Entanglement Oscillations near a Quantum Critical Point

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We study the dynamics of entanglement in the scaling limit of the Ising spin chain in the presence of both a longitudinal and a transverse field. We present analytical results for the quench of the longitudinal field in the critical transverse field which go beyond current lattice integrability techniques. We test these results against a numerical simulation on the corresponding lattice model finding extremely good agreement. We show that the presence of bound states in the spectrum of the field theory leads to oscillations in the entanglement entropy and suppresses its linear growth on the time scales accessible to numerical simulations. For small quenches, we exactly determine these oscillatory contributions and demonstrate that their presence follows from symmetry arguments. For the quench of the transverse field at zero longitudinal field, we prove that the Rényi entropies are exactly proportional to the logarithm of the exponential of a time-dependent function, whose leading large-time behavior is linear, hence, entanglement grows linearly. We conclude that, in the scaling limit, linear growth and oscillations in the entanglement entropies can not be simply seen as consequences of integrability and its breaking, respectively.

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Introduction.—Over the past two decades, one-dimensional many-body quantum systems far from equilibrium have become ubiquitous laboratories for scrutinizing fundamental aspects of statistical mechanics. Out-of-equilibrium protocols featuring unitary dynamics, such as quantum quenches, have been commonly employed to test relaxation and thermalization hypotheses [1,2] in experimentally realizable setups [3–6]. A powerful theoretical device that tests whether a physical system can eventually approach equilibrium is represented by its entanglement dynamics [7]. In $1+1$ dimensions, the linear-in-time increase of the entanglement entropies is a signature that local observables relax exponentially fast [8–10] and thermalize [11,12]. This characteristic growth has been further conjectured to be a generic feature of integrable models [13], where it has been analyzed within a quasiparticle picture, inspired by conformal field theory [7] and free fermion calculations [14]. In such a framework, entangled quasiparticle pairs propagate freely in space-time and generate linearly growing entropies. Minimal models, with random unitary dynamics, that show analogous entanglement growth, have also been studied [15,16] in connection with quantum chaos [17,18] and nonintegrable systems.

Nonetheless, there exists a vast class of one-dimensional systems that escape this paradigm and fail to relax at large times after the quench. They have been observed both in earlier studies [19] and in actual recent experiments [20], see, also, [21]. Within a qualitative quasiparticle picture [22], absence of thermalization has been associated with integrability breaking interactions and confinement. Numerical studies in the Ising spin chain [22] and its scaling limit [23–26] indicated that, in the presence of a longitudinal field entanglement, growth is strongly suppressed, while local observables feature persistent oscillations whose frequencies coincide with the meson masses. A similar lack of relaxation has also been found later in a variety of physical models spanning from gauge theories [27–30] and fractons [31] to Heisenberg magnets [32] and systems with long-range interactions [33]. However, despite a large number of numerical investigations, most of the understanding of whether and how local observables will equilibrate after a quench remains at a phenomenological level. This is mainly because no lattice integrability technique [34] is available to systematically analyze these strongly interacting systems.

In this Letter, we put forward a unified picture to perturbatively address questions about entanglement dynamics and relaxation in gapped $1+1$ dimensional systems close to a quantum critical point (QCP). In particular, for the first time, we provide an analytical grasp on how entanglement growth can be so dramatically different depending on the nonequilibrium protocol considered. The formalism combines the perturbative approach of [35,36] with the mapping in the scaling limit between

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powers of the reduced density matrix and correlation functions of a local field, called the branch point twist field [37,38]. For massive systems, this mapping has been successfully employed at equilibrium [39] and, also, recently, in a time-dependent context [40]. Crucially, its conclusions do not rely on any *a priori* assumption about the space-time evolution of the quasiparticles.

Focusing on the illustrative example of a quench of the longitudinal field in the ferromagnetic Ising spin chain, we will provide examples of how bound state formation and symmetries of the twist field are responsible for slow relaxation of local observables and oscillations in the entanglement entropies. Although derived through field theory techniques, our results are numerically tested in the lattice model in the scaling limit, and very good agreement is found.

Model.—Consider the ferromagnetic Ising spin chain defined by the Hamiltonian

$$H_{\text{lattice}} = -\sum_{n\in\mathbb{Z}} [\sigma_n^x \sigma_{n+1}^x + h_z \sigma_n^z + h_x \sigma_n^x]. \quad (1)$$

The Ising chain is a prototype of a quantum phase transition with spontaneous breaking of $\mathbb{Z}_2$ symmetry and is critical for $h_z = 1$ and $h_x = 0$. At criticality, the low energy excitations are massless free Majorana fermions described by a conformal field theory with central charge $c = 1/2$ [41]. Within the renormalization group framework, near the QCP, expectation values of local operators in the spin chain can be calculated from the relativistic quantum field theory (QFT) action

$$\mathcal{A}_0 = \mathcal{A}^{\text{CFT}} - \lambda_1 \int dx dt \varepsilon(x,t) - \lambda_2 \int dx dt \sigma(x,t), \quad (2)$$

which is the celebrated Ising field theory (IFT) [42–44]. In Eq. (2), the conformal invariant action $\mathcal{A}^{\text{CFT}}$ is perturbed by the $\mathbb{Z}_2$ even field ε (energy), which is the continuum version of the lattice operator σ_n^z, and the $\mathbb{Z}_2$ odd field σ (spin), which is, instead, the continuum version of the order parameter σ_n^x. The coupling constant λ_1 is proportional to the deviation of the transverse field from its critical value $h_z - 1$, while λ_2 is proportional to the longitudinal field h_x. At the QCP, the scaling dimension of σ is $\Delta_\sigma = 1/8$ and that of ε is $\Delta_\varepsilon = 1$.

Let $|\Omega\rangle$ be the ground state of the Hamiltonian H of the field theory (2). Following a widely studied nonequilibrium protocol, dubbed quantum quench, at time $t = 0$, one of the two coupling constants λ_i ($i = 1, 2$) is modified according to $\lambda_i \to \lambda_i + \delta_\lambda$. The evolution of the prequench ground state $|\Omega\rangle$ is governed by the perturbed Hamiltonian

$$G(t) \coloneqq H + \theta(t)\delta_\lambda \int dx \Psi(x), \quad (3)$$

Ψ being either the spin or the energy field and $\theta(t)$, the Heaviside step function. This dynamical problem is analytically not solvable in general [35]. Thus, to provide theoretical predictions for local observables and entanglement entropies following a quench, one sets up a perturbative expansion in the relative quench parameter $(\delta_\lambda/\lambda_i) \ll 1$.

Perturbation theory.—We revisit and extend the perturbative approach to the quench problem [35] to include entanglement calculations. In a relativistic scattering theory, it is possible to consider a basis of in and out states, denoted by $|\alpha\rangle^{\text{in-out}}$, which are multiparticle eigenstates of the Hamiltonian $H = G(-\infty)$. In particular, a single-particle eigenstate of H has energy $e(p) = \sqrt{m_0^2 + p^2}$, where m_0 is its prequench mass and p the momentum. Similar eigenbases are constructed for the post-quench Hamiltonian $H_{\text{post}} \coloneqq H + \delta_\lambda \int dx \Psi(x) = G(\infty)$. In this case, the energy of a single-particle state will be denoted by $\tilde{e}(p) = \sqrt{m^2 + p^2}$, being m its postquench mass.

The initial state $|\Omega\rangle$ can be formally expanded into the basis of the out-states of H_{post} as: $|\Omega\rangle = \sum_\alpha c_\alpha |\alpha\rangle_{\text{post}}^{\text{out}}$. Assuming, for simplicity, a unique family of particles, $|\alpha\rangle_{\text{post}}^{\text{out}}$ is then the multiparticle out-state $|p_1, \ldots, p_n\rangle_{\text{post}}^{\text{out}}$ while the symbol $\sum_\alpha$ is a shorthand notation for the Lorentz invariant integration measure in $1+1$ dimensions [45]. The overlap coefficients c_α are the elements of the scattering matrix for the quench problem in Eq. (3). At first-order in perturbation theory, one has [46,47]

$$|\Omega\rangle = |\Omega\rangle_{\text{post}} + 2\pi\delta_\lambda \sum_{\alpha\neq\Omega} \frac{\delta(P^\alpha)}{E^\alpha} (F_\alpha^\Psi)^* |\alpha\rangle_{\text{post}}^{\text{out}} + \mathcal{O}(\delta_\lambda^2). \quad (4)$$

In Eq. (4), E^α and P^α are the prequench energy and momentum of the state $|\alpha\rangle^{\text{out}}$; $\delta(x)$ is the Dirac delta and the function F_α^Ψ is the form factor: $F_\alpha^\Psi \coloneqq \langle\Omega|\Psi(0,0)|\alpha\rangle^{\text{in}}$, calculated in the prequench theory. From the expansion in Eq. (4), it is straightforward to derive the postquench evolution of a local operator Φ

$$\begin{aligned}\langle\Omega|\Phi(0,t)|\Omega\rangle &= {}_{\text{post}}\langle\Omega|\Phi(0,0)|\Omega\rangle_{\text{post}} \\ &\quad + 4\pi\delta_\lambda \sum_{\alpha\neq\Omega} \frac{\delta(P^\alpha)}{E^\alpha} \text{Re}[e^{-itE_{\text{post}}^\alpha} (F_\alpha^\Psi)^* \langle\Omega|\Phi(0,0)|\alpha\rangle_{\text{post}}^{\text{out}}] \\ &\quad + \mathcal{O}(\delta_\lambda^2), \end{aligned} \quad (5)$$

with now E_{post}^α the energy of the state $|\alpha\rangle_{\text{post}}^{\text{out}}$. At $\mathcal{O}(\delta_\lambda)$, one can replace $|\alpha\rangle_{\text{post}}^{\text{out}}$ by $|\alpha\rangle^{\text{out}}$ inside the sum in Eq. (5) and, by using known properties of the form factors, it is also possible to relax the ordering prescription on the momenta of the out-states [48]. We will denote, then, by $\sum'_\alpha$, a Lorentz invariant integration over the prequench multiparticle states with unrestricted momenta [49]. The leading order correction to the one-point function of a local operator after a quench is, therefore, [35,36]

$$\begin{aligned}\langle\Omega|\Phi(0,t)|\Omega\rangle\\ &= {}_{\text{post}}\langle\Omega|\Phi(0,0)|\Omega\rangle_{\text{post}}\\ &\quad+4\pi\delta_\lambda{\sum_{\alpha\neq\Omega}}'\frac{\delta(P^\alpha)}{E^\alpha}\mathrm{Re}[e^{-itE^\alpha_{\text{post}}}(F^\Psi_\alpha)^*F^\Phi_\alpha]+\mathcal{O}(\delta_\lambda^2).\end{aligned}\tag{6}$$

Now, consider a semi-infinite spatial bipartition of the Hilbert space of the QFT associated to the quench problem (3). In particular, let $\mathcal{L}$ be the semi-infinite negative real line and $\mathcal{R}$ the semi-infinite positive real line and denote by $\rho_{\mathcal{R}}(t) \coloneqq \mathrm{Tr}_{\mathcal{L}}[e^{-iH_{\text{post}}t}|\Omega\rangle\langle\Omega|e^{iH_{\text{post}}t}]$, the reduced density matrix after the quench obtained tracing over the left degrees of freedom. In QFT, half-space Rényi entropies after a quench $S_n(t) \coloneqq (1/1-n)\log[\mathrm{Tr}_{\mathcal{R}}\rho^n_{\mathcal{R}}(t)]$ are related to the one-point function of the twist field $\mathcal{T}_n$ [37,38] by

$$S_n(t)=\frac{1}{1-n}\log\,[\epsilon^{\Delta_{\mathcal{T}_n}}\langle\Omega|\mathcal{T}_n(0,t)|\Omega\rangle].\tag{7}$$

In Eq. (7), ϵ is a short distance cutoff and $\Delta_{\mathcal{T}_n}=(c/12)(n-n^{-1})$ is the scaling dimension of the twist field at the QCP [50–52]. The von Neumann entropy $S(t)$ is defined through the limit $S(t)\coloneqq\lim_{n\to1}S_n(t)$.

In writing Eq. (7), a new difficulty arises: the expectation value of the twist field has to be calculated in an n-fold replicated QFT, and the time evolution of the twist field after the quench is governed by the replicated Hamiltonian $\sum_{r=1}^n H^{(r)}_{\text{post}}=\sum_{r=1}^n(H^{(r)}+\delta_\lambda\int dx\Psi^{(r)})$, r being the replica index. However, when calculating the overlaps ${}^{\text{out}}_{\text{post}}\langle\alpha|\Omega\rangle$ at first order in δ_λ, the sum over the replica trivializes since the perturbing field $\Psi^{(r)}$ has nonvanishing matrix elements only between particles in the same copy. Therefore, one gets Eq. (4) with a prefactor n in front of the sum, which only involves states within one particular replica, for instance the first. By repeating the derivation of Eq. (6) now, we conclude that the leading order expansion of the twist field one-point function after a quench is

$$\begin{aligned}\langle\Omega|\mathcal{T}_n(0,t)|\Omega\rangle\\ &= {}_{\text{post}}\langle\Omega|\mathcal{T}_n(0,0)|\Omega\rangle_{\text{post}}\\ &\quad+4\pi n\delta_\lambda{\sum_{\substack{\alpha\neq\Omega;\\ \alpha\in 1\text{st rep.}}}}'\frac{\delta(P^\alpha)}{E^\alpha}\mathrm{Re}[e^{-itE^\alpha_{\text{post}}}(F^\Psi_\alpha)^*F^{\mathcal{T}_n}_\alpha]+\mathcal{O}(\delta_\lambda^2),\end{aligned}\tag{8}$$

where, as indicated, the sum only contains states in the first replica. Similar to the discussion around Eq. (6), $F^{\mathcal{T}_n}_\alpha$ in Eq. (8) denotes the prequench twist-field matrix element $F^{\mathcal{T}_n}_\alpha=\langle\Omega|\mathcal{T}_n(0,0)|\alpha\rangle^{\text{in}}$.

Longitudinal field quench.—We examine a quench along the vertical axis of the phase diagram of the IFT depicted in Fig. 1. This quench, cf. Eq. (2), involves a sudden change of

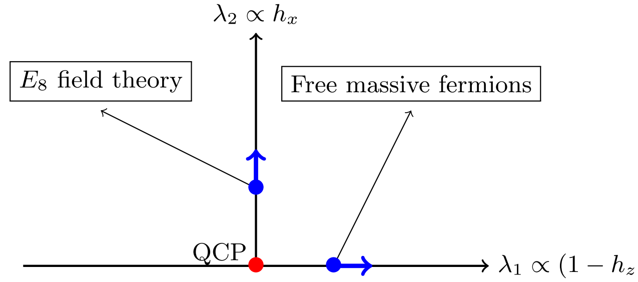

FIG. 1. Phase diagram of the IFT, described by the action (2). We consider applications of the perturbation theory to a quench of the longitudinal field $h_x \propto \lambda_2$, while keeping the transverse field $(1-h_z)\propto\lambda_1$ at its critical value, i.e., $\lambda_1=0$. In the scaling limit, the prequench theory is integrable and corresponds to the E_8 field theory. For $\lambda_2=0$, the prequench theory can be mapped to noninteracting fermions with mass λ_1.

the coupling $\lambda_2\to\lambda_2+\delta_\lambda$ while keeping $\lambda_1=0$. In the lattice model described by Eq. (1), it modifies the longitudinal field $h_x\to h_x+\delta_{h_x}$ at fixed transverse field $h_z=1$. In the presence of a longitudinal field, the Ising spin chain is strongly interacting and the perturbative approach is the only analytical device for studying entanglement dynamics.

From a QFT perspective, at $\lambda_1=0$ and $\lambda_2\neq0$, both the prequench and postquench theories are integrable. The spectrum contains eight stable particles [42,53], whose masses are in correspondence with the components of the Perron-Frobenius eigenvector of the Cartan matrix of the Lie algebra E_8. We will refer to such a field theory, in short, as the E_8 field theory, see Fig. 1. The masses of the eight particles have been partially measured experimentally [54] and numerically estimated in the scaling limit using matrix product states [55]. In the E_8 field theory, both the spin operator and the twist field couple to the eight one-particle states. Equations (6) and (8), predict, in this case, that at $\mathcal{O}(\delta_\lambda)$, the one-point function of the spin and the entanglement entropies must oscillate in time without relaxing. The first-order result for the order parameter [36] is reobtained in [56]. For the time evolution of the entanglement entropies, perturbation theory, combined with Eq. (7), gives at large times

$$\begin{aligned}S_n(t)-S_n(0)&\overset{t\gg1}{=}\frac{\delta_\lambda}{\lambda_2}\left[\frac{2n\mathcal{C}_\sigma}{1-n}\sum_{a=1}^{8}\frac{\hat{F}^\sigma_a\hat{F}^{\mathcal{T}_n}_a}{r_a^2}\cos(mr_at)\right.\\ &\quad\left.+\frac{1}{1-n}\frac{\Delta_{\mathcal{T}_n}}{2-\Delta_\sigma}\right]+\mathcal{O}(\delta_\lambda^2).\end{aligned}\tag{9}$$

where the coefficient [63] $\mathcal{C}_\sigma=-0.065841\ldots$, and the (real) normalized prequench one-particle form factors of the spin field [64], $\hat{F}^\sigma_a$, and the twist field [65], $\hat{F}^{\mathcal{T}_n}_a$ are also summarized in [56]. The universal ratios r_a in Eq. (9) are the masses of the particles in the E_8 field theory normalized by the mass of the lightest particle, whose value after the quench is m. It is finally possible [56] to extrapolate the results for the Rényi entropies to $n \to 1$ and predict the long-time limit of the von Neumann entropy. There are subleading corrections in time to Eq. (9) of order $t^{-3/2}$ (but of leading order in δ_λ) which are discussed in [56].

The field theoretical result in Eq. (9) can be tested against numerical simulations through matrix product states in the Ising spin chain near the QCP. For finding the initial state and for the time evolution, we use the iTEBD algorithm [66,67] extrapolated to the scaling limit, details are given in [56]. In a nonequilibrium protocol, the longitudinal field is quenched from h_x to $h_x + \delta_{h_x}$ with $\delta_{h_x}/h_x = \delta_\lambda/\lambda_2 = -0.04$, 0.05. Because of the absence of visible linear growth of the entanglement entropies [56], the simulation can reach large enough time to carry out a Fourier analysis. The nonuniversal mass coupling relation is obtained by fitting the numerical data for the order parameter to the theoretical curve given in [56], and we have $m \approx 5.42553(h_x + \delta h_x)^{8/15}$, consistent with earlier estimates [55,68].

According to Eq. (9), in the scaling limit, with time measured in units of m^{-1}, the time evolution of entanglement entropies should follow a universal curve. The numerical results for real time evolution in the scaling region are summarized in Fig. 2 for the von Neumann and the second Rényi entropy, showing excellent agreement with theoretical predictions obtained from Eq. (9). The curves for the entanglement entropies have been shifted vertically by an empirical value to account for higher order corrections [40] to the twist field postquench expectation value, cf. Eq. (39) in [56]. In Fig. 3, we also show the numerical Fourier spectrum of the von Neumann entropy calculated from extrapolated data up to $mt = 170$. The Fourier transform was carried out with respect to the rescaled time mt, therefore, the main frequency is at $\tilde{\omega} = 1$ for both quenches. The various peaks are related to the mass ratios of different particles summarized in [56]. For infinite time, the one particle peaks would be δ-function peaks, but for finite time they have finite height. The height ratios are related to form factors of the longitudinal field and the twist fields through Eq. (9). The horizontal line in Fig. 3 related to the lightest particle is set by hand, the ones related to m_2, m_3, and m_4 are calculated from the form factors given in [56].

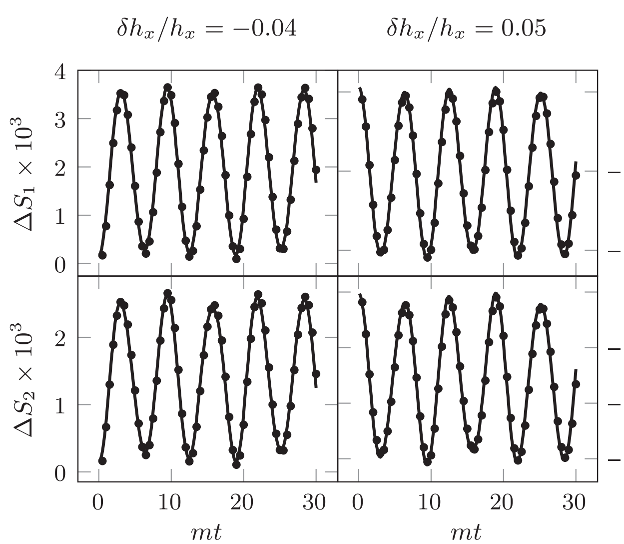

FIG. 2. The time evolution of the von Neumann entropy (top) and the second Rényi entropy (bottom) differences $\Delta S_n = S_n(t) - S_n(0)$ for quenches with $\delta h_x/h_x = -0.04$ (left) and $\delta h_x/h_x = 0.05$ (right). The dots are the extrapolated iTEBD data. Lines are the theoretical prediction from Eq. (9) ($n \to 1$ limit for von Neumann), up to the first four particles in the sum, and incorporating the two particle contributions given in [56].

Transverse field quench.—Now, we consider a quench of the transverse field $h_z \to h_z + \delta_{h_z}$ for longitudinal field $h_x = 0$. In the IFT, see Fig. 1, this protocol displaces along the horizontal axis of the phase diagram: $\lambda_1 \to \lambda_1 + \delta_\lambda$, modifying the mass of the Majorana fermion [9,69]. The ground state $|\Omega\rangle$ of the prequench theory can be expanded in the postquench quasiparticle basis as

$$|\Omega\rangle = \exp\left[\int_0^\infty \frac{\mathrm{d}p}{2\pi\tilde{e}(p)} \tilde{K}(p) a^\dagger_{\mathrm{post}}(-p) a^\dagger_{\mathrm{post}}(p)\right] |\Omega\rangle_{\mathrm{post}}, \tag{10}$$

where the function $\tilde{K}(p)$ is given in [69] and $a^\dagger_{\mathrm{post}}(p)$ are postquench fermionic creation operators. Because of the properties of the free fermionic form factors, the expectation value of the twist field exponentiates

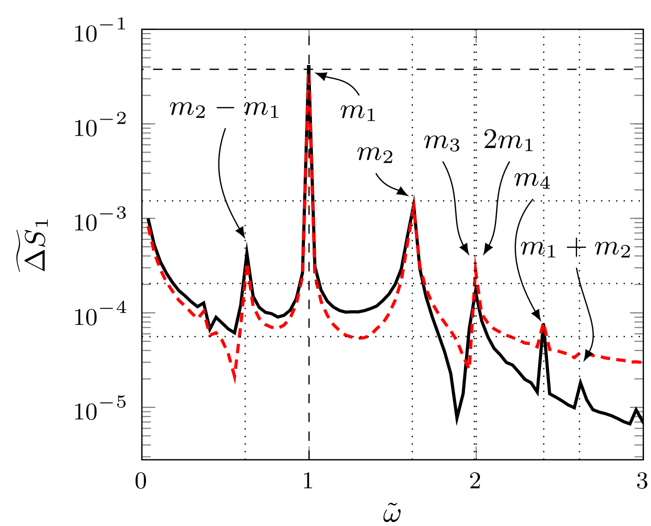

FIG. 3. Numerical Fourier transform of the variation of the von Neumann entropy (related to the variable mt) for quenches with $\delta h_x/h_x = 0.05$ (solid) and $\delta h_x/h_x = -0.04$ (dashed). Vertical lines indicate different frequencies. The horizontal lines mark the peaks corresponding to the masses of the four lightest particles. From top to bottom they correspond to m_1, m_2, m_3, and m_4, respectively. The dashed horizontal line is set by hand, and the three dotted horizontal lines were calculated from the ratios of the one-particle form factors based on Eq. (9).

$$\frac{\langle\Omega|\mathcal{T}_n(0,t)|\Omega\rangle}{{}_{\rm post}\langle\Omega|\mathcal{T}_n(0,0)|\Omega\rangle_{\rm post}} = \exp\left[\sum_{k,l=0}^{\infty} D^c_{2k,2l}(t)\right]. \qquad (11)$$

For a proof of Eq. (11), we refer to the Supplemental Material [56]. The amplitudes $D^c_{2k,2l}(t)$ contribute at leading order $[(\delta_\lambda/\lambda_1)]^{k+l}$ in perturbation theory and can be systematically computed. Differently from Eq. (9), the oscillatory first-order term is $O(t^{-3/2})$ for large time, while in [40], it was shown that $D^c_{2,2}(t) = [(\delta_\lambda/\lambda_1)]^2[-|A|t + \mathcal{O}(1)]$. In the absence of interactions, exponentiation of second order contributions then leads to linear growth of the Rényi entropies as a by-product of relaxation of the twist-field one-point function.

Discussion.—Absence of relaxation of the order parameter and persistent oscillations in the entanglement entropies have been observed previously in several numerical investigations of the Ising spin chain and its scaling limit [22–26]. In this Letter, we formulated a new first principle perturbative approach which quantitatively explains these phenomena. Persistent oscillations in the one-point function of a local observable are only possible if it can create a single quasiparticle excitation of the postquench Hamiltonian. This is a necessary condition that is never satisfied in absence of interactions, as also emphasized in [35]. By mapping entanglement entropies into correlation functions of a local field, the twist field, we then provided an analogous criterion for understanding when entanglement growth can slow down.

An important question is whether the exponentiation of higher orders in perturbation theory will damp the oscillatory first-order result for local observables derived in Eqs. (6) and (8). The time-scale for this to happen defines the relaxation time which is model dependent and relates specifically to the analytic structure of the overlaps with the initial state [70,71]. For instance, see Eq. (11), for mass quenches in free theories, the relaxation time can be calculated starting from the second order in perturbation theory. To test the robustness of the first-order result in an interacting theory, we performed, in [56], additional numerical simulations. They indicate that, along the E_8 line, see Fig. 1, the order parameter σ^x does not relax and the entanglement entropies still show long-living oscillations also when the quench parameter $\delta h_x/h_x$ is of order one. Remarkably then, even after a large longitudinal field quench, the late-time dynamics continues to be qualitatively captured by first-order perturbation theory.

The formalism in this Letter allows calculating entanglement entropies in the scaling limit for any interacting massive theory without fine-tuning of the initial state. *A priori*, this includes nonintegrable models, even if the development of a perturbation series will generally be more challenging. It could be adapted to quench protocols in absence of translation invariance [72], and we believe that it will be useful for other one-dimensional systems [28,33,73] that show similar long-living oscillations.

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