gamma_4 \not\triangleq \Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}$. Considering the Majorana structure of Γ_j in Table II, Eq. (13) translates to an alternating pattern of $\bar{C}_{q,\infty}$ as q is varied in a given symmetry class, with complementary $\bar{C}_{q,\infty}$ values for classes with opposite $s\mathcal{N}_{\mathrm{loc}}$.

In Fig. 2, we show the numerically obtained value of $\bar{C}_\infty M/4$ for various k and q. The alternating pattern expected from Eq. (13) is clearly visible [panels (a) to (c)]. While the numerical value of the nonzero plateau differs from expectation when $q \ll k$ (and $k - q \ll k$, not shown), Eq. (13) gives an accurate prediction when $\binom{k}{q}/\binom{k}{\lfloor k/2 \rfloor}$ is close to one (with $\lfloor \ldots \rfloor$ the floor function). To investigate this further, in panels (d) and (e), we show $C_\infty M/4$ versus k. The growth with k is slower for $q = 5$ than for $q = 3$, which is in turn slower than the almost linearly growing $q = 1$ case [46].

To summarize, we have shown that supersymmetry is (almost) always present in the SYK model with generic four-body interactions. It is only absent in those classes without particle-hole symmetry, i.e., in classes AI and AII. The type of SUSY, in particular the number $\mathcal{N}_{\mathrm{loc}}$ of local supercharges and their symmetry properties, follow a pattern that finds a natural interpretation when $\Gamma_{j\leq\mathcal{N}_{\mathrm{loc}}}$ are viewed as emergent Majorana fermions in a one-dimensional topological phase with SYK model boundary physics. These SUSY features all link directly to features in time-dependent correlation functions of fermion parity-odd observables. For q-body retarded correlation functions, this includes the shape of the ramp in the short-time regime, due to a link between $\mathcal{N}_{\mathrm{loc}}$ and the Dyson index β, and the value of the long-time plateau due to the imprint of how Γ_j transforms under $T_\pm$ on its microscopic Majorana structure. These q-body correlation functions, even with large q, can be measured in digital quantum simulation of the SYK model [55]. The single-particle Green's function ($q = 1$) is accessible through scanning tunneling spectroscopy [20,21]. We stress that the features in the correlation functions are dynamical consequences of SUSY, which are less frequently considered than ground-state consequences [56,57].

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