## Comment on "Effective Confining Potential of Quantum States in Disordered Media"

In the Letter [1], the inverse of the landscape function $u(x)$ introduced in Ref. [2] was shown to play the role of an effective potential. This leads to the following estimation of the integrated density of states (IDoS), in one dimension,

$$\mathcal{N}_{\mathrm{ADJMF}}(E)=\frac{1}{\pi}\int_{u(x)>1/E}dx\sqrt{E-1/u(x)}. \qquad (1)$$

We consider here two disordered models for which we obtain the distribution of $u(x)$ and argue that the precise spectral singularities are not reproduced by Eq. (1).

*Pieces model.*—We consider the Schrödinger Hamiltonian $H=-d^2/dx^2+\sum_n v_n\delta(x-x_n)$, where the positions of the $\delta$ potentials are independently and uniformly distributed on $[0,L]$ with mean density $\rho$. The landscape function, which solves $Hu(x)=1$, is thus parabolic on each free interval. In the limit $v_n\to+\infty$ ("pieces model"), intervals between impurities decouple and IDoS per unit length is $N(E)=\lim_{L\to\infty}(1/L)\mathcal{N}(E)=\rho/[e^{\pi\rho/\sqrt{E}}-1]$ [3]. We compare it with Eq. (1). Assuming now ordered positions, $x_1<x_2<\cdots$, we have $u(x)=(1/2)(x-x_{n-1})(x_n-x)$ for $x\in[x_{n-1},x_n]$. We first study its distribution $P(u)=\langle\delta(u-u(x))\rangle$. The disorder average can be replaced by a spatial average, $P(u)=\rho^2\int_0^\infty d\ell e^{-\rho\ell}\int_0^\ell dx\,\delta[u-x(\ell-x)/2]$, leading to

$$P(u)=4\rho^2K_0(\rho\sqrt{8u}), \qquad (2)$$

where $K_\nu(z)$ is the MacDonald function. Denoting by $\theta_H(x)$ the Heaviside function, we can now deduce the estimate $N_{\mathrm{ADJMF}}(E)=(1/\pi)\langle\sqrt{E-1/u}\,\theta_H(E-1/u)\rangle$:

$$N_{\mathrm{ADJMF}}(k^2)=\frac{k}{\pi}\int_\xi^\infty dt\sqrt{t^2-\xi^2}K_0(t)\ \ \text{for}\ \xi=\frac{\rho\sqrt{8}}{k}. \qquad (3)$$

For $k=\sqrt{E}\gg\rho$, we get $N_{\mathrm{ADJMF}}(k^2)\simeq k/\pi$, as it should. For low energy, $k\ll\rho$, one gets $N_{\mathrm{ADJMF}}(k^2)\simeq(k/2)\times\exp\{-\sqrt{8}\rho/k\}$, which is a rather poor approximation of the Lifshitz tail $N(k^2)\simeq\rho\exp\{-\pi\rho/k\}$: the coefficient in the exponential is underestimated and the preexponential function incorrect, thus overestimating the IDoS by an exponential factor.

*Supersymmetric quantum mechanics.*—We consider the Hamiltonian [4] $H=Q^\dagger Q$, where $Q=-\partial_x+m(x)$. The analysis is more simple for boundary conditions $\psi(0)=0$ and $Q\psi(L)=0$, leading to the Green's function $G(x,y)=\langle x|H^{-1}|y\rangle=\psi_0(x)\psi_0(y)\int_0^{\min(x,y)}dz\psi_0(z)^{-2}$, where $\psi_0(x)=\exp\{\int_0^x dtm(t)\}$. We study $u(x)=\int_0^L dy\,G(x,y)$, when $m(x)$ is a Gaussian white noise with $\langle m(x)\rangle=\mu g$ and $\langle m(x)m(x')\rangle_c=g\,\delta(x-x')$, thus $B(x)=\int_0^x dt\,m(t)$ is a Brownian motion (BM) with drift $\mu$ [in Ref. [5], the more regular case with $m(x)$ being a random telegraph process was considered, leading to the same low energy properties]. We have

$$u(x)=e^{B(x)}\left\{\int_0^x dy\,e^{B(y)}\int_0^y dz\,e^{-2B(z)}+\int_0^x dy\,e^{-2B(y)}\int_x^L dz\,e^{B(z)}\right\}\equiv u_<(x)+u_>(x). \qquad (4)$$

The cases $\mu\geq0$ and $\mu<0$ are very different: numerical simulations show that the first moments of $\ln u(x)$ grow with $x$ for $\mu\geq0$ [in particular, $\langle\ln u(x)\rangle\simeq\mu gx+\mathrm{cst}$ for $\mu>0$], while they remain uniform (apart near boundaries) for $\mu<0$. We first discuss the term $u_>(x)=\int_x^L dy\,G(x,y)$ of Eq. (4), which is the product of two *independent* exponential functionals of the BM $u_>(x)\overset{\text{(law)}}{=}(4/g^2)Z_{gx}^{(-\mu)}\tilde{Z}_{g(L-x)/4}^{(-2\mu)}$, where $Z_L^{(\mu)}=\int_0^L dt\,e^{-2\mu t+2W(t)}$, $W(t)$ being a Wiener process (a normalized BM with no drift). The $n$th moment of $Z_L^{(\mu)}$ is $\sim e^{2n(n-\mu)L}$ [6], thus $\langle u_>(x)^n\rangle\sim\exp\{\frac{1}{2}n^2g(L+3x)+n\mu g(L+x)\}$, which suggests a log-normal tail. For $\mu\geq0$, there is no limit law and $u_>(x)$ grows exponentially, hence the bound of the landscape approach is useless. For $\mu<0$, $1/Z_\infty^{(-\mu)}$ is distributed by a Gamma law [6] and we get the exact distribution of $u_>(x)$ for $x$ & $L-x\to\infty$:

$$P_>(u)=\frac{2g^{-3|\mu|}u^{-1-3|\mu|/2}}{\Gamma(|\mu|)\Gamma(2|\mu|)}K_{|\mu|}\left(\frac{2}{g\sqrt{u}}\right)\underset{u\to\infty}{\sim}u^{-1-|\mu|}. \qquad (5)$$

$u_<(x)=\int_0^x dy\,G(x,y)$ should have the same statistical properties as confirmed numerically. Although $u_>(x)$ and $u_<(x)$ are correlated, the distribution of their sum is expected to present the same power law tail $P(u)\sim u^{-1-|\mu|}$, what we checked numerically.

We now apply Eq. (1): for $\mu\geq0$, $u(x)$ has not limit law when $x$ and $L-x\to\infty$ and the distribution of $W=1/u(x)$ converges to $\delta(W)$, hence $N_{\mathrm{ADJMF}}(E)=\sqrt{E}/\pi$. For $\mu<0$, we get $N_{\mathrm{ADJMF}}(E)=(1/\pi)\int_{1/E}^\infty duP(u)\times\sqrt{E-1/u}\sim E^{|\mu|+1/2}$ for $E\to0$, while the exact IDoS behaves as $N(E)\sim E^{|\mu|}$ [7]. Hence, Eq. (1) predicts a power law with an incorrect exponent, i.e., underestimates the IDoS.

For boundary conditions $\psi(0)=\psi(L)=0$, we have also obtained $P(u)\sim u^{-1-|\mu|}$ and $N_{\mathrm{ADJMF}}(E)\sim E^{|\mu|+1/2}$, independently of the sign of $\mu$ in this case.

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